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Large fluctuations and fluctuational transitions in systems driven by colored Gaussian noise: A high-frequency noise

M. I. Dykman*
Institute of Semiconductors, Academy of Sciences of the Ukrainian S.S.R., Kiev, U.S.S.R.
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The quasistationary distribution and the transition probabilities for systems driven by weak noise are analyzed to logarithmic accuracy. The problem is reduced to the boundary-value problem for the set of ordinary differential equations. The form of the equations is determined by the shape of the power spectrum of noise. They are solved in the limiting cases of small and large times characterizing noise correlations, and also for a high-frequency narrow-band noise. The dependences of the logarithm of the distribution on the parameters and the coordinate of the system are quite specific in the latter case. The logarithms of the transition probability and of the quasistationary distribution in the point separating the ranges of attraction to different stable states differ from each other.

I. INTRODUCTION

Many physical systems can be described as dynamical systems driven by external random forces that result, say, from coupling to surroundings. Features of fluctuations in a system depend substantially on the character of a force. The latter is often modelled by white Gaussian noise, i.e., it is assumed to be δ-correlated in time and to have a frequency-independent power spectrum, respectively. A much more realistic model of a random force is colored Gaussian noise; the correlation time of such noise is finite and the power spectrum $\Phi(\omega)$ has a pronounced dependence on frequency. The latter model holds, in particular, for the systems coupled to a bath; random forces result here from thermal fluctuations, thus they are Gaussian and have finite correlation time. In recent years a number of papers dealing with systems subjected to colored noise have appeared (see Refs. 1–5 and references therein).

The important characteristic of a noise-driven system is its stationary distribution over the phase space (the probability density). When noise is sufficiently weak, this distribution has maxima in the stable equilibrium positions of the system and for Gaussian noise is Gaussian near the maxima, where the equations of motion can be linearized. Far from the stable states it is substantially non-Gaussian, generally speaking. It is formed here by large fluctuations, i.e., by large "outbursts" of noise. In systems with two or more stable states, large fluctuations can give rise also to transitions between the states. For weak external noise the transition probabilities are exponentially small; they are much smaller than the reciprocal relaxation time of the system t_r^{-1} and the reciprocal noise-correlation time t_c^{-1}.

A convenient approach to the analysis of large fluctuations in systems driven by Gaussian noise is based[6] on Feynman's idea[7] of the direct interrelation between the probability densities of the paths (the realizations in time) of the dynamical variables of the system and of the paths of the noise $f(t)$. This interrelation arises from the fact that each path $f(t)$ results in the respective path of the dynamical variables. As a consequence, the probability density of reaching a given point in the phase space of the system at a given instant is determined by the probability density of that path $f(t)$ which brings the system to this point at this instant [by the integral of the probability density functional for the noise over proper $f(t)$, cf. Sec. II]. For a point lying far from the stable states the probability densities for all proper $f(t)$ are very small and differ substantially for different $f(t)$. The probability density of reaching such a point is determined then, with an accuracy to the preexponential factor, by the probability density of the most probable suitable path $f(t)$, i.e., by the optimal fluctuation (optimal outburst) of the noise. In Ref. 6 the outlined approach was developed and explicit expressions for the transition probabilities were obtained for systems driven by white noise; the possibility of generalization to the case of an arbitrary Gaussian noise was stressed (see also Ref. 8).

In the present paper we consider the stationary (quasistationary) distribution far from the stable states and the fluctuational transitions, and analyze their features due to the nonmonotonic power spectrum of the driving noise. Aiming at simplicity we assume the system to be described by one dynamical variable x that satisfies the equation

$$\dot{x} = -U'(x) + f(t) \quad [\dot{x} \equiv dx/dt,\ U'(x) \equiv dU(x)/dx] \ , \tag{1}$$

where $f(t)$ is weak zero-mean Gaussian noise with the frequency-dependent power spectrum

$$\Phi(\omega) = \int_{-\infty}^{\infty} dt \exp(i\omega t)\phi(t), \quad \phi(t) = \langle f(t)f(0)\rangle \ . \tag{2}$$

The quantity $\Phi(\omega)$ is a noise characteristic that is often determined experimentally, so it is important to express the characteristics of fluctuations in a noise-driven system in terms of $\Phi(\omega)$.

In Sec. II the quasistationary distribution of the system subjected to colored noise is analyzed to logarithmic accuracy. The problem is reduced to the boundary-value problem for the set of ordinary differential equations which is solved in limiting cases. The transition probabilities are shown in Sec. III to be determined by the same equations, but the boundary conditions are quite different here. In Sec. IV the equations are solved for the case of a narrow-band high-frequency noise and the features of the probability distribution and the transition are revealed. Section V contains concluding remarks. In the Appendix an alternative formulation of the problems in question is given for the case when $\Phi^{-1}(\omega)$ is a polynomial in ω^2.

II. GENERAL EXPRESSION FOR THE QUASISTATIONARY DISTRIBUTION

We shall suppose that the potential of the system $U(x)$ has two minima as shown in Fig. 1 (the generalization to the case of several minima is straightforward). If the random force $f(t)$ is weak, the system placed initially, e.g., at $x<x_s$ [x_s is the local maximum of $U(x)$] with an overwhelming probability will approach the left minimum of the potential x_1 (the stable state 1). This occurs within a characteristic relaxation time t_r, which equals $1/U''(x_1)$ for sufficiently simple potentials. Then the system will fluctuate about x_1 due to noise. The characteristic correlation time of these fluctuations is obviously $\sim\max(t_r,t_c)$, where t_c is the correlation time of noise. Within the time $\sim\max(t_r,t_c)$ the system "forgets" its initial state and the quasistationary distribution $p_1(x)$ is worked out in the range of attraction to x_1, i.e., in the range $x<x_s$ (and sometimes out of this range; see Sec. IV). This distribution varies over the time W_{12}^{-1}, where W_{ij} is the probability of the noise-induced switching from the ith stable state to the jth one.

Weakness of noise means that the fluctuational mean-square-root displacement of the system is small as compared with the characteristic distances $|x_{1,2}-x_s|$ and that the transition probabilities

$$W_{ij} \ll t_c^{-1}, t_r^{-1}, \quad i,j=1,2 \ . \tag{3}$$

These criteria are fulfilled provided the characteristic noise intensity D, equaling the maximum value of the power spectrum of noise is small,

$$D=\Phi_{\max}(\omega), \quad D\ll 1 \ . \tag{4}$$

D is assumed the smallest parameter of the theory.

When (3) is fulfilled the quasistationary distribution $p_i(x)$ can be expressed easily in terms of the probability density $w(x_b,x_a,t_b-t_a)$ of the transition of the system from the point x_a occupied at the instant t_a to the point x_b at the instant t_b. It follows from the above arguments that in the range

$$t_r, t_c \ll t \ll W_{ij}^{-1} \tag{5}$$

the value of $w(x,x_a,t)$ for x and x_a lying within the range of attraction and not too far from one and the same stable state i is independent of t and the initial coordinate x_a and just gives the quasistationary probability distribution,

$$\begin{aligned} &p_i(x)=w(x,x_a,t) \ , \\ &(x_a-x_s)(-1)^i \gg (Dt_r)^{1/2}|x_i-x_s| \ . \end{aligned} \tag{6}$$

[According to Eqs. (1), (2), and (4) $D^{1/2}$ determines the fluctuational "spreading" of the coordinate x. If the initial position x_a is too close to x_s the system can go to the states 1 and 2 with the probabilities of the same order of magnitude, so this case is excluded when expressing the distribution in terms of $w(x,x_a,t)$.]

The quantity $w(x_b,x_a,t_b-t_a)$ can be written in the form of a path integral,

$$w(x_b,x_a,t_b-t_a)=\int_{x(t_a)=x_a}\mathcal{D}f(t)\mathcal{P}[f(t)]\delta(x(t_b)-x_b) \times\left[\int\mathcal{D}f(t)\mathcal{P}[f(t)]\right]^{-1} . \tag{7}$$

The first integral here is calculated supposing the system to be located in the point x_a at the instant t_a. Equation (7) expresses the obvious fact that $w(x_b,x_a,t_b-t_a)$ is the integral over all realizations of the force $f(t)$ which transfer the system from x_a to x_b over the time t_b-t_a. The weighting factor $\mathcal{P}[f(t)]$ gives the probability of a realization.

By making use of the expressions given in Ref. 7 one can show the probability density functional $\mathcal{P}[f(t)]$ for the zero-mean Gaussian noise with the power spectrum (2) to be of the form

$$\mathcal{P}[f(t)]=\exp\left[-\frac{1}{2D}\int dt\, f(t)F\left[-i\frac{d}{dt}\right]f(t)\right] , \quad F(\omega)=D/\Phi(\omega) \ . \tag{8}$$

In obtaining (8) it was taken into account that $\Phi(\omega)=\Phi(-\omega)$ for a stationary random process [for such

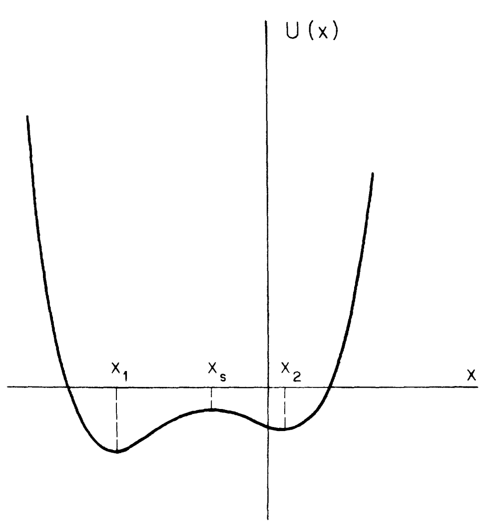

FIG. 1. The double-well potential. The points x_1, x_2, and x_s are the positions of the stable states 1,2 and of the unstable stationary state s, respectively.

a process $\langle f(t)f(t')\rangle$ depends only on $t-t'$], and $\Phi(\omega)$ and its derivatives were assumed smooth [in essence, it was supposed that $\Phi^{-1}(\omega)$ could be expanded in a series in ω^2 converging for finite ω]. The operator $F(-id/dt)$ is then self-adjoint within the class of sufficiently smooth functions $f(t)$ which vanish for $t\rightarrow\pm\infty$ and for which the integral in Eq. (8) exists. Only smooth $f(t)$ are of physical interest, because noise generated by real physical sources does not have singularities. Since the power spectrum $\Phi(\omega)$ is positive for all ω, the argument of the exponential in Eq. (8) is negative for all $f(t)$ (cf. the Appendix).

We note, in particular, that for the widely investigated model[1-5] where the noise correlator decays exponentially in time,

$$\phi(t)=\frac{D}{2t_c}\exp(-|t|/t_c),\quad \Phi(\omega)=D/(1+\omega^2t_c^2)\ , \tag{9}$$

the operator $F(-id/dt)$ is obviously of the form of the Hamiltonian of a quantum oscillator.

The paths $f(t)$ contributing to (7) are determined by the equation of motion (1) with the proper boundary conditions, thus they are independent of the noise intensity D. Therefore in the limit of small D the functional $\mathcal{P}[f(t)]$ is exponentially small for such $f(t)$. As a consequence, the distribution $p_i(x)=w(x,x_a,t)$ is exponentially small as well (for $|x-x_i|$ exceeding greatly the root-mean-square displacement $\propto D^{1/2}$). To find $p_i(x)$ to logarithmic accuracy in D it suffices to find $\mathcal{P}[f(t)]$ for the most probable proper path $f(t)$. The expression for $p_i(x)$ is then of the form

$$p_i(x)=\text{const}\times\exp[-R_i(x)/D]\ , \tag{10}$$

where $R_i(x)$ is given by the solution of the variational problem

$$\begin{aligned}
&R_i(\bar{x})=\min\mathcal{R}_i[f;\bar{x},\bar{t}]\ ,\\
&\mathcal{R}_i[f;\bar{x},\bar{t}]=\tfrac{1}{2}\int_{-\infty}^{\infty}dt\, f(t)F\left[-i\frac{d}{dt}\right]f(t)\\
&\qquad+\int_{-\infty}^{\bar{t}}dt\,\lambda(t)[\dot{x}+U'(x)-f(t)]\ ,\\
&\dot{x}+U'(x)-f(t)=0\ ,
\end{aligned} \tag{11}$$

with the boundary conditions

$$\begin{aligned}
&x(\bar{t})=\bar{x},\ \ x(t)\rightarrow x_i\ \ \text{for } t\rightarrow-\infty\ ,\\
&f(t)\rightarrow 0\ \ \text{for } t\rightarrow\pm\infty\ .
\end{aligned} \tag{12}$$

The functional $\mathcal{R}_i[f;\bar{x},\bar{t}]$ should be minimized with respect to both $f(t)$ and $x(t)$ independently. The first term in $\mathcal{R}_i[f;\bar{x},\bar{t}]$ determines the probability density (8) for a given realization $f(t)$. The second term reflects the interrelation (1) between $x(t)$ and $f(t)$, which is taken into account by using the method of undetermined coefficients; $\lambda(t)$ is just such a coefficient. The instant $\bar{t}$ in (11) at which the system reaches the given point $\bar{x}$ is arbitrary, $R_i(\bar{x})$ is independent of $\bar{t}$. This is a consequence of quasistationarity [cf. Eq. (6) where the distribution in a given point is independent of the moment of reaching this point also].

The boundary conditions (12) for $t\rightarrow-\infty$ correspond to the above physical picture: the system stays for a long time in the point x_i with $f(t)=0$ (with an accuracy to small terms $\propto D^{1/2}$) prior to the optimal fluctuation driving the system to $\bar{x}$ starts. When calculating the distribution $p_i(\bar{x})$, the motion of the system after it has reached the given $\bar{x}$ is out of interest, cf. (6), so $f(t)$ dies out for $t>\bar{t}$. We note that this decay influences the behavior of $f(t)$ for $t<\bar{t}$ via the continuity conditions, i.e., the finiteness of the noise correlation time gives rise to certain "postaction" [in the case of white noise $f(t)$ becomes zero abruptly, $f(t)=0$ for $t>\bar{t}$].

Since $F(\omega)$ is obviously, from Eqs. (4) and (8), independent of the noise intensity D, the function $R_i(x)$ (11) is independent of D as well. The dependence of $p_i(x)$ on D (10) is thus of the activation type, and $R_i(x)$ may be called the "activation energy" of reaching the point x from the stable state i.

Analysis of the variational equations

The set of variational equations describing the extreme paths $x(t),f(t)$ follows from (11) to be of the form

$$\begin{aligned}
&F\left[-i\frac{d}{dt}\right]f(t)-\lambda(t)=0\ \ \text{for } t\le\bar{t}\ ,\\
&F\left[-i\frac{d}{dt}\right]f(t)=0\ \ \text{for } t>\bar{t}\ ,\\
&\dot{\lambda}(t)-U''(x)\lambda(t)=0\ \ \text{for } t\le\bar{t}\ ,\\
&\dot{x}(t)+U'(x)-f(t)=0\ \ \text{for } t\le\bar{t}\ .
\end{aligned} \tag{13}$$

Note that the highest-order derivative in $F(-id/dt)f(t)$ should not be continuous at $t=\bar{t}$.

Equations (13) with the boundary conditions (12) make up the boundary-value problem, which can be solved numerically for arbitrary potential of the system $U(x)$ and power spectrum of noise $\Phi(\omega)$. A simple procedure can be proposed when $F(\omega)\propto\Phi^{-1}(\omega)$ is a polynomial in ω^2 (of a degree N). Since the function $U'(x)$ is linear in $x-x_i$ near the equilibrium position x_i, the set (13) is linear here as a whole and it is possible to seek the solution for $f(t),\lambda(t),x(t)-x_i$ at $t\rightarrow-\infty$ in this case in the form of $\exp(\alpha t)$ with positive $\text{Re}\alpha$. The values of α can be obtained from the secular equation

$$\begin{aligned}
&\alpha_0=U''(x_i)\ ,\\
&F(-i\alpha_n)=0,\quad \text{Re}\alpha_n>0\ \ (n=1,2,\ldots,N)\ .
\end{aligned} \tag{14a}$$

The resulting solution for $f(t),\lambda(t),x(t)$ contains $N+1$ coefficients. They can be determined from the condition that $x(\bar{t})=\bar{x}$ and from N relations between $f(t),df/dt,\ldots,d^{2N-1}f/dt^{2N-1}$ for $t=\bar{t}$. The relations follow from the continuity of these functions and from the fact that the solution of Eq. (13) for $t>\bar{t}$ is of the form

$$f(t)=\sum_{n=1}^{N}\{A_n\exp[\alpha_n(t-\bar{t})]+B_n\exp[-\alpha_n^*(t-\bar{t})]\}\ ,\quad t\ge\bar{t}\ \ (\text{Re}\alpha_n>0)\ . \tag{14b}$$

Indeed, since $f(t)$ should vanish as $t\to\infty$, all A_n in (14b) should equal zero, and this just gives N relations between $f(t)$ and its derivatives for $t=\bar{t}$. The numerical realization of this procedure will be demonstrated elsewhere.

In some limiting cases Eqs. (13) can be solved analytically. We shall consider first the case when the noise correlation time t_c is small as compared with the relaxation time $t_r\sim 1/U''(x_i)$ [there may be several times characterizing the noise correlator $\phi(t)=\langle f(t)f(0)\rangle$, say, the reciprocal decrement and the period of oscillations of $\phi(t)$, see Sec. IV; all of them are assumed small now]. To zeroth order in t_c/t_r the optimal path is given by the expressions

$$f(t)=2U'(x),\quad \lambda(t)=2F(0)U'(x)\ ,$$
$$\dot{x}=U'(x)[x\equiv x(t)],\quad t\le\bar{t},x(\bar{t})=\bar{x}\ . \tag{15}$$

The dependence (15) of x on t is obviously just opposite to that for the system in the absence of noise, when $\dot{x}=-U'(x)$.

At first glance the corrections to (15) should be $\sim t_c^2/t_r^2$, since $F(-id/dt)$ in (13) is a series in $t_c^2d^2/dt^2$. However, there is a correction $\sim t_c/t_r$ which comes from the continuity of $f(t)$ at the moment $\bar{t}$. To find it we use the relation

$$f(t)=\frac{1}{D}\int_{-\infty}^{\infty}dt'\phi(t-t')\lambda(t') \tag{16a}$$

which follows from Eq. (13) with allowance for (2),(8), if one sets

$$\lambda(t)=0\quad\text{for } t>\bar{t} \tag{16b}$$

[note that $\phi(t)/D$ is independent of the noise intensity]. The function $\phi(t)$ is localized within the domain $|t|\lesssim t_c\ll t_r$. Therefore for $t<\bar{t}\,(\bar{t}-t\gg t_c)$, where $\lambda(t)$ is comparatively smooth, $\dot{\lambda}/\lambda\sim t_r^{-1}$, Eq. (16a) results in $f(t)=\lambda(t)/F(0)$ in agreement with (15). The discontinuity of $\lambda(t)$ at $t=\bar{t}$ obvious from Eqs. (15), (16b) gives rise to a "fast" addition δf to $f(t)$,

$$f(t)=2U'(x(t))-\delta f(|t-\bar{t}|)\quad\text{for } t-\bar{t}<0\ ,$$
$$f(t)=\delta f(t-\bar{t})\quad\text{for } t-\bar{t}>0\ , \tag{17}$$
$$\delta f(t)=2F(0)U'(\bar{x})\int_0^{\infty}dt_1\phi(t+t_1)/D\ .$$

Substituting (15), (17) into (11) and replacing $U'(x)$ in Eq. (15) for $\dot{x}$ by $U'(x)-\delta f(|t-\bar{t}|)$ one obtains the following expression for the effective activation energy of reaching a given point $\bar{x}$:

$$R_i(\bar{x})=2F(0)[U(\bar{x})-U(x_i)]+F(0)[U'(\bar{x})]^2\tilde{t}_c\ ,$$
$$\tilde{t}_c=2F(0)\int_0^{\infty}dt\,t\phi(t)/D,\quad |\tilde{t}_c|\ll t_r\ . \tag{18}$$

The first term in $R_i(\bar{x})$ (18) is the well-known result for white-noise-driven systems. The second term gives the first-order correction due to noise correlations. For a particular case of noise with the correlation function of the form (9) $\tilde{t}_c=t_c$ and Eq. (18) goes over into the result of Ref. 3 obtained by a substantially different way. We note that in the general case $\tilde{t}_c$ may be positive or negative depending on the character of $\phi(t)$. Therefore weak correlation of noise either compresses (localizes) the distribution $p_i(x)$ by increasing $R_i(x)$ far from the equilibrium position x_i for $\tilde{t}_c>0$, or widens the distribution for $\tilde{t}_c<0$.

In the opposite case of large correlation time of noise, $t_c\gg t_r$, the system follows $f(t)$ "adiabatically" (cf. Ref. 4). To zeroth order in t_r/t_c we can neglect dx/dt in the latest equation in (13) [for $U''(x)>0$] and replace $\phi(t-t')$ by $\phi(0)$ in (16a). Then

$$R_i(\bar{x})=\frac{D}{2\phi(0)}(U_m')^2,\quad t_c\gg t_r \tag{19}$$

where $|U_m'|$ is the maximum value of $|U'|$ in the interval $(x_i,\bar{x})$. It should be noted, that in the case when U'' changes sign in this interval there arise nonanalytic $[\sim(t_r/t_c)^{2/3}]$ corrections to (19).

III. TRANSITION PROBABILITIES

To logarithmic accuracy the probability W_{ij} of the transition from the ith stable state to the jth one is determined by the optimal fluctuation of noise which drives the system from x_i to x_j. The transition takes place provided the system stays at x_j when the fluctuation has finished, i.e., $f(t)$ and its derivatives have become zero (with an accuracy to terms $\propto D^{1/2}$). The transition occurs also, if $f(t)$ and its derivatives become zero for $x(t)$ lying anywhere in the range of attraction to the state j, including the boundary point x_s [for $|x(t)-x_s|\lesssim(Dt_r)^{1/2}|x_i-x_j|$ the probability of the transition to x_j is $\sim\frac{1}{2}$].

It is just the point x_s where the extreme path of the system described by Eq. (13) should finish in the problem of the transition probability. Indeed, in this point $U'=0$, so there occurs here the slowing down of the motion of the system for $f(t)=0$. The conditions of approaching x_s by the system and of vanishing of $f(t)$ and its derivatives are fulfilled self-consistently as $t\to\infty$, since $U''(x_s)<0$ and thus $\lambda(t)\to 0$ for $x\to x_s$, $t\to\infty$. Therefore the expression for the transition probability is of the form

$$W_{ij}=\text{const}\times\exp(-R_{is}/D)\ ,$$
$$R_{is}=\min\mathcal{R}_i[f;x_s,\bar{t}\to\infty]\ . \tag{20}$$

An alternative proof of Eq. (20) for the important case when $F(\omega)$ is a polynomial in ω^2, so that $x(t)$, $f(t)$, and the derivatives of $f(t)$ can be considered as the components of a Markov process, is given in the Appendix.

The activation dependence of the transition probabilities on the intensity of Gaussian noise was obtained earlier in Refs. 2, 6, and 8. The above path-integral method, which is the generalization of the method,[6] differs from that in Ref. 2 (which is based on the path-integral formulation as well). At the same time the main terms in the functional[2] determining the extreme paths can be obtained from $\mathcal{R}_i$ (11) by eliminating $f(t)$ with the aid of Eq. (16a) and by replacing $\lambda(t)$ by $-i\lambda(t)$. The boundary-value problem (12), (13), and (20) was not formulated in Ref. 2. We stress that as it follows from Eqs. (11), (12), and (20) $R_{is}\ge R_i(x_s)$ in the general case, i.e., the transition probability is not given to logarithmic accuracy by the occupation of the saddle point, it is exponentially smaller for colored noise. The example is given in Sec. IV.

It follows from Eqs. (10)–(13), (15), (19), and (20) that in the limiting cases $t_c/t_r \ll 1$ and $t_c/t_r \gg 1$ the activation energy of the transition $R_{is} \approx R_i(x_s)$. For small t_c/t_r the corrections to R_{is} of the first order in t_c/t_r, obviously from (18), vanish. To the second order in t_c/t_r

$$R_{is} \approx R_i(x_s) \approx 2F(0)[U(x_s)-U(x_i)] + F''(0)\int_{x_i}^{x_s} dx\, U'(x)[U''(x)]^2 ,$$
$$F''(\omega)=\frac{d^2F(\omega)}{d\omega^2}, \quad |F''(0)/F(0)| \ll t_r^2 . \tag{21}$$

For noise with the correlator of the form (9) Eq. (21) gives the known result (cf. Refs. 2 and 3): $F''(0)/F(0)=2t_c^2$ here. In the general case the sign of $F''(0)$ is arbitrary, so the correlation of noise can cause increase or decrease of the activation energy of the transition R_{is}. We note that the signs of the corrections to R_{is} and to $R_i(x)$ $(x \neq x_s)$ can differ from each other [cf. Eq. (26) below].

IV. LARGE FLUCTUATIONS CAUSED BY A NARROW-BAND NOISE

In many physical systems the power spectrum of the external noise $\Phi(\omega)$ contains a relatively narrow peak with a half-width Γ small compared with the position of the maximum ω_0,

$$\Gamma \ll \omega_0 . \tag{22}$$

Incoherent light filtered by a narrow-band color filter is an example of the respective noise. Another well-known example is thermal noise of any resonant system.

We shall model the power spectrum of a narrow-band noise by the expression

$$\Phi(\omega)=4\Gamma\tilde{D}/[(\omega^2-\omega_0^2)^2+4\Gamma^2\omega^2] . \tag{23}$$

This expression gives the power spectrum of the white-noise-driven harmonic oscillator which obeys the following equation of motion:

$$\ddot{f}+2\Gamma\dot{f}+\omega_0^2 f=\xi(t) ,$$
$$\langle \xi(t)\xi(t')\rangle = 4\Gamma\tilde{D}\delta(t-t') . \tag{24}$$

Here Γ has the meaning of the friction coefficient of the oscillator, while ω_0 is the eigenfrequency of the latter. For an oscillator in thermal equilibrium with a bath the characteristic noise intensity

$$\tilde{D}=\omega_0^2\phi(0) \tag{25a}$$

equals temperature in energy units. The effective noise intensity D introduced in (4) is proportional to $\tilde{D}$,

$$D=\frac{\tilde{D}}{\Gamma(\omega_0^2-\Gamma^2)} \quad \text{for } \omega_0^2 \geq 2\Gamma^2 ,$$
$$D=\frac{4\Gamma\tilde{D}}{\omega_0^4} \quad \text{for } \omega_0^2 \leq 2\Gamma^2 . \tag{25b}$$

The correlator $\phi(t)$ of the noise under consideration has two characteristic times, Γ^{-1} and ω_0^{-1}. So, fluctuations in a system driven by such a noise depend on interplay of three characteristic times, i.e., on three dimensionless parameters: Γ/ω_0, Γt_r, and $\omega_0 t_r$. For slow relaxation of the system the probability distribution $p_i(x)$ and the transition probability W_{ij} are given by Eqs. (10), (18), (20), and (21) with

$$\tilde{t}_c=-(\omega_0^2-4\Gamma^2)/2\Gamma\omega_0^2 ,$$
$$F''(0)/F(0)=-4(\omega_0^2-2\Gamma^2)/\omega_0^4 , \tag{26}$$
$$F(0)=\omega_0^4 D/4\Gamma\tilde{D} \quad (\Gamma t_r \gg 1, \quad \omega_0 t_r \gg 1) .$$

The dependence of the main terms in the arguments $R_i(x)/D$ and R_{is}/D of the exponentials determining $p_i(x)$ and W_{ij} on noise parameters in the case (26) is given by $\omega_0^4/4\Gamma\tilde{D}$ according to (18) and (21) [i.e., by $\Phi^{-1}(0)$, that is quite natural in view of the system being most sensitive evidently to fluctuations with frequencies $\omega \lesssim t_r^{-1}$ for broad smooth $\Phi(\omega)$]. The signs of the leading-order corrections to $R_i(x)$ and R_{is} are given by those of $\tilde{t}_c$ and $F''(0)$, respectively. They depend on the ratio ω_0/Γ and are obviously, from (26), opposite for $2\Gamma^2 < \omega_0^2 < 4\Gamma^2$.

The explicit form of the coefficient $D/\phi(0)$ in the expression (19) for $R_i(x)$ in the case of fast relaxation, $\Gamma t_r \ll 1$, $\omega_0 t_r \ll 1$, is evident from Eqs. (25a) and (25b) (cf. Ref. 9).

The most interesting and nontrivial limiting case is that of the narrow-band noise (22) and intermediate relaxation time t_r,

$$\Gamma \ll t_r^{-1} \ll \omega_0 . \tag{27}$$

This case is specific for colored noise with two substantially different characteristic times. It turns out, in particular, that due to the resonant character of noise, fluctuations with frequencies $\omega \approx \omega_0 \gg t_r^{-1}$ influence the system strongly here.

When (27) is fulfilled we can seek the solution of Eq. (13) in the form of a superposition of fast oscillating and smooth terms,

$$\begin{bmatrix} x(t) \\ \lambda(t) \\ f(t) \end{bmatrix} = \begin{bmatrix} x_{\rm sm}(t) \\ \lambda_{\rm sm}(t) \\ f_{\rm sm}(t) \end{bmatrix} + \begin{bmatrix} x_+(t) \\ \lambda_+(t) \\ f_+(t) \end{bmatrix} e^{i\omega_0 t} + \begin{bmatrix} x_-(t) \\ \lambda_-(t) \\ f_-(t) \end{bmatrix} e^{-i\omega_0 t} . \tag{28}$$

The complex amplitudes $x_\pm, \lambda_\pm, f_\pm$, as well as the main terms in $x_{\rm sm}, \lambda_{\rm sm}, f_{\rm sm}$, are supposed to vary slowly in time, i.e., their variation over the time $\sim\omega_0^{-1}$ is assumed small [there are also terms in x, λ, f which oscillate at frequencies $2\omega_0, 3\omega_0, \ldots$, but they are $\sim(\omega_0 t_r)^{-1} \ll 1$ and thus can be neglected]. The equations for the amplitudes $x_\pm, \lambda_\pm$ follow from (13) to be algebraic,

$$x_{\pm}(t)=\pm(i\omega_0)^{-1}f_{\pm}(t)\ ,$$
$$\lambda_{\pm}(t)=\pm(i\omega_0)^{-1}U''_{\pm}\lambda_{\rm sm}(t)\ ,$$
$$U''_{\pm}\equiv U''_{\pm}(x_{\rm sm},x_+,x_-)=\frac{\partial^2 U_{\rm sm}}{\partial x_{\rm sm}\partial x_{\mp}}\ , \qquad (29)$$
$$U_{\rm sm}\equiv U_{\rm sm}(x_{\rm sm},x_+,x_-)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\, U(x_{\rm sm}+x_+e^{i\psi}+x_-e^{-i\psi})\ ,$$

while those for $f_{\pm}$ are the second-order differential equations,

$$\ddot{f}_{\pm}-\Gamma^2 f_{\pm}=-\Gamma^2\lambda_{\pm}\ . \qquad (30)$$

In obtaining (29) and (30) we have taken into account only resonant terms and neglected $\dot{x}_{\pm},\dot{\lambda}_{\pm}$ and $\ddot{f}_{\pm},\Gamma^2\dot{f}_{\pm}$ as compared with $\omega_0 x_{\pm},\omega_0\lambda_{\pm}$ and $\omega_0\dot{f}_{\pm}$, respectively. The terms $U''_{\pm}$ are obviously the components of $U''(x)$ oscillating as $\exp(\pm i\omega_0 t)$, while $U_{\rm sm}$ is the smoothly varying part of the potential $U(x)$.

The smooth parts of x,λ,f obey the equations

$$\dot{x}_{\rm sm}=-U'_{\rm sm}+f_{\rm sm},\quad \dot{\lambda}_{\rm sm}=U''_{\rm sm}\lambda_{\rm sm}\ ,$$
$$f_{\rm sm}=(4\Gamma^2/\omega_0^2)\lambda_{\rm sm}\ ,$$
$$U'_{\rm sm}\equiv U'_{\rm sm}(x_{\rm sm},x_+,x_-)=\frac{\partial U_{\rm sm}}{\partial x_{\rm sm}}\ , \qquad (31)$$
$$U''_{\rm sm}\equiv U''_{\rm sm}(x_{\rm sm},x_+,x_-)=\frac{\partial^2 U_{\rm sm}}{\partial x^2_{\rm sm}}\ .$$

The complex amplitudes $f_{\pm}$ obviously, from (30), vary over the time $\sim\Gamma^{-1}\gg t_r$. This concerns also $x_{\pm}$ according to (29). The force component $f_{\rm sm}$ is seen from (31) to contain the very small factor Γ^2/ω_0^2 and turns out to be small (see below), so it may be neglected in a number of cases, and Eq. (31) for $x_{\rm sm}$ becomes then an algebraic equation

$$U'_{\rm sm}(x_{\rm sm},x_+,x_-)=0\quad (U''_{\rm sm}>0)\ . \qquad (32)$$

This equation corresponds to the smooth part of the coordinate $x_{\rm sm}$ following adiabatically the slowly varying amplitude of the fast oscillations for $\Gamma\ll t_r^{-1}$. In essence, fast oscillations of $x(t)$ with slowly varying amplitude change the effective potential for smooth motion $U_{\rm sm}$, and the smooth component of x occupies the minimum of this potential. The motion here is analogous to some extent to that of a particle in a fast oscillating potential considered by Kapitza.[10] In our case, however, the fast oscillating part of the coordinate is not small. In addition, we have "double" adiabaticity: oscillations are much faster while their amplitude is much slower than the relaxation rate of $x_{\rm sm}$.

A. "Activation energy" $R_i(\bar{x})$ for a narrow-band noise

The activation energy $R_i(\bar{x})$ of reaching a given point $\bar{x}$ is determined from the boundary condition (12), which takes the following form in the variables (28):

$$x_{\rm sm}(0)+2x_+(0)=\bar{x}$$
$$[x_+(0)=x_-(0)\equiv x_+^*(0),\quad \bar{t}=0]\ . \qquad (33)$$

Without loss of generality we have supposed here that $\bar{x}$ is reached at $t=0$; $x_-(0)$ can be made equal to $x_+(0)$ by shifting the time origin in (28) by $\Delta t\sim 1/\omega_0$.

Equations (32) and (33) express $x_{\pm}(0),x_{\rm sm}(0)$ in terms of $\bar{x}$. To find $f(t)$, $\lambda(t)$, and $x(t)$ as a whole we note that for $t\le 0$ the function $\lambda_{\rm sm}(t)$ obviously from (31) and (32) increases with the increment $U''_{\rm sm}>0$, while for $t>0$ one can set $\lambda_{\rm sm}(t)=0$ [cf. Eq. (16b)]. Equation (16a) can be used then, and allowing for discontinuity of $\lambda_{\rm sm}(t)$ and for the interrelation (30) between $f_{\pm}$ and $\lambda_{\pm}$ we arrive at the expression

$$f_{\pm}(t)=\tfrac{1}{2}\Gamma\int_{-\infty}^{0}dt'\exp(-\Gamma|t-t'|)\lambda_{\pm}(t')$$
$$\pm\frac{i\Gamma}{2\omega_0}\lambda_{\rm sm}(0)\exp(-\Gamma|t|)\ .$$

Since $|\lambda_{\pm}(t)|\propto|\lambda_{\rm sm}(t)|$ increase rapidly compared with $f_{\pm}(t)$ for $t\le 0$, the integral here can readily be evaluated to give

$$f_{\pm}(t)=\pm\frac{i\Gamma}{2\omega_0}\lambda_{\rm sm}(0)\left[1-\frac{U''_{\pm}(0)}{U''_{\rm sm}(0)}\right]\exp(-\Gamma|t|)$$
$$[U''_{+}(0)=U''_{-}(0)]\ , \qquad (34)$$

where $U''_{\pm}(0),U''_{\rm sm}(0)$ are the values of $U''_{\pm},U''_{\rm sm}$ for $t=0$, i.e., for the values of $x_{\rm sm},x_{\pm}$ given by (32) and (33). Since $f_{\pm}(0)=\pm i\omega_0 x_{\pm}(0)$, Eqs. (32) and (34) express $\lambda_{\rm sm}(0)$ [and hence $\lambda(t),f(t),x(t)$ allowing for (28),(29)] in terms of $\bar{x}$. Note that $\lambda_{\rm sm}(0)\sim\omega_0/\Gamma$, so $f_{\rm sm}\sim\Gamma/\omega_0\ll 1$ and the neglect of $f_{\rm sm}$ in Eq. (31) for $\dot{x}_{\rm sm}$ is reasonable.

Equations (28), (29), and (34) result in the following expression for the activation energy $R_i(\bar{x})$ (11):

$$R_i(\bar{x})=(2\omega_0^2/\Gamma)x_+^2(0)\ , \qquad (35)$$

where $x_+(0)$ is related to $\bar{x}$ via Eqs. (32) and (33).

The dependence of $R_i(\bar{x})$ on the parameters of noise is given by the coefficient ω_0^2/Γ in (35). We note that the ratio $R_i(\bar{x})/D=2\omega_0^4x_+^2(0)/\tilde{D}$, which gives the argument of the exponential in the expression for the distribution $p_i(\bar{x})$ (10), is independent of the half-width Γ of the peak in the power spectrum $\Phi(\omega)$ for a fixed noise-intensity parameter $\tilde{D}$ (e.g., for a fixed temperature in the case of thermal noise).

The dependence of R_i on $\bar{x}$ for smooth single-well potentials is simple, but rather specific. For example, for a parabolic potential

$$U(x)=\tfrac{1}{2}a_2x^2 \qquad (36a)$$

we have from Eqs. (32), (33), and (35)

$$R(\bar{x})=(\omega_0^2/2\Gamma)\bar{x}^2\ \text{ with } x_{\rm sm}(0)=0,\quad x_+(0)=\tfrac{1}{2}\bar{x}. \qquad (36b)$$

The curvature a_2 of the potential (36a) does not enter $R(\bar{x})$ at all, so the result (36b) differs qualitatively from that for white-noise-driven systems [in the case (36a) Eqs. (13) are linear and can be solved exactly with the result coinciding with (36b) in the limit (27)]. We note that Eq. (36b) holds also for an arbitrary symmetric anharmonic potential $U(x)=\sum_n a_{2n}x^{2n}$, if $U''(x)>0$ for all x.

For bistable potentials the dependence of R_i on $\bar{x}$ turns out to be more complicated. We shall illustrate it by taking the simple model potential

$$U(x)=-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4 \quad (37)$$

as an example. The stable states and the saddle point are given here by

$$x_1=-1,\quad x_2=1,\quad x_s=0\ . \quad (38)$$

The solution of Eqs. (32) and (33) is of the form

$$x_+(0)=\tfrac{1}{10}[2\bar{x}\pm(10-6\bar{x}^2)^{1/2}]\ ,$$
$$x_{\rm sm}(0)=\tfrac{1}{5}[3\bar{x}\mp(10-6\bar{x}^2)^{1/2}]\quad \text{for } x_+^2(0)<\tfrac{1}{6}\ , \quad (39a)$$

$$x_+(0)=\tfrac{1}{2}\bar{x},\quad x_{\rm sm}(0)=0\quad \text{for } x_+^2(0)>\tfrac{1}{6}\ . \quad (39b)$$

The inequalities here follow from the condition $U''_{\rm sm}>0$. The dependence of $x_+(0)$ on $\bar{x}$ given by (39a) is plotted in Fig. 2.

Equations (35) and (39a) give $R_i(\bar{x})$ for not-too-large deviations $|\bar{x}-x_i|$. The upper sign in (39a) refers to $R_1(\bar{x})$, while the lower one refers to $R_2(\bar{x})$. Obviously, $R_1(\bar{x})=R_2(-\bar{x})$. If the system occupies initially, e.g., the state 1, then for sufficiently small $|\bar{x}-x_1|\ll 1$ we have $x_+(0)\sim\bar{x}-x_1$, $x_{\rm sm}(0)-x_1\sim(\bar{x}-x_1)^2$, and $R_1(\bar{x})\propto(\bar{x}-x_1)^2$, cf. Figs. 2 and 3. Note that $x_{\rm sm}(0)-x_1>0$, i.e., the center of vibrations of $x(t)$ shifts towards $x=0$ irrespective of the sign of $\bar{x}-x_1$. This shift makes it impossible for the system to reach the range of sufficiently large $x_1-\bar{x}$ $[\bar{x}<-(\tfrac{5}{3})^{1/2}]$ when moving along the branch (39a),

$$\left|\frac{\partial x_{\rm sm}(0)}{\partial\bar{x}}\right|\to\infty\quad \text{for } \bar{x}\to\pm x_0,\quad x_0=\sqrt{5/3}\ . \quad (40a)$$

In the range $|\bar{x}|\approx x_0$ the above theory is inapplicable obviously. The relation $U''_\pm=\partial U'_{\rm sm}/\partial x_\mp$ combined with the conditions $x_+(0)=x_-(0)$, $U''_+(0)=U''_-(0)$ and with Eqs. (32) and (33) show that

$$\frac{\partial x_{\rm sm}(0)}{\partial\bar{x}}=-\frac{U''_+(0)}{U''_{\rm sm}(0)-U''_+(0)}\ . \quad (40b)$$

Thus $|\partial x_{\rm sm}(0)/\partial\bar{x}|\to\infty$ when $|U''_{\rm sm}(0)-U''_+(0)|\to 0$. In this case $\lambda_{\rm sm}(0)$ is seen from Eqs. (29) and (34) to diverge as well. Therefore the higher-order terms in $\Gamma/\omega_0,(\omega_0 t_r)^{-1}$ should be taken into account when $|U''_{\rm sm}(0)-U''_\pm(0)|$ is small.

For $\bar{x}$ lying below $-x_0$ the function $R_1(\bar{x})$ is given by the solution (39b). So, in the region $\bar{x}\approx -x_0$ $R_1(\bar{x})$ changes sharply [discontinuously to zeroth order in $\Gamma/\omega_0,(\omega_0 t_r)^{-1}$], cf. Fig. 3. Similar "jumps" occur in $R_i(\bar{x})$ for other potentials where $U''_{\rm sm}(0)-U''_\pm(0)$ vanishes at some $\bar{x}$.

One more feature of $R_1(\bar{x})$ obvious from (35) and (39a) lies in $R_1(\bar{x})$ spreading on the domain of attraction to the stable state 2, i.e., on the range $\bar{x}>0$. Reaching a point $\bar{x}>0$ by the system does not imply the transition to the state 2, since for not-too-large $\bar{x}$ the center $x_{\rm sm}$ of the vibrations of $x(t)$ lies for all t in the range of attraction to x_1 [i.e., $x_{\rm sm}(t)<0$], so the system returns to the initial state 1 as $f(t)$ dies out after $x_{\rm sm}(t)+2x_+(t)$ has reached $\bar{x}$. For the potential (37) the quasistationary distribution $p_1(\bar{x})$ given by (10), (35), and (39a) extends up to $\bar{x}=\sqrt{2/3}$ [where $U''_{\rm sm}(0)=0$ for the solution (39a)], the value of $R_1(\sqrt{2/3})$ coincides with the activation energy of the transition from the state 1.

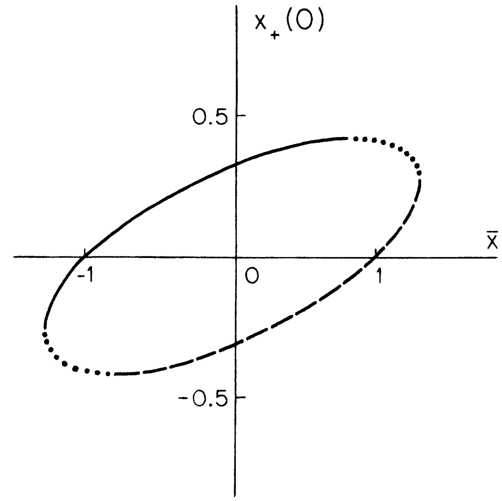

FIG. 2. The dependence (39a) of $x_+(0)$ on $\bar{x}$ for the double-well potential (37) $[\tfrac{1}{2}|x_+(0)|$ is the amplitude of vibrations of the system when the latter reaches a point $\bar{x}]$. Solid and dashed lines refer to the initial states 1 $(x_1=-1)$ and 2 $(x_2=1)$. The dotted lines show the ranges where the interrelation (39a) between $x_+(0)$ and $\bar{x}$ is physically meaningless.

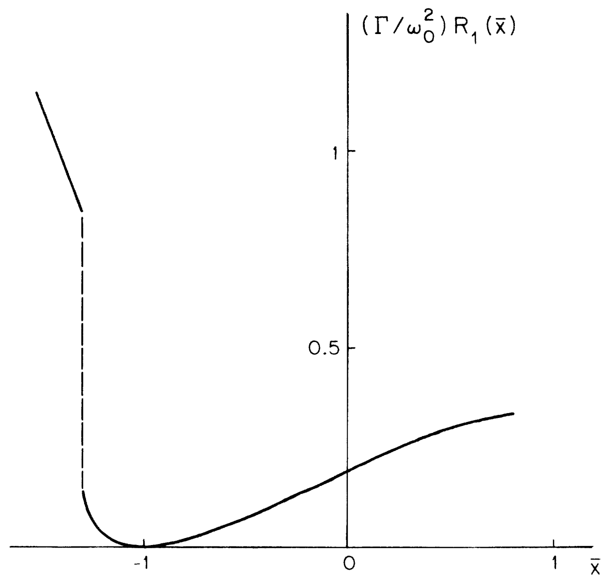

FIG. 3. The "activation energy" $R_1(\bar{x})$ of reaching a point $\bar{x}$ from the stable state 1 $(x_1=-1)$ for the potential (37). In the region of the "step" at $\bar{x}=-(5/3)^{1/2}$ the corrections to Eqs. (35) and (39a) for $R_1(\bar{x})$ should be taken into account. $R_1(\bar{x})$ is shown dashed here.

B. Activation energies of the transitions

Just as in the case considered above the motion along the extreme path resulting in a transition between stable states presents itself as a superposition (28) of a smooth motion and fast oscillations. However, Eq. (32) does not hold for the whole path, since $|\lambda_{\rm sm}(t)|$ increases for $U''_{\rm sm}>0$, and thus the boundary condition (20) $f(t)\to 0$ [and hence $\lambda(t)\to 0$] as $t\to\infty$ is not fulfilled for the solution (32).

The pattern of the motion is as follows. Initially the system is close to a stable state i and $f(t),\lambda(t)$ are very small [and $|\lambda(t)|\ll|f(t)|$]. The complex amplitudes $f_\pm(t)$ increase with the increment Γ, and their increase is followed adiabatically by the amplitudes $x_\pm(t)=\pm(i\omega_0)^{-1}f_\pm(t)$ and by the smooth component of the coordinate $x_{\rm sm}$ which is given by (32). At some instant $t^{(0)}$ (we set $t^{(0)}$ equal to zero) $x_\pm(t)$ reach the values $x_\pm^{(0)}$ which are given by the equations

$$U'_{\rm sm}(x_{\rm sm},x_+^{(0)},x_-^{(0)})=0\ ,\qquad U''_{\rm sm}(x_{\rm sm},x_+^{(0)},x_-^{(0)})=0 \tag{41}$$

[we suppose that Eq. (41) has unique solution for $x_{\rm sm}$ lying between x_i and x_s, and that $x_-^{(0)}=x_+^{(0)}$].

In the region $\Delta|x_+|\equiv|x_+|-|x_+^{(0)}|\sim\Gamma^{2/3}$ [$x_{\rm sm}$ differs from its value given by (41) by $\sim\Gamma^{1/3}$ in this region] the adiabaticity is broken and the term $\dot{x}_{\rm sm}$ in (31) is substantial. Within a time $\Delta t\sim\Gamma^{-1/3}$ the value of $|x_+|$ reaches its maximum and begins to decrease, while $x_{\rm sm}$ goes over from the adiabatic branch (32) to that solution of Eq. (31) for $x_{\rm sm}$ which is close to the unstable quasistationary solution given by $U'_{\rm sm}=0$ with $U''_{\rm sm}<0$ ($f_{\rm sm}$ plays a role as the "stabilizing" factor for this solution). Along the respective path $x_{\rm sm}$ approaches obviously the saddle point x_s as $|x_+|\to 0$ [note that $U'(x_s)=0$, $U''(x_s)<0$], while $|\lambda_{\rm sm}|$ falls down with the decrement $|U''_{\rm sm}|$. In its maximum $|\lambda_{\rm sm}|\propto\omega_0^2\Gamma^{-2/3}$. The latter estimate follows from the expression

$$f_\pm(t)=\frac{\Gamma}{2}\int_{-\infty}^{\infty}dt'\lambda_\pm(t')\exp(-\Gamma|t-t'|)\ , \tag{42}$$

which gives the solution of Eq. (30) for the case under consideration, if one takes into account the relations (29) between $\lambda_\pm(t),f_\pm(t)$ and $\lambda_{\rm sm}(t),x_\pm(t)$ and the fact that the main contribution to (42) comes from the "critical" region $|t'|\lesssim\Gamma^{-1/3}\ll\Gamma^{-1}$.

Allowing for Eqs. (11), (20), (28), and (42) we have

$$R_{is}=\tfrac{1}{2}\int_{-\infty}^{\infty}dt[f_+(t)\lambda_-(t)+f_-(t)\lambda_+(t)] =\frac{2\omega_0^2}{\Gamma}|x_+^{(0)}|^2\ . \tag{43}$$

It follows from Eqs. (41) and (43) that for the noise under consideration the activation energy of the transition R_{is} exceeds the activation energy of reaching the saddle point $R_i(x_s)$. In particular, for the system with the potential of the form (37)

$$R_1(x_s)=R_2(x_s)=\tfrac{1}{5}\omega_0^2\Gamma^{-1},\quad R_{1s}=R_{2s}=\tfrac{1}{3}\omega_0^2\Gamma^{-1}\ .$$

We note that for this particular system the characteristic scales for the motion in the critical region, $|x_+|\approx|x_+^{(0)}|=1/\sqrt{6}$, differ from those given above for the general case, since $U'''_{\rm sm}$ is very small in this region. Nevertheless the time scale is small as compared with Γ^{-1} and therefore Eq. (43) holds.

V. CONCLUSION

It follows from the above results that the system is influenced effectively not only by fluctuations with frequencies $\omega\lesssim t_r^{-1}$, but also by high-frequency fluctuations. In the case of a broad smooth power spectrum of noise $\Phi(\omega)$ the latter cause the corrections to the quasistationary distribution and the transition probabilities. The signs of these corrections depend on the shape of $\Phi(\omega)$. In the important case when $\Phi(\omega)$ has the sharp peak at frequency $\omega_0\gg t_r^{-1}$ the high-frequency fluctuations play the dominant role. The logarithms of the quasistationary distribution of the system, $-R_i(x)/D$, and of the transition probability, $-R_{is}/D$, are proportional here not to $\Phi^{-1}(0)$ [as for smooth $\Phi(\omega)$], but to $\Gamma t_r\Phi^{-1}(0)\ll\Phi^{-1}(0)$.

The dependences of $R_i(x)$ on the parameters of the system and on the coordinate x are quite different in these cases. In particular, $R_i(x)$ can vary extremely sharply within a narrow range of x (can have "steps") for the high-frequency noise. Qualitative features of the systems driven by such noise are also the difference between $R_i(x_s)$ and R_{is} and the spreading of the quasistationary distribution for one stable state over a part of the range of attraction to another stable state (these features are interconnected with each other and can be manifested for other types of colored noise as well; the point x_s separates the ranges of attraction to different states). Thus "colorfulness" of driving noise enriches substantially the pattern of fluctuations in a system. The character of fluctuations depends strongly on the shape of the power spectrum of noise.

APPENDIX

If the normalized reciprocal power spectrum of noise $F(\omega)=D/\Phi(\omega)$ is a polynomial in ω^2 of degree N, the colored noise $f(t)$ is the component of a Markov process. The full set of components is $\{f,f^{(1)},\dots,f^{(N-1)}\}$ ($f^{(n)}\equiv d^nf/dt^n$), and the stochastic differential equation for the process is of the form

$$L(-id/dt)f(t)=\xi(t),\quad \langle\xi(t)\xi(t')\rangle=D\delta(t-t')\ ,$$
$$L(\omega)=L_0\prod_{n=1}^{N}(i\omega+\alpha_n),\quad L_0=F^{1/2}(0)\prod_{n=1}^{N}|\alpha_n|^{-1} \tag{A1}$$
$$[F(-i\alpha_n)=0,\quad {\rm Re}\alpha_n>0]\ .$$

Here $\xi(t)$ is white Gaussian noise and $-i\alpha_n$ are the roots of the polynomial $F(\omega)$ with positive real parts; cf. (14a). The correspondence of $f(t)$ as given by (A1) to our initial colored noise is obvious, since the process described by (A1) is Gaussian and its power spectrum is seen from (A1) to equal $D/|L(\omega)|^2$, which coincides with (2) according to (8).

Instead of the initial system (1) driven by colored noise $f(t)$ we can consider now the auxiliary multidimensional system with the set of $N+1$ dynamical variables $\zeta(t)=\{x(t),f(t),\ldots,f^{(N-1)}(t)\}$, which is driven by white noise $\xi(t)$. In the absence of noise the system is described by Eq. (1) and by the equation $L(-id/dt)f(t)=0$. If the potential $U(x)$ in (1) is bistable, the system has two stable and one unstable stationary states where $f=f^{(1)}=\cdots=f^{(N-1)}=0$ and x equals x_1, x_2, and x_s, respectively. The ranges of attraction to different stable states are separated by the separating hypersurface which contains the unstable stationary point.

When white noise $\xi(t)$ is applied, the motion of the auxiliary system is a Markov process. Large fluctuations for such a process in the limit of small noise intensity D may be analyzed by the path-integral method[6] (cf. Sec. II) or by solving the respective Fokker-Planck equation or the equation for the mean first passage time in the eikonal approximation (see, e.g., Refs. 3 and 11). The results coincide with each other (cf. Refs. 8 and 12). Allowing for the relationships (1),(A1) between $f(t)$, $x(t)$, and $\xi(t)$, and for the well-known form[7] of the probability density functional for the white noise $\xi(t)$,

$$\mathcal{P}[\xi(t)]=\exp\left[-\frac{1}{2D}\int dt\,\xi^2(t)\right], \tag{A2}$$

and applying the arguments used in Sec. II [in particular, substituting $L(-id/dt)f(t)$ instead of $\xi(t)$ into the argument of the exponential (A2) for the extreme path] one obtains, that to logarithmic accuracy the probability of reaching a point $(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})$ by the auxiliary system is given by $\exp[-(1/D)R_i'(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})]$, where R_i' is the solution of the following variational problem:

$$\begin{aligned}
&R_i'(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})=\min\mathcal{R}_i'[f;\bar{x},\bar{t}]\,,\\
&\mathcal{R}_i'[f;\bar{x},\bar{t}]=\tfrac{1}{2}\int_{-\infty}^{\bar{t}}dt[L(-id/dt)f(t)]^2\\
&\qquad+\int_{-\infty}^{\bar{t}}dt\,\lambda(t)[\dot{x}+U'(x)-f(t)]\,,\\
&x(-\infty)=x_i,\quad f(-\infty)=\cdots=f^{(N-1)}(-\infty)=0\,,\\
&\bar{x}(\bar{t})=\bar{x},\quad f(\bar{t})=\bar{f},\ldots,f^{(N-1)}(\bar{t})=\overline{f^{(N-1)}}\,.
\end{aligned} \tag{A3}$$

To compare (A3) with Eq. (11) we note that, if we are interested in the distribution over $\bar{x}$ only, the multidimensional probability distribution should be integrated over $\bar{f},\ldots,\overline{f^{(N-1)}}$. To logarithmic accuracy the result is given by the extremum of R_i' with respect to $\bar{f},\ldots,\overline{f^{(N-1)}}$, i.e., by R_i' for $\bar{f},\ldots,\overline{f^{(N-1)}}$ such that

$$\frac{\partial R_i'}{\partial\bar{f}}=\cdots=\frac{\partial R_i'}{\partial\overline{f^{(N-1)}}}=0\,. \tag{A4}$$

This condition is satisfied automatically for the function $R_i\equiv R_i(\bar{x})$ which is the solution of the variational problem

$$\begin{aligned}
&R_i=\min(\mathcal{R}_i'+\mathcal{R}_i'')\,,\\
&\mathcal{R}_i''=\tfrac{1}{2}\int_{\bar{t}}^{\infty}dt[L(-id/dt)f(t)]^2\,,\\
&f(\infty)=\cdots=f^{(N-1)}(\infty)=0\,,
\end{aligned} \tag{A5}$$

with unfixed $f(\bar{t}),\ldots,f^{(N-1)}(\bar{t})$; the extreme path of $\mathcal{R}_i''$ is given by the solution of the equation $L(-id/dt)f(t)=0$, so $R_i=R_i'$ for R_i' given by (A3) and (A4). Integrating the term

$$\tfrac{1}{2}\int_{-\infty}^{\infty}dt[L(-id/dt)f(t)]^2$$

in $\mathcal{R}_i'+\mathcal{R}_i''$ by parts we see that the expression (A5) for R_i coincides with Eq. (11). It is obvious from (A3) and (A5), in particular, that $R_i'>0$, $R_i>0$.

The probability W_{ij} of the transition between stable states of the $(N+1)$-dimensional auxiliary Markov system is given to logarithmic accuracy by the probability of reaching the separating hypersurface[6] (cf. also Ref. 13). As soon as the system reaches this hypersurface the driving large fluctuation of $\xi(t)$ is switched off [$\xi(t)$ should not be continuous in contrast with $f(t)$] and the system goes then to another stable state "by itself" with a probability $\sim\frac{1}{2}$.

To find W_{ij} we should minimize R_i' with respect to the position of the final point $(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})$ on the separating hypersurface. According to (A3) the variation of R_i' when this point is shifted is of the form

$$\delta R_i'=\lambda\delta\bar{x}+\sum_{m=0}^{N-1}\delta\overline{f^{(m)}}\sum_{n=0}^{N}\sum_{k=0}^{N-m-1}l_n l_{k+m+1}(-1)^k\times\overline{f^{(n+k)}}\,, \tag{A6}$$

where

$$l_n=\frac{(-i)^n}{n!}(d^nL(\omega)/d\omega^n]_{\omega=0}\,. \tag{A7}$$

We shall consider $\delta R_i'$ for the relationship between the components of the shift $\delta\bar{x},\delta\bar{f},\ldots,\delta\overline{f^{(N-1)}}$ which corresponds to the small shift of the system in the absence of noise,

$$\frac{\delta\bar{x}}{-U'(\bar{x})+\bar{f}}=\frac{\delta\bar{f}}{\overline{f^{(1)}}}=\cdots=\frac{\delta\overline{f^{(N-2)}}}{\overline{f^{(N-1)}}}=-\frac{\delta\overline{f^{(N-1)}}}{\sum_{n=0}^{N-1}(l_n/l_N)\overline{f^{(n)}}}\,. \tag{A8}$$

Taking into account the relationship

$$\begin{aligned}
&H=0\,,\\
&H=-\frac{1}{2}\left[\sum_{n=0}^{N}l_nf^{(n)}(t)\right]^2+\sum_{n=0}^{N}\sum_{m=1}^{N}\sum_{k=0}^{N-m}(-1)^k l_n l_{k+m}f^{(n+k)}(t)f^{(m)}(t)-\lambda(t)[U'(x)-f(t)]
\end{aligned} \tag{A9}$$

[H is obviously, from Eqs. (13), the integral of motion, and $H=0$ due to the boundary conditions (12) and (A3)], we obtain from (A6) and (A8)

$$\delta R_i' = -\frac{1}{2}\left[\sum_{n=0}^{N} l_n f^{(n)}(\bar{t})\right]^2 \delta\bar{x} \Big/ [-U'(\bar{x})+\bar{f}] \; . \qquad (A10)$$

When the point $(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})$ lies on the separating hypersurface, the ratio (A8) is positive for the system shifting towards the saddle point $(x_s,0,\ldots,0)$ [all paths starting on this hypersurface go to the saddle point for $\xi(t)=0$]. Therefore the activation energy R_i' decreases according to (A10) as the end point of the extreme path $(\bar{x},\bar{f},\ldots,\overline{f^{(N-1)}})$ shifts towards the saddle point, and thus the transition probability is determined by the path which ends in the saddle point. So, we arrive at the expression (20) for W_{ij}.

We note in conclusion that the function $R_i'(x,f,\ldots,f^{(N-1)})$ can be considered as the mechanical action of an $(N+1)$-dimensional system with the coordinates $x,f,\ldots,f^{(N-1)}$. The function H (A9) may be shown in a standard way[10] to be the energy of this system. The system is obviously, from (A3), conservative, so H is the integral of motion. The condition $H=0$ reflects the independence of R_i' on time. Since the function $R_i'(x,f,\ldots,f^{(N-1)})$ can have several local minima for a given x, the derivative of the activation energy of reaching a point x, $R_i(x)=\min R_i'(x,f,\ldots,f^{(N-1)})$, can be discontinuous (this happens for those x where the lowest minima of R_i' are of the same depth).

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*Present address: Department of Physics, University of Lancaster, Lancaster LA1 4YB, United Kingdom.

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