Efficient box-counting determination of generalized fractal dimensions

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An alternative algorithm for the numerical analysis of fractal structures and measures is presented, which consumes computer time and memory only quasiproportionally to the size of the input data set. This efficient tool is applied to various deterministic and random multifractals, in particular to the growth probability measures of diffusion-limited aggregation clusters in two- and three-dimensional embedding space.

I. INTRODUCTION

In the last two decades it has become clear that nontrivial self-similar objects, the so-called fractals,[1] play a major role in many physical situations like nonlinear dynamics,[2] critical phenomena,[3] and growth processes.[4] The crucial feature of such structures is that they are (often in a subtle and statistical way) invariant under contractions or dilations in the sense of symmetry operations.

Fractals embedded in Euclidean space E^d usually do not have a finite d-dimensional volume and cannot be described in terms of traditional geometry. In order to assign well-defined "contents" to these objects, the D-dimensional Hausdorff measure has to be used,[5] where $D \in [0,d]$ is the Hausdorff dimension. While the rigorous determination of D is almost always impossible for a fractal point set $\mathcal{X} \subset E^d$, a very good estimate can be achieved, in general, by computing the so-called *box-counting* (or capacity) *dimension* D_B (Ref. 6).

It is defined as

$$D_B = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} , \tag{1}$$

where ϵ is the lattice constant of d-cubic covers of $\mathcal{X}$ and $N(\epsilon)$ is the number of cubes contained in the minimal cover.

The single number D_B may be sufficient for a geometrical characterization of the fractal set considered, but physical processes or probability and weight distributions that take place on such a structure have to be described by multifractal measures.[7] The concept of multifractality has turned out to be of considerable value for the quantitative investigation of the growth zone of aggregates,[4,8] of the current distribution of critical percolation clusters,[9] or the distribution of local strain in breaking solids.[10] The formalism is based on the so-called *generalized* (box-counting) *fractal dimensions* $D_B(q)$, $q \in \mathbb{R}$,[11] defined by

$$D_B(q) = \lim_{\epsilon \to 0} \frac{1}{q-1} \frac{\ln \sum_{\nu=1}^{N(\epsilon)} [p_\nu(\epsilon)]^q}{\ln \epsilon} . \tag{2}$$

Here the index ν labels the individual boxes of the ϵ cover and $p_\nu(\epsilon)$ denotes the relative weight of the νth box: In practice the measure under investigation is represented by a finite ensemble of points (time series, collection of particle positions in a computer-simulated aggregate, etc.) in E^d, whose number N may have to be very large in order to catch the intricacy of the fractal. Within this approximation $p_\nu(\epsilon)$ is given by

$$p_\nu(\epsilon) = \frac{N_\nu(\epsilon)}{N} , \tag{3}$$

where $N_\nu(\epsilon)$ is the number of points falling into the νth box. The definition of generalized box-counting dimensions obviously includes the ordinary one, i.e., $D_B = D_B(0)$.

Let us emphasize here that the spectrum of generalized fractal dimensions $D_B(q)$ quantifies the nonuniformity of the measure defined on the fractal set $\mathcal{X}$. If, on the other hand, p were chosen as the $D_B(0)$-dimensional Hausdorff measure [assigning to each subset $\mathcal{A} \subset \mathcal{X}$ its fraction of $D_B(0)$-dimensional volume], then $D_B(q) = D_B(0)$ for all $q \in \mathbb{R}$. In the following we will refer to this uniform measure as the trivial one.

A very suggestive way to treat multifractals is provided by the thermodynamic formalism,[7] which resolves these measures into homogeneously scaling components located on subsets of individual ordinary box-counting dimension $f(\alpha)$. The entire range $f(\alpha)$ of dimensions (or scaling indices) can be constructed easily via Legendre transformation once the $D_B(q)$ are known.

So "all" we need are efficient numerical schemes for computing the generalized box-counting dimensions from a given pile of data encoding the multifractal phenomenon in question. At first sight it seems quite easy to devise such schemes in a straightforward manner starting from the clear-cut recipe as expressed in Eqs. (2) and (3). The problem is, however, that consumption of computer resources, i.e., memory and CPU time, grows nonlinearly with the number N of data points available.[12] Several computational procedures have been reported in the recent literature,[13] but even advanced algorithms consume CPU time of order N^2.[14] Attempts to reduce this time via Monte Carlo techniques[15] may entail intolerable statistical errors.

In summary, the box-counting approach to multifractal analysis has been regarded, until now, as cumbersome in general, and as even unfeasible in the case of intricate subsets of higher-dimensional Euclidean space ($d \geq 3$) (see, for example, Schuster in Ref. 2). The main objective of the present paper is to demonstrate that this opinion has to be revised. In Sec. II we will construct an efficient box-counting algorithm, which reduces the memory costs to order N and limits the CPU time to order $N \log_2 N$. In Sec. III our method is first tested with various standard objects of Euclidean geometry and then applied to several deterministic and stochastic fractal structures and measures, namely, the Kiesswetter curve, the Julia set associated with the Cayley problem, and diffusion-limited aggregation (DLA) clusters in two- and three-dimensional embedding space. In particular, this computational scheme enables us to determine the spectrum of scaling indices of the growth zones associated with 3D DLA.

II. COUNTING BOXES THE ECONOMICAL WAY

In the following we describe the basic ideas instrumental to improved box-counting strategies and give a detailed formulation of the actual algorithm applied to the examples presented in Sec. III. Our approach is a spinoff of an extensive investigation of multifractal growth phenomena as simulated and analyzed with the aid of parallel computer networks (transputer cluster).[16]

Let $\mathcal{S} \subset E^d$ be a given point set consisting of the vectors $s(i)$, $i=1,\ldots,N$. The components of these vectors are denoted by $s_\delta(i)$, $\delta=1,\ldots,d$. Our task is to determine the generalized box-counting dimensions $D_B(q)$ as defined in Eq. (2). We first introduce the scales

$$\epsilon_{\max} = \max_\delta \{ \max_i [s_\delta(i)] - \min_j [s_\delta(j)] \} , \tag{4}$$

$$\epsilon_{\min} = \min_{i,j} [\| s(i) - s(j) \|] . \tag{5}$$

$\epsilon_{\max}$ is used to confine our data set to the unit d cube Q. This is achieved by proper choice of the origin and by normalization of the resulting non-negative coordinates, i.e.,

$$\tilde{s}_\delta(i) \equiv s_\delta(i) / (\epsilon_{\max} + \Delta) , \tag{6}$$

where Δ is a small positive number guaranteeing that $\tilde{s}_\delta(i) \in [0,1]$. Note that this affine mapping does not alter the dimensional properties of the point set in question.

Next we construct a sequence of rational decompositions of the unit hypercube, where the coarseness decreases roughly exponentially and is bounded from below by $\epsilon_{\min}$. Given an arbitrary positive integer M, let

$$k_m \equiv \mathrm{int}(\epsilon_{\min}{}^{-m/M}), \quad m=0,1,\ldots,M . \tag{7}$$

At the mth stage, Q is decomposed into $(k_m)^d$ d-cubic boxes of edge $\epsilon_m = 1/k_m$, which satisfies $\epsilon_{\min} \leq \epsilon_m \leq 1$. The fact that $M+1$ grid sizes $\{\epsilon_m\}$ are nearly equidistantly distributed on a logarithmic scale is very useful for the statistical analysis of the box-counting results.

Now the boxes in the mth family could be labeled in the usual way by the integer vector $a \in \mathbb{Z}^d$ with $0 \leq a_\delta < k_m$. One of the crucial steps in our new approach is to map these boxes instead, in a one-to-one manner, onto *a linear list.* This map can be chosen as

$$f(a) = \sum_{\delta=1}^{d} (k_m)^{\delta-1} a_\delta . \tag{8}$$

Note that $0 \leq f(a) \leq (k_m)^d - 1$.

What we really need to know are the pertinent box addresses $b(i)$ of the points $\tilde{s}(i)$ of our normalized data set $\tilde{\mathcal{S}}$. These addresses are simply given by

$$b(i) \equiv b(m;i) = f(\mathrm{int}(k_m \tilde{s}(i))) = \sum_{\delta=1}^{d} (k_m)^{\delta-1} \mathrm{int}(k_m \tilde{s}_\delta(i)) . \tag{9}$$

The next important step is to reshuffle this linear list until the resulting list $\hat{b}(i)$ displays ascending order, i.e.,

$$\hat{b}(1) \leq \hat{b}(2) \leq \cdots \leq \hat{b}(N) . \tag{10}$$

All the necessary ingredients for calculating $D_B(q)$ from Eqs. (2) and (3) are now obtained straightforwardly: $N(\epsilon_m)$ is just the number of different blocks of identical elements in $\hat{b}(i)$, and the $N_\nu(\epsilon_m)$ are given by the lengths of these blocks. The actual addresses are irrelevant in this context.

Introducing the "partition function,"

$$Z_m(q) \equiv \sum_{\nu=1}^{N(\epsilon_m)} [p_\nu(\epsilon_m)]^q , \tag{11}$$

the generalized box-counting dimension $D_B(q)$ is approximatically determined by the slope of $\ln Z_m(q)$ versus $\ln \epsilon_m$. The accuracy of this estimate can be significantly improved by employing a hierarchical cluster analysis[17] that computes the particular subset of the $M+1$ pivotal values $Z_m(q)$, which achieves the best linear fitting. This technique is also an important part of our strategy, because generally the subboxes will not necessarily be centered around points of the fractal.

Let us summarize our box-counting method by formulating a specific realization in an algorithmic way.

Input: (i) Data matrix $\tilde{S}$ with elements $\tilde{s}_\delta(i)$,
$i=1,\ldots,N$;
$\delta=1,\ldots,d$; where $0 \leq \tilde{s}_\delta(i) < 1$.
(ii) Lower bound $\epsilon_{\min}$ for box partition.
(iii) Positive integer M controlling of pivotal points.
(iv) Moment parameter $q \neq 1$.

Output: Generalized fractal dimension $D_B(q)$.

(a) $e \leftarrow 1; m \leftarrow 0; r \leftarrow (\epsilon_{\min})^{-1/M}$.

(b) While $1/e > \epsilon_{\min}$ do

(b1) $k \leftarrow \mathrm{int}(e)$.

(b2) $x_m \leftarrow -\ln k$.

(b3) Evaluate $b(i) = \sum_{\delta=1}^{d} k^{\delta-1} \mathrm{int}(k \cdot \tilde{s}_\delta(i))$ for $i=1,\ldots,N$.

(b4) Sort list $b(i)$.

(b5) $j \leftarrow 2$; $a \leftarrow b(1)$; $n \leftarrow 1$; $Z_m(q) \leftarrow 0$.

(b6) While $j \leq N$ do

(b6a) if $b(j) \neq a$, then $Z_m(q) \leftarrow Z_m(q) + (n/N)^q$;

$a \leftarrow b(j)$; $n \leftarrow 1$; else $n \leftarrow n+1$ *fi*

(b6b) $j \leftarrow j+1$ od

(b7) $y_m \leftarrow \ln[(Z_m(q)+(n/N)^q]/(q-1)$.

(b8) $m \leftarrow m+1$.

(b9) $e \leftarrow e \cdot r$ od.

(c) Classwise linear regression with random point selection for abcisses x_m and ordinates y_m to determine $D_B(q)$.

The special case $q=1$ can be treated along the same lines.

The computer resources needed to run this algorithm are easily estimated: The determination of the box addresses requires $O(N)$ computations including Nd multiplications. As a matter of fact, most of the CPU time consumed is used for sorting these addresses. Employing the highly efficient Quick-sort method, it is possible, however, to limit this time and, as a consequence, the total time required to $O(N \log_2 N)$. [Note that the determination of $\epsilon_{\min}$ requires $O(N^2)$ steps, if its value is not known beforehand from the data-generating procedure. But this computation has to be performed only once anyway, independent of the number of pivotal points, and is rewarded by significantly enhanced precision for $D_B(q)$].

As for the memory demand, we come to a similar conclusion: Besides the input data array, only one further list of length N is needed for storage of the box addresses, and another return stack of length $\log_2 N$ for executing the sorting algorithm.

This overall improved performance of our technique is a solid basis for box-counting multifractal analysis of phenomena even in high-dimensional space, where very large data sets have to be handled.

III. TESTS AND APPLICATIONS

In the following we first test our method using various simple Euclidean manifolds, like lines, planes, cubes, and 2D tori. After that, the technique is applied to certain deterministic and stochastic fractal phenomena, respectively: the Kiesswetter curve, a Newtonian Julia set, and diffusion-limited aggregates in two- and three-dimensional embedding space. The calculations have been done both on an IBM/370 machine and with the aid of a parallel computer network (transputer cluster). In the latter case software parallelizing concepts like "farming"[16] have been used.

A. Standard objects

If the Euclidean test objects for our improved box-counting algorithm—straight lines or planes, for example—are generated pointwise on a regular d-dimensional grid, then we will numerically recover their exact topological dimensions. If, on the other hand, the data set representing the manifold in question is generated randomly and off-lattice, one usually has a hard time in determining its dimension by box counting. This familiar problem is well suited to put our method to the test.

Therefore, the point sets characterizing lines, planes, and cubes in E^3 have been produced with the help of the IMSL uniform random number generator. The results for $D_B(0)$ as obtained by running our algorithm with $M=60$ are presented in Table I.

The convergence with growing N is very good for the line, while it is somewhat slower for plane and cube. This may also have to do with the random number generator; nevertheless, the results are quite accurate for $N > 5 \times 10^4$.

The point set representing a two-torus in E^3, i.e., a "doughnut manifold," has been produced quasistochastically by using two rationally independent winding numbers $\omega_1 = 1$ and $\omega_2 = \frac{1}{2}(\sqrt{5}-1)$. The performance of our method is again very satisfactory (see Table I).

As the way our data sets are created simulates the trivial measure on the Euclidean objects considered, we have also computed $D_B(3)$ in all cases. The results of this further test are also listed in Table I and conform to the analytical requirement $D_B(q) = D_B(0)$ for all $q \in \mathbb{R}$.

Our box-counting algorithm has been applied to torus manifolds also in a different context: A very appealing approach to the analysis of seemingly irregular time series is provided by the reconstruction technique proposed by Packard *et al.*[18] and Takens,[19] which reveals

TABLE I. Numerical determination of generalized dimensions $D_B(0)$ and $D_B(3)$ for the trivial measure on Euclidean objects ($\epsilon_{\max} = 0.5$, $\epsilon_{\min} = 0.05$; all standard deviations ≤ 0.02).

Number of points N	$D_B(0)$ Line	Area	Cube	Torus	$D_B(3)$ Line	Area	Cube	Torus
1000	0.991	1.808	2.657	1.61	0.961	1.562	2.166	1.750
2000	1.000	1.929	2.759	1.80	0.984	1.703	2.438	1.793
5000	1.000	1.973	2.881	1.924	0.990	1.821	2.546	1.863
10 000	1.000	1.996	2.854	1.948	0.997	1.941	2.665	1.909
20 000	1.000	1.999	2.888	1.97	0.998	1.943	2.713	1.941
30 000	1.000	2.000	2.928	1.985	0.998	1.959	2.777	1.998
60 000	1.000	2.000	2.931	1.999	0.998	1.978	2.794	1.991
90 000			2.978	2.01			2.832	1.998
Topological dimension	1	2	3	2	1	2	3	2

whether a deterministic (strange) attractor in pseudospace is controlling the variation of the signal in time. The evaluation of the fractal dimensions of such an attractor is again very difficult, especially when high-dimensional embedding pseudospace is required (for details, see Ref. 2).

Standard box-counting techniques are rather ineffective even in case of simple quasiperiodic time series generated by superposition of signals with frequencies ω_1 and ω_2, respectively. The corresponding "attractor" is simply a two-torus. Our improved method has been applied to various artificial quasiperiodic time series by Teuber *et al.*[20] in order to obtain standards for dimensional analysis of real time series encountered in acoustical chaos. Regarding the computation of the correlation dimension $D_B(2)$, this technique has also been compared with the Grassberger-Procaccia algorithm:[6] For the above-mentioned two-torus and $N=90\,000$ the latter algorithm yields $D_B(2)=2.08$, while our box-counting technique finds the value 2.01.

B. Deterministic fractals

The first nonstandard tests object for our method are the so-called Kiesswetter curves of degree $k\in\mathbb{N}$.[21] Such a curve is the graph of the function $F^{(k)}(x)$, which is defined and bounded on [0,1] and satisfies

$$F^{(k)}\left[\frac{s}{k}\right]=f_s,\quad s=0,1,\ldots,k\ . \tag{12}$$

The f_s are arbitrary parameters for $s=1,\ldots,k-1$, while one fixes $f_0=0$ and $f_k=1$. Dilational invariance is enforced by the condition

$$F^{(k)}(x)=f_i+(f_{i+1}-f_i)F(kx-i)\ ,\quad x\in\left[\frac{i}{k},\frac{(i+1)}{k}\right]\ , \tag{13}$$

where $i=0,1,\ldots,(k-1)$.

Choosing $k=4$ and $f_1=-0.5$, $f_2=0$, $f_3=0.5$, one produces a special realization, which has $D_B(0)=1.5$ (Ref. 21) and is depicted in Fig. 1.

This curve has recently been used by Dubuc *et al.* for an assessment of box-counting methods.[22] The numerical estimate for $D_B(0)$ obtained by them with standard techniques can be significantly improved by our new algorithm in combination with a classwise linear regression analysis based on random point selection.[17] The results are compared in Table II.

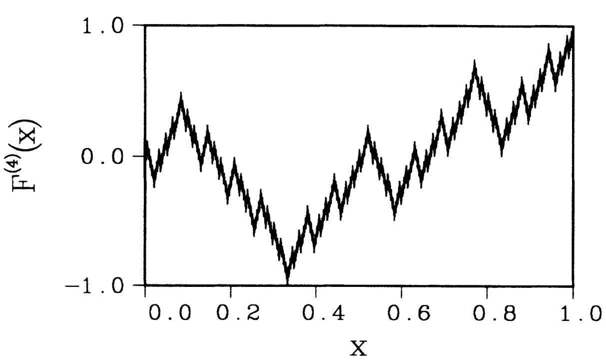

FIG. 1. The Kiesswetter curve of degree 4 produced by the parameter choice described in the text.

TABLE II. Box-counting dimension $D_B(0)$ of Kiesswetter curve ($\epsilon_{max}=0.25$, $\epsilon_{min}=0.01$; standard deviation ≤ 0.015).

N	Improved box counting	Standard box counting (Ref. 22)	Analytical result
16 384	1.435	1.39	1.5
65 536	1.461		

Next we report on the application of the improved box-counting algorithm, to a complicated multifractal, namely, the "natural" measure on the Julia set associated with the Cayley problem:

Let

$$F(z)=z^3-1,\quad z\in\mathbb{C}\ . \tag{14}$$

The roots of F are

$$z_k=e^{i(2\pi/3)k},\quad k=0,1,2\ . \tag{15}$$

Denote by $A(z_k)$ the basin of attraction of z_k with respect to the Newtonian iterated map defined by F, namely,

$$N(z)=z-\frac{F(z)}{F'(z)}=\tfrac{2}{3}z+\frac{1}{3z^2}\ . \tag{16}$$

The Julia set J is a strange repellor of N and constitutes the *simultaneous* boundary of the basins of attractions of the z_k,[23] i.e.,

$$J=\partial A(z_0)=\partial A(z_1)=\partial A(z_2)\ . \tag{17}$$

Let I denote the inverse orbit of 0 under N:

$$I=\{z\in\mathbb{C}|N^l(z)=0,\quad l\in\mathbb{N}\}\ . \tag{18}$$

Then it can be shown[23] that

$$J=\bar{I}(\text{closure of } I)\ . \tag{19}$$

Equation (18) induces a method for producing iteratively a dense subset of J. As a matter of fact, this technique

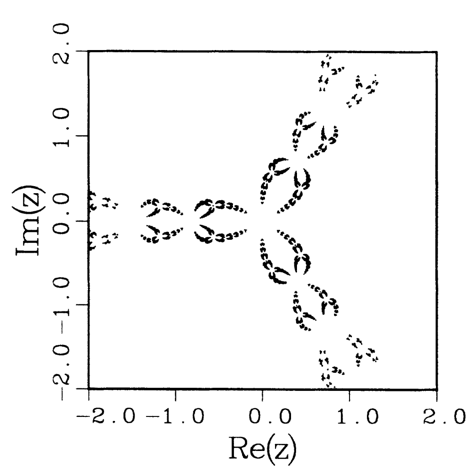

FIG. 2. Inverse-orbit generation of the Newtonian Julia set associated with Cayley's problem.

TABLE III. Numerical evaluation of selected generalized dimensions for the nontrival measure μ defined on the Julia set J ($\epsilon_{max}=0.396$, $\epsilon_{min}=0.0049$; all standard deviations ≤ 0.025).

N	$D_B(0)$	$D_B(1)$	$D_B(2)$
20 000	1.334	1.209	1.128
133 326	1.393	1.232	1.151
333 308	1.399	1.209	1.128
666 614	1.416	1.234	1.153
1 000 000	1.420	1.228	1.132

simultaneously defines a measure μ on the Julia set. μ can be shown to represent the unique measure with maximal entropy.[24] Figure 2 gives an impression of the multifractal emerging by iteration.

Using large samples of points generated by the inverse-orbit method, we have analyzed the dimensional properties of μ with our box-counting algorithm. The results for $D_B(0)$, $D_B(1)$, and $D_B(2)$ as shown in Table III are in good agreement with the analytical approximation by Nauenberg and Schellnhuber based on a forward manifold iteration scheme.[25] The numerical analysis of Julia-set measures may be further improved by combining our method with the strategies proposed by Saupe.[26]

C. Multifractal properties of DLA clusters

Some years ago Witten and Sander introduced a very successful model for diffusion-limited aggregation,[27] which is easily implemented on a computer and displays interesting features. Extensive studies have revealed that the aggregates grown according to the rules of this model are random fractals; for recent reviews see Refs. 4 and 28.

We have numerically generated and analyzed DLA clusters consisting of approximately 5×10^4 particles in two- and three-dimensional embedding space. Using the efficient computational technique described above, we find $D_B(0)=1.65$ and $D_B(0)=2.3$ for $d=2$ and 3, respectively. These results differ slightly but definitely from the familiar values obtained for the mass-scaling dimension, namely, $D_m=1.70$ for $d=2$ and $D_m=2.51$ for $d=3$.[28]

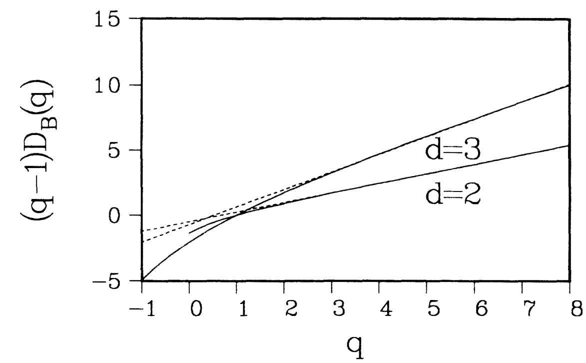

FIG. 3. Solid lines: improved box-counting results for modified generalized dimensions $(q-1)D_B(q)$, characterizing the multifractal growth probability measure of DLA clusters for $d=2$ and $d=3$, respectively. Dashed lines: Linear approximations according to Ref. 29.

It is well known by now that the growth-site probability distribution $\{p_i\}_{i\in A}$ of DLA clusters is a multifractal measure. We denote by $\mathcal{A}$ the perimeter of the aggregate and by p_i the probability that site i becomes part of the cluster in the next time step. Knowledge of this measure, which is completely characterized by the infinite hierarchy of generalized fractal dimensions $D_B(q)$, $q\in\mathbb{R}$, is crucial to the understanding of the growth phenomenon under consideration.

We have used our box-counting algorithm for computing $D_B(q)$, in particular, on the interval $q\in[0,8]$ and compared the results with recently published values obtained by different methods.[29,30] The growth-zone data sample was generated by launching random-walking test particles, which may collide with the quenched aggregate. The collision record directly yields the desired probability measure.

The growth-probability results for DLA clusters in two- and three-dimensional embedding space are shown in Table IV and Figs. 3 and 4, respectively. Our numerical values regarding $d=2$ are in good agreement with singular analytical results stating that the entropy dimen-

TABLE IV. Selected generalized dimensions $D_B(q)$ for the multifractal growth probability measure of DLA for $d=2,3$ (size of cluster: 5×10^4 particles; number of tracer particles: 10^5). ($\epsilon_{max}=706$, $\epsilon_{min}=12.9$; all standard deviations $\leq$ than 0.02).

	$D_B(q)$	
q	$d=2$	$d=3$
−1		2.696
0	1.352	2.191
1	1.012	1.950
2	0.902	1.812
3	0.849	1.725
4	0.849	1.669
8	0.567	1.470

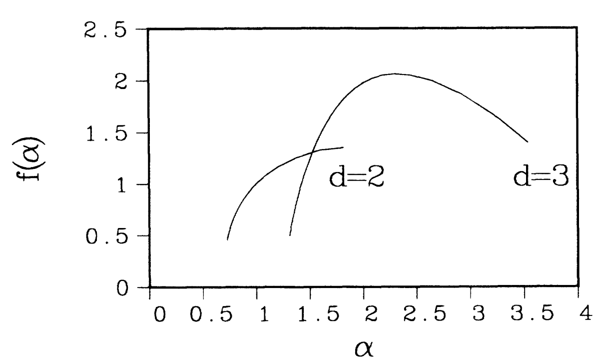

FIG. 4. Box-counting determination of spectrum of scaling indices $f(\alpha)$ for $d=2$ and 3, respectively.

sion $D_B(1)$ has to be precisely 1 for any connected object in E^2 (Ref. 31) (see Table IV) and that the lower boundary of the support of $f(\alpha)$ satisfies $\alpha_{\min}=D_B^{\text{cluster}}(0)-1$ (Ref. 32) (see Fig. 4).

From Fig. 4 one can see that we find for the entity $(q-1)D_B(q)$ in the regime $q \geq 2$ a quasilinear behavior, similar to the one reported by Halsey *et al.*;[29] in the interval $0 \leq q \leq 2$ we are able to provide for an independent confirmation of the results of Armitrano *et al.*,[30] who find nonlinear functional dependence by using an electrostatic analogon and solving numerically the Laplace equation by standard Green's techniques.

We are not aware of any previous box-counting determinations of the multifractal growth probability measure of 3D DLA. However, interesting analytical approximations were presented quite recently by Wang *et al.*,[33] who employ a kinetic renormalization group approach. These authors find, for example, $D_B(1)=2.21$, a result that differs somewhat from the value 2 predicted on general grounds. Our numerical result for $D_B(1)$ (see Table IV) fits much better.

IV. CONCLUDING REMARKS

In the present paper we have demonstrated that box-counting analysis of multifractal phenomena is quite generally possible if efficient computational strategies are employed. The costs of the particular algorithm presented here depends quasilinearly on the size of the input data set. This high performance is achieved by a philosophy similar to the one underlying the definition of Lebesgue integration: Instead of searching the almost everywhere empty Euclidean space for sample points, we set out with the data and determine their coordinates. This means that our technique is range oriented as opposed to the previous domain-oriented ones. Applications of the improved algorithm to random multifractals have produced several novel results.

After completion of this work we came across a recent paper by Liebovitch and Toth,[34] who have independently developed a similar box-counting method for the computation of the capacity dimension. In fact, the basic idea instrumental to efficient box counting is rather evident, but its full potential has to be explored now.

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