Master equation for the logistic map

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A master equation is constructed that provides a stochastic description underlying the logistic map. In an appropriate macroscopic limit, the underlying master map (equation) yields the logistic map. It also describes intrinsic fluctuations associated with the logistic map. When the logistic map parameters are chosen so that the map produces a chaotic trajectory, the variance of the associated fluctuations diverges. This means that the distribution function determined by the master map becomes very broad and that the logistic map no longer results from averaging with respect to the master map distribution function. Numerical examples of this behavior and its interpretation are discussed.

I. INTRODUCTION

There is a long tradition behind the description of macroscopic dissipative processes by phenomenological equations, e.g., hydrodynamics, electrical circuits, and mass action chemical reactions. It is now widely appreciated[1–8] that a complete macroscopic description of these processes must include intrinsic molecular fluctuations as well as the deterministic macrovariables, both of which reflect underlying microscopic dynamics.[1,2,8] Indeed, these fluctuations provide the basis for our understanding of light scattering,[9] electrical noise, and other noise measurements for macroscopic systems.[10] Recently, the effect of dynamical chaos on these ideas was explored in detailed and general ways.[11]

A general approach to the effect of chaos on macrovariable fluctuations is provided by the master equation idea.[10–20] Given a macrovariable dynamics, a master equation is constructed such that an appropriate macroscopic limit yields the macrovariable dynamics as well as a description of the fluctuations. The construction process involves the underlying physics.[8] For chemical reactions it is very well established how this construction works,[8,10,16,17] but for hydrodynamics it is not so straightforward, so that, to date, there is no single master equation for all fluid density regimes.[10,13–15]

The purpose of this paper is to exhibit the effect of chaos on the description of intrinsic fluctuations in a very simple setting, the logistic map. The fluctuations described in this paper are intrinsic fluctuations and must be carefully distinguished from externally introduced fluctuations. A number of authors have explored the consequences of introduced external fluctuations on the behavior of the chaotic systems. See, for example, Refs. 21–24. Below we will address the appropriateness of these earlier studies in the context established here (and in Ref. 11) that results from the interplay of chaos and intrinsic fluctuations on the description of chaotic dynamics.

In classical physics, chaos is characterized by sensitive dependence of trajectories on initial conditions.[25] This idea is made quantitative by the Liapunov exponent.[25] A positive Liapunov exponent implies chaos. As was shown elsewhere,[11,25,26] the value of the Liapunov exponent is related to the Jacobi matrix for the macrovariable dynamics, and the Jacobi matrix also determines the time evolution of the fluctuations.[11] The salient consequence recently discovered[11] is that a positive Liapunov exponent (chaos) for the macrovariable dynamics implies a divergence of the covariance matrix for the fluctuations. Moreover, this circumstance implies that the macroscopic limit procedure breaks down so that the macrovariable equations no longer follow from the underlying master equation.[11] Instead, the distribution function determined by the master equation becomes very broad and the dynamical description is only correctly given entirely at the master equation level. Clearly, it is inappropriate to introduce external fluctuations[21–24] into macrovariable equations that are no longer valid; instead, external fluctuations will have to be introduced into the master equation level of description.

Because relatively few scientist are fluent in master equation ideas, this paper has been written to present a very simple example of the essential ideas and their consequences. This objective is realized by constructing a master map (equation) for the logistic map. We have chosen the logistic-map paradigm because it is virtually the simplest example possible and because of its fundamental and historical significance with respect to chaos.[27]

In Sec. II, a master map for the logistic map is constructed. In Sec. III, the Liapunov exponent concept is developed for both the logistic map and the master map. In Sec. IV, the breakdown of the macroscopic limit is presented, along with an account of the numerical evidence. In Sec. V, concluding remarks are offered.

II. LOGISTIC MAP AND MASTER MAP

The logistic map is given by[27]

$$x_{n+1}=4\lambda x_n(1-x_n)\ , \tag{1}$$

in which the x's are mapped from the unit interval onto the unit interval and the tunable parameter λ is taken from the unit interval as well. It is now well understood[27] that for $\lambda < 0.25$ the logistic map has a globally stable fixed-point attractor at $x=0$; for $0.25 < \lambda < 0.75$, $x=0$ becomes unstable and $x=1-1/4\lambda$ becomes the globally stable fixed-point attractor; and for $0.75 < \lambda$, there are no fixed-point attractors. Instead, a sequence of bifurcations yielding 2^k cycles ensues up to about $\lambda = 0.892\,48\ldots$. Beyond this value there are regions of chaos interspersed with windows of all possible cycles not expressible as 2^k. At $\lambda=1$, the chaotic attractor covers the entire unit interval.

For the sake of the presentation in this paper, we take the perspective[11] that Eq. (1) is a macrovariable map. This means that we are thinking of the x's as describing a macroscopic amount of something (rescaled to the unit interval). In fact, in the original context of population biology, the x's represented populations of a species from generation to generation. Therefore we expect intrinsic fluctuations to be associated with the macrovariable x that represent variations induced at the level of individual organisms. We may achieve a more refined description by introducing a master map (equation) that describes the process at the more microscopic level of individuals and yields the logistic map as its moment map in the macroscopic limit.

The master map is constructed as follows. We first rewrite Eq. (1) on the real numbers between 0 and N:

$$y_{n+1} = 4\lambda y_n (N - y_n)/N \; . \tag{2}$$

This map takes real numbers from the interval 0 to N onto the same interval. Clearly, in the limit $N \rightarrow \infty$, $y_n/N \rightarrow x_n$. This limit is what we will call the macroscopic limit. Next, we introduce the probability distribution for the population in the nth generation $W(q,n)$, in which q only takes on integer values from 0 to N. The restriction of the argument of W to the integers introduces an effective noise level of size $1/N$. Of course, as $N \rightarrow \infty$, this noise vanishes. The master map determines how the probability distribution changes from generation to generation. With this simple example, we are attempting to exhibit a more general phenomenon[11] that occurs in real physical systems, e.g., chemical reactions. For them, the underlying physics determines the form of the master equation, whereas in this simple example we are free to choose any one of many possible constructions since we are dealing with a paradigm and not with a real process.

Initially, we construct it in the following simple way:

$$W(q,n+1) = \int_0^N dq' \delta_N (q - 4\lambda q'(N-q')/N) W(q',n) \; , \tag{3}$$

in which $\delta_N(\;)$ is not precisely a Dirac delta function, but instead picks out the largest integer value q contained in $4\lambda q'(N-q')/N$, and, therefore, has some dispersion of order $1/N$. A computer program realization of this mapping may be found in the Appendix.

In the literature,[22,23] this equation is known as the noisy Frobenius-Perron equation and has been introduced in the context of adding external noise to the logistic map. Consequently, it would appear as though the two problems of intrinsic noise and external noise reduce to an identical analysis. There are two important reasons why this is not so, however. The first reason is that the particular master map picked in Eq. (3) is quite arbitrary, i.e., many master maps can be constructed that reduce to the logistic map in the macroscopic limit ($N \rightarrow \infty$), and the one we have picked is the same as the noisy Frobenius-Perron equation by accident. The expression $4\lambda q'(N-q')/N$ in Eq. (3) can be augmented by any function of q' of an order higher than N^{-1} and there will accrue no difference in the macroscopic limit. Had I chosen such an expression at the outset, there would be no cause for confusion with the noisy Frobenius-Perron equation. The second reason is deeper. As is shown below and elsewhere,[11] the consequence of chaos on the macroscopic limit is to invalidate the logistic map as a stable contracted description of the behavior of the underlying master map. This consequence is caused by the intrinsic noise, the variance of which becomes enormous. The addition of external noise cannot be made at the level of the logistic map in this situation, as it is in Ref. 22 and 23, but must be made at the master map level instead.

III. LIAPUNOV EXPONENTS

For the logistic map, the Liapunov exponent Λ may be computed from the formula[28]

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left| \frac{df}{dy_i} \right| \; , \tag{4}$$

where f is given by

$$f = 4\lambda y (N-y)/N \; . \tag{5}$$

The quantity df/dy_i is the Jacobi matrix for a one-dimensional map. For a map in r dimensions, the Jacobi matrix is $r \times r$ dimensional. If we represent the r variables of an r-dimensional map by $\mathbf{z}$, then the Jacobi matrix $\mathbf{J}$ determines how a small deviation from $\mathbf{z}$ will map:

$$\Delta \mathbf{z}(n+1) = \mathbf{J}(n) \Delta \mathbf{z}(n) \; . \tag{6}$$

If we have some way of averaging over these deviations, e.g., there is an underlying master map probability distribution, then the covariance of the fluctuations (deviations) defined by

$$C_{ij} = \langle \Delta z_i \Delta z_j \rangle \tag{7}$$

satisfies the mapping[26]

$$\mathbf{C}(n+1) = \mathbf{J}(n) \mathbf{C}(n) \mathbf{J}^{\dagger}(n) \; , \tag{8}$$

in which $\mathbf{J}^{\dagger}$ is the adjoint of $\mathbf{J}$. We have shown how to define a Liapunov exponent for this covariance equation[11,26] and have proven the identity that for the one-dimensional case it has exactly twice the value determined from Eq. (4). This factor-of-2 relationship also holds in r dimensions.[11,26]

The impact of this identity is that the covariance of the fluctuations becomes very large when the Liapunov exponent is positive. For the present situation in which we have a one-dimensional map, the covariance matrix simply degenerates into the variance for the fluctuation (deviation) for y_n, i.e., Δy_n, which satisfies the degenerate version of Eq. (6):

$$\Delta y_{n+1} = \frac{df}{dy_n} \Delta y_n \ . \tag{9}$$

If we now use the master map probability distribution to determine the averaged variance, i.e., $\langle \Delta y_n \Delta y_n \rangle$, then we see that df/dy_n determines both the Liapunov exponent for the logistic map [Eq. (4)] and the growth of the variance [Eq. (9)].

The dual role of the Jacobi matrix in these considerations is crucial and occurs in a much broader class of systems than represented by the simple example in this paper.[11]

IV. MASTER MAP→LOGISTIC-MAP TRANSITION

We will now describe the consequences of the preceding results. The reader may wish to avail himself/herself of the advantages of following our remarks with a computer simulation. In this way the reader can see firsthand the numerical evidence we ourselves have seen for the remarks that follow (see also the figures in Ref. 23). The program in the Appendix will provide assistance.

Succinctly put, we find that when the Liapunov exponent is not positive, it is always possible to choose N sufficiently large that the distribution function determined by the master map follows the logistic map's output with a very sharp distribution. However, when the Liapunov exponent for the logistic map is positive, no matter how large an N is chosen, the distribution function determined by the master map becomes very broad and after only a few iterations neither its mean nor its maximum bear any relationship to the output of the logistic map. In the first case, i.e., nonpositive Liapunov exponent, averaging over the distribution function will produce a mean value for y which is precisely equal to the output of the logistic map in the macroscopic limit. This is a consequence of averaging Eq. (2) and finding that the right-hand side, the average of a nonlinear expression, can be replaced with very high accuracy by the nonlinear expression of the average

$$\langle y_{n+1} \rangle = 4\lambda \langle y_n (N - y_n) \rangle / N = 4\lambda \langle y_n \rangle (N - \langle y_n \rangle) / N \ . \tag{10}$$

It is this replacement that breaks down in the second case, i.e., positive Liapunov exponent, for which the distribution is too broad,

$$4\lambda \langle y_n (N - y_n) \rangle / N \neq 4\lambda \langle y_n \rangle (N - \langle y_n \rangle) / N \ . \tag{11}$$

The following remarks elaborate the content of the preceding paragraph in much more detail. Because there are so many special cases, some involving complicated transient behavior, far too many figures would have been required for this paper. The reader may reproduce those cases of interest to him on a computer. Our remarks are presented in the order of increasing complexity. In each case, unless stated otherwise, all of the initial probability is placed in bin no. 25, i.e., $W(25,1)=1$.

(1) *$N=100$, $\lambda=0.3$.* The logistic map output approaches the attractor at $x=\frac{1}{6}$. The distribution function stays sharp and follows the logistic map output with a precision of $\frac{1}{100}$.

(2) *$N=400$, $\lambda=0.3$.* This is the same as case (1), except the precision is now $\frac{1}{400}$.

(3) *$N=100$, $\lambda=0.8$.* The logistic map output approaches the two-cycle (0.513 045, 0.799 455). The distribution function stays sharp and follows the logistic map output with a precision of $\frac{1}{100}$. This means that the distribution function also describes a two-cycle.

(4) *$N=400$ $\lambda=0.8$.* This is the same as case (3), except the precision is now $\frac{1}{400}$.

(5) *$N=100$, $\lambda=0.865$.* The logistic map output approaches the four-cycle (0.413 233, 0.838 951, 0.467 488, 0.861 343). The distribution function stays sharp and follows the logistic-map output with a precision of $\frac{1}{100}$. This means that the distribution function also describes a four-cycle. The distribution is sharper around the two larger cycle values than it is around the two smaller values. If we reduce the bin subdivision parameter from 50 to 10, then the results are qualitatively the same although somewhat less smooth.

(6) *$N=400$, $\lambda=0.865$.* This is the same as case (5), except the precision is now $\frac{1}{400}$.

(7) *$N=100$, $\lambda=0.886$.* The logistic-map output approaches the eight-cycle (0.3642, 0.8206, 0.5216, 0.8843, 0.3625, 0.8190, 0.5254, 0.8837). The approach to this eight-cycle takes several hundred iterations before the fourth digit of accuracy is obtained. This contrasts markedly with the preceding examples. The $\frac{1}{100}$ precision of the master map produces a distribution function that cannot follow this eight-cycle. Instead, the distribution settles down on a four-cycle associated with the values (0.36, 0.82, 0.52, 0.88). While the distribution is still quite sharp around the two larger values, it is rather broad around the two lower values and is in fact bimodal around both 0.36 and 0.52. Thus, if we were to use the average of y determined by this distribution function, it would describe a four-cycle that is not identical with any four-cycle of the logistic map for any value of λ. This consequence of the noise in the master map output can be eliminated by reducing the noise level by increasing N (see below). It is also noteworthy that the apparent four-cycle reached by the master map distribution function is reached in relatively few iterations, i.e., in much less than 100 iterations.

(8) *$N=400$, $\lambda=0.886$.* This is the same as case (7), except the precision is now $\frac{1}{400}$. This precision is still not good enough because a precision of at least $\frac{1}{1000}$ is needed in order to distinguish each separate eight-cycle point.

(9) *$N=4000$, $\lambda=0.886$.* Now the precision is high enough for the distribution function to follow the logistic-map output. The time required by the computer, however, has grown enormous.

(10) *$N=400$, $\lambda=0.886$, $W(208,1)=1$.* This time we have started the master map distribution function with all of its probability on one of the eight-cycle points (0.5216, $400\times 0.5216=208.64$). We must also initialize the logistic map with $y=208$. The behavior of the master map distribution function is now dramatically different. It can follow the eight-cycle accurately. For

the large cycle values (0.88) it is extremely sharp, and it is somewhat less sharp for the smallest values (0.36), but for the intermediate values (0.52) it is rather broad. Nevertheless, the distribution function is clearly peaked at 0.5216 and 0.5254, respectively, in spite of the fact that its rather broad structures for these two-cycle points overlap greatly. By contrasting the results for cases (7)–(10), we see that the outcome depends crucially on the initial conditions. It is clear that a trajectory with a transient will create a growth in the master map distribution function that will persist even after a stationary state has been reached, whereas the absence of such a transient permits the master map distribution function to follow the logistic map even when the noise level would indicate that there is insufficient precision. Even though by starting on one of the eight-cycle points, the probability distribution follows the logistic-map output, averages with respect to this distribution do not satisfy Eq. (10). It is simply the peaks of the distribution that follow the logistic-map output. The averages satisfy Eq. (11) for $N=400$. By increasing N, Eq. (10) is approached more and more accurately.

(11) $N=100$, $\lambda=0.9$. The value of λ implies chaos for the logistic map. The Liapunov exponent for this λ is 0.183. Scrutiny of an attractor plot for the logistic map[29] shows that the attractor for $\lambda=0.9$ is made up of two disjoint regions. One region covers the x interval from about 0.3 to about 0.6 whereas the other region covers the x interval from about 0.8 to about 0.9. The invariant measure on these regions is not uniform.[29,30] The logistic-map output quickly reaches the attractor and then jumps about chaotically on the attractor. The master map distribution function, however spreads out on the two attractor regions and alternately hops from one to the other. After only a few dozen iterations, the distribution function reaches a steady two-cycle behavior. This two-cycle is between two broad subdistributions. Any average over this behavior would look like a two-cycle. Equation (11) is strongly the case. Most remarkable of all, however, is the fact that the union of the two subdistributions very closely matches the invariant measure for the logistic map attractor. By increasing N, this match gets better. Thus we see that the master map's probability distribution may be identified with a noisy average of the invariant measure for the deterministic logistic map.

(12) $N=100$, $\lambda=0.95$. The value of λ implies chaos for the logistic map. The Liapunov exponent for this lambda is 0.435. Scrutiny of an attractor plot for the logistic map[29] shows that the attractor for $\lambda=0.95$ is now made up of just one region. Once again the master map probability distribution quickly approaches a steady distribution that closely matches the invariant measure for the logistic map with $\lambda=0.95$. Since the Liapunov exponent here is bigger than in case (11), the steady distribution is reached correspondingly more quickly. Again, by increasing N, the correspondence with the invariant measure is improved. Because there is now only one region covered by the very broad distribution function, or by the invariant measure for that matter, an average over the master map distribution function yields a simple fixed value that bears no resemblance to the chaotic trajectory of the logistic map.

(13) $N=100$, $\lambda=0.96$. This value of λ produces a three-cycle for the logistic map (0.1494, 0.4879, 0.9594). This is in one of the periodic windows of the attractor plot.[29] The corresponding Liapunov exponent is -0.0044. This is clearly not chaotic. However, the transient for the approach to the three-cycle causes the master map distribution to grow very broad before the three-cycle attractor is reached. Consequently, the distribution ends up steady and broad and looking very much like the invariant measure for a chaotic λ just below 0.96. Because there is again only one region covered by the very broad distribution function, an average over the master map distribution function yields a simple fixed value that bears no resemblance to the three-cycle trajectory of the logistic map.

(14) $N=300$, $\lambda=0.96$, $W(146,1)=1$. This is the same as case (13) except that the initialization of both the master map and the logistic map has been switched to a three-cycle point (0.4879). The distribution function now attempts to follow the logistic map's three-cycle and does so quite sharply for over a dozen internations, but by iteration 15 it has developed a broad background. Nevertheless, a steady distribution is not the outcome as it was in case (13). A three-cycle distribution results in which each of the three subdistributions has three peaks and a broad background spanning the space between the peaks. The peak corresponding to the cycle point of the logistic map is the largest in each case, and accounts for 99% of the total probability, whereas the background accounts for less than 0.1% of the probability, while the two lesser peaks make up the remainder, about 1%. This three-cycle distribution is reached in about two dozen iterations.

(15) $N=100$, $\lambda=1$. For the logistic map, this is the strongest chaos with a Liapunov exponent of 0.693. The master map probability distribution rapidly spreads over the entire unit interval and assumes a form very similar to the invariant measure for the logistic map.[29,30] Because the entire unit interval is covered by the distribution function, an average over the master map distribution function yields a simple fixed value that bears no resemblance to the chaotic trajectory of the logistic map. By increasing N, the correspondence between the probability distribution for the master map and the invariant measure for the logistic map gets better.

V. CONCLUDING REMARKS

These examples demonstrate how very different the logistic map and the master map behave. When λ is less than 0.892 48. . . , the Liapunov exponent for the logistic map is less than zero and the master map can produce a distribution function that follows the behavior of the logistic map as accurately as desired provided N is taken sufficiently large. Indeed, in the macroscopic limit they have identical behavior. In practice, however, N may have to be enormous in order to have the master map distribution follow a 2^k cycle with $k>5$. For λ greater than

0.892 48. . . , chaos ensues for most λ values. For the chaotic λ's, the master map distribution grows broad and no longer follows the behavior of the logistic map. In fact, it goes rapidly to a steady distribution (or to a low cycle distribution of subdistributions). This steady distribution has a very accurate resemblance to the invariant measure on the attractor for the logistic map. It is made more accurate by increasing N, which is tantamount to decreasing the noise. For λ values corresponding with periodic windows, the Liapunov exponent is again less than zero. Nevertheless, points not on the periodic attractors tend to exponentiate part until their iterates reach the periodic attractors. The master map distribution function in this situation depends strongly on the initial conditions. A trajectory with a transient will result in a broad and steady distribution. By starting on a cycle point, however, a periodic distribution will result that follows the logistic map and also possesses a low level but broad background.

The point of this paper is to take the view that this simple paradigm represents physical reality as regards the relationship between a macrovariable description and an underlying microscopic or master equation description.[11,26] In this view, the master map is viewed as physical reality whereas the logistic map is viewed as a contracted description created by averaging with respect to the master map's distribution function. As long as the distribution is sharply peaked, Eq. (10) may be expected to hold and one can have faith in the contracted description. But when the distribution is not sharply peaked, Eq. (11) must hold instead and the contraction is no longer valid. This circumstance occurs whenever the contracted description (the logistic map) predicts chaos, because of the connection of the Liapunov exponent to the Jacobi matrix and the connection of the Jacobi matrix to the covariance of the fluctuations.[11,26] Therefore, under these circumstances, the only recourse is to abandon the logistic map and to use the master map instead. This consequence is of special significance when it comes to introducing external fluctuations into consideration.[21–24] Since the logistic map is no longer valid, external fluctuations must be introduced into the master map in order to properly see their effects. The chaos of the logistic map is a mathematical artifact of an equation that no longer has physical significance. It is, nevertheless, significant that the invariant measure for the chaotic attractor produced by the logistic map is so similar to the stationary probability distribution produced by the master map, in the low noise limit. This fact is readily explained by looking at the equation for the invariant measure on the logistic map's attractor (the Frobenius-Perron equation):[23,28]

$$P(x)=\int_0^1 dz\,\delta(x-4\lambda z(1-z))P(z)\ , \tag{12}$$

in which the kernel, $\delta(\ )$, is a genuine Dirac delta function [cf. Eq. (3)]. Clearly, the invariant probability distribution determined by Eq. (3) is equivalent to P in the macroscopic limit. The importance of these thoughts in realistic physical contexts has been explored elsewhere.[11,26,31]

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APPENDIX

This program produces a plot of the master map distribution function. In fact, it yields the scaled logarithm of the distribution because the distribution function ranges over many orders of magnitude, but it can easily be modified to yield the distribution function directly if desired. It also plots the position of the output from the logistic map as a pair of points just below and above the distribution function plot. This makes it easy to compare the two types of output. We develop the program in the following simple steps. The restriction of q to the integers may remind one of maps on the integers,[32] but here this restriction only occurs in the master map:

```
FOR q = 0 to N
  IF W(q,N,1) > 0 then
    FOR m = 0 to N
      IF m = INT[4λq(N − q)/N] then
        LET W(m,n + 1,2) = W(m,n,2) + W(q,n,1)
      END IF
    NEXT m
  END IF
NEXT q .                                            (A1)
```

In this mapping, a third variable, taking on the values 1 or 2, has been introduced into W. After completing the FOR-NEXT cycle above, the W values are updated for another cycle of Eq. (A1) by the FOR-NEXT cycle below:

```
FOR q = 0 to N
  LET W(q,n + 1,1) = W(q,n + 1,2)
  LET W(q,n + 1,2) = 0
NEXT q .                                            (A2)
```

This master map does not achieve our desired result very effectively. In fact, it produces a probability distribution that follows the modified, diophantine logistic map,

$$y_{n+1}=\mathrm{INT}\ (4\lambda y_n(N-y_n)/N)\ , \tag{A3}$$

precisely if $W(q,0,1)=1$ when the initial value of y for Eq. (A3) is $y_0=q$. Since we really want to follow Eq. (2) with a master map and not Eq. (A3), we must make our master map a bit more complicated. The desired result is achieved by subdividing the integer interval between q and $q+1$ into a large number of equal subintervals and distributing the probability uniformly among them before mapping the probability into the next generation. This is achieved by the following mapping:

```
FOR q = 0 to N − 1
  IF W(q,n,1) > 0 then
    FOR m = 0 to N
       FOR j = 1 to 50
         IF m = INT {4λ(q + j/50)[N − (q + j/50)]/N} then
            LET W(m,n + 1,2) = W(m,n,2) + W(q,n,1)/50
         END IF
       NEXT j
    NEXT m
  END IF
NEXT q
LET q = N
IF W(N,n,1) > 0 then
  LET W(0,n + 1,2) = W(0,n,2) + W(N,n,1)
END IF .                                                        (A4)
```

This FOR-NEXT cycle must be followed by Eq. (A2). In this example, 50 subdivisions have been invoked. It should be clear that in the macroscopic limit this fine subdivision will have no effect on the logistic-map limit. However, it will just as clearly affect the fluctuations because probability in the integer interval from q to $q+1$ will now potentially end up in more than one integer valued probability bin, $W(m,n+1,2)$. In this way, the master map follows the behavior of of the logistic map, Eq. (2), with a noise level of $1/N$. A version of the entire program, written in the true-basic language of Kemeny and Kurtz,[33] follows for a Macintosh II computer.

```
REM LOGISTIC MAP MASTER MAP
SET WINDOW 0,640,0,460
DIM W(0 to 10000,2)
PRINT "What is N?"
INPUT N
PRINT "What is lambda?"
INPUT lambda
LET i = 0
LET s = 0
MAT REDIM W(0 to N,2)
LET W(25,1) = 1
LET y = 25
DO
    FOR q = 0 to N − 1
     IF W(q,1) > 0 then
        FOR m = 0 to N
           FOR j = 1 to 50
             IF m = INT {4*lambda*(q + j/50)*[N − (q + j/50)]/N} then
                LET W(m,2) = W(m,2) + W(q,1)/50
```

```
            END IF
          NEXT j
        NEXT m
      END IF
    NEXT q
    LET q = N
    IF W(N,1) > 0 then
      LET W(0,2) = W(0,2) + W(N,1)
    END IF
    LET y = 4*lambda*y*(N - y)/N
    LET i = i + 1
    CLEAR
    FOR m = 0 to N
      PLOT 100 + m, 300 + 10*log[W(m,2) + 1e - 50]
      LET s = s + W(m,2)
    NEXT m
    PLOT 100 + y, 50
    PLOT 100 + y, 350
    PRINT i, "sum = ";s, "lambda = ";lambda, "N = ";N
    PRINT y
    FOR m = 0 to N
      LET W(m,1) = W(m,2)
      LET W(m,2) = 0
    NEXT m
    LET s = 0
LOOP
END
```

In this program, the initial probability is entirely put into bin no. 25. This is an arbitrary choice. The subdivision by 50 is also aribtrary. Becuase the mapping is now embedded in a DO-LOOP, we no longer need the reference to n and $n+1$ inside W. The variable s checks the normalization requirement after each mapping interation.

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