Family of exact solutions for the Coulomb potential perturbed by a polynomial in *r*

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A method based on supersymmetric quantum mechanics is given for obtaining exact solutions of the potential $V(r)=\alpha/r+p_1r+p_2r^2+p_3r^3+p_4r^4$ where α and the p's are parameters, provided certain relations are satisfied between the parameters. Detailed results are given for three specific cases. The potential in question gives rise to some very interesting shapes (double-well, etc.). The applicability of the shifted $1/N$ expansion method to such potential shapes is examined by comparing eigenenergies obtained by this method with the exact ones obtained from supersymmetric considerations. It is found that in certain situations, the shifted $1/N$ expansion method may give poor or erroneous results. Applicability of the proposed method to potentials involving higher powers of r is also discussed.

I. INTRODUCTION

The Coulomb potential perturbed by a term or terms involving various powers of r occurs in several physical contexts and such potentials have been investigated by a number of workers. The potential

$$V(r)=\frac{\alpha}{r}+p_1 r,\quad \alpha<0 \tag{1}$$

where α and p_1 are parameters, corresponds to a spherical Stark effect in hydrogen. This potential also occurs in the context of quarkonium and similar bound-state problems in particle physics, and has been studied by a number of works with a variety of techniques.[1–20] The potential

$$V(r)=\frac{\alpha}{r}+p_2 r^2 \tag{2}$$

may be considered to correspond to a spherical quadratic Zeeman effect and has been examined.[21] The ion-sphere model used in plasma-physics problems[22–24] also has the same potential form. A generalization of the above two potentials,

$$V(r)=\frac{\alpha}{r}+p_n r^n \tag{3}$$

has also been investigated.[25–28]

Gupta and Khare[29] suggested

$$V(r)=\frac{\alpha}{r}+p_1 r+p_2 r^2 \tag{4}$$

as a quark confining potential on the basis of the ^3P_J splittings of charmonium levels. This potential or its special cases have been studied by several authors.[30–40] Potentials of the form

$$V(r)=-\frac{Z}{r}\sum_{k=0}^{\infty}V_k(\lambda r)^k , \tag{5}$$

where λ is the screening parameter, have also been investigated.[41–43]

Recently Dutra[44] has obtained an exact solution for the potential

$$V(r)=\frac{\alpha}{r}+p_1 r+p_2 r^2+p_3 r^3+p_4 r^4 , \tag{6}$$

where α and the p's are parameters, proved two of the parameters depend on the other three parameters through certain relations.

Adhikari, Dutt, and Varshni[45] have considered a more general potential

$$V(x)=\sum_{n=3}^{2N}b_{n-2}x^{n-2}+\alpha/x+l(l+1)/x^2,\quad b_{2N-2}>0 ,$$

in which x refers to either the one- or the three-dimensional variable. Numerical results have been obtained for a tenth-degree even-power polynomial potential ($N=6$, $\alpha=0$).

There have been a number of investigations on even-power polynomial potentials. References may be found in Adhikari, Dutt, and Varshni[45] An additional recent reference is that of Kaushal.[46]

In the present paper we give a general method using supersymmetric quantum mechanics (SUSYQM) for obtaining a family of exact solutions for the potential (6) subject to certain relations between the parameters. We may note here that a special case of the potential (6), namely the Coulomb potential ($p_1=p_2=p_3=p_4=0$) has been treated by SUSYQM by other workers.[47–49] Khare and Sukhatme[49] have produced a family of phase equivalent potentials to the Coulomb potential, some of which have shapes that bear a similarity to some of the shapes produced by Eq. (6). Here we are interested in the case when p_1, p_2, p_3, and p_4 are not equal to zero. Depending on the values of the parameters, the potential (6) gives rise to a variety of interesting shapes (e.g., double-well, etc.). The fact that we are able to obtain an exact eigenenergy for one of the levels for such potential shapes provides us with an opportunity to examine the applicability of the shifted $1/N$ expansion method[50–54] to such potential shapes. The shifted $1/N$ expansion method has proved to be quite successful for a variety of potentials with simple shapes. It is of obvious interest to examine how well it does for more complicated shapes. The plan of the paper is as follows. In Sec. II we present a general method of obtaining exact solutions for the potential (6) and we illustrate it by three cases. In Sec. III we apply the shifted $1/N$ expansion to this potential. The numerical results are presented and discussed in Sec. IV. In Sec. V it is shown that the proposed method can also be extended for potentials which have terms in higher powers of r. Throughout the paper, we shall use atomic units in which $2m=\hbar=e=1$.

II. EXACT SOLUTIONS FOR THE POTENTIAL (6) FROM SUSYQM

In one dimension the Hamiltonian of SUSYQM is given by

$$H^S=\{Q^\dagger,Q\}=\begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} , \tag{7}$$

where

$$H_\pm=-\frac{d^2}{dx^2}+V_\pm(x) , \tag{8}$$

$$V_\pm(x)=W^2(x)\pm\frac{dW(x)}{dx} . \tag{9}$$

$W(x)$ is called the superpotential and $Q,Q^\dagger$ the supercharges, whose explicit forms are given below:

$$Q=(p+iW)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \tag{10}$$

$$Q^\dagger=(p-iW)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} . \tag{11}$$

The relations obeyed by Q, $Q^\dagger$, and H^S are the following:

$$[H^S,Q]=[H^S,Q^\dagger]=0 ,$$

$$Q^2=(Q^\dagger)^2=0 .$$

The eigenstates of H^S are

$$\phi^n(x)=\begin{bmatrix} \phi_+^n(x) \\ \phi_-^n(x) \end{bmatrix} . \tag{12}$$

If supersymmetry is unbroken the ground-state energy is zero and the ground-state wave functions are of the form

$$\begin{bmatrix} \phi_+^0(x) \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ \phi_-^0(x) \end{bmatrix} , \tag{13}$$

depending on the normalizability of $\phi_+^0(x)$ or $\phi_-^0(x)$. Now if $|\varphi\rangle$ is a ground state then

$$Q|\varphi\rangle=Q^\dagger|\varphi\rangle=0 . \tag{14}$$

From (10) and (11) it follows that

$$\phi_\pm^0(x)=\exp\left[\pm\int^x W(t)dt\right] . \tag{15}$$

Now we consider the potential (6). The effective potential corresponding to (6) can be written as

$$V^{\text{eff}}(r)=\frac{l(l+1)}{r^2}+\frac{\alpha}{r}+p_1r+p_2r^2+p_3r^3+p_4r^4 , \tag{16}$$

and the Schrödinger equation, as

$$-\frac{d^2\psi}{dr^2}+\left[\frac{l(l+1)}{r^2}+\frac{\alpha}{r}+p_1r+p_2r^2+p_3r^3+p_4r^4-E\right]\psi=0 . \tag{17}$$

Following the standard method of constructing exact solutions of the Schrödinger equation from supersymmetric considerations,[55,56] we take the superpotential in the following form:

$$W=a+\frac{b}{r}+cr+dr^2+\sum_{i=1}^{n_1}\frac{2g_ir}{1+g_ir^2}+\sum_{i=1}^{n_2}\frac{h_i}{1+h_ir} . \tag{18}$$

While n_2 can have any positive integral value, n_1 is restricted to integral values up to and including 4 as each g_i puts a constraint on the parameters $\alpha,p_1,\ldots$. Writing

$$V^{\text{eff}}(r)-E=W^2+W'-E_s , \tag{19}$$

where E_s denotes the supersymmetric energy, we have, from Eq. (18), after some manipulations,

$$\begin{aligned} V^{\text{eff}}(r)+E_s-E={}&\frac{b(b-1)}{r^2}+\frac{2ab}{r}+r(2ac+2bd+2d)+r^2(c^2+2ad)+2cdr^3+d^2r^4+4a\sum_{i=1}^{n_1}\frac{g_ir}{1+g_ir^2} \\ &+4b\sum_{i=1}^{n_1}\frac{g_i}{1+g_ir^2}+4n_1c-4c\sum_{i=1}^{n_1}\frac{1}{1+g_ir^2}+4dn_1r-4d\sum_{i=1}^{n_1}\frac{r}{1+g_ir^2}+2a\sum_{i=1}^{n_2}\frac{h_i}{1+h_ir} \\ &+\frac{2b}{r}\sum_{i=1}^{n_2}h_i-2b\sum_{i=1}^{n_2}\frac{h_i^2}{1+h_ir}+2cn_2-2c\sum_{i=1}^{n_2}\frac{1}{1+h_ir}+2dn_2r-2d\sum_{i=1}^{n_2}\frac{1}{h_i}+2\sum_{i=1}^{n_2}\frac{d/h_i}{1+h_ir} \\ &+4\sum_{i=1}^{n_1}\sum_{\substack{j=1\\ j\neq i}}^{n_2}\frac{g_ig_j}{g_i-g_j}\left[\frac{1}{1+g_jr^2}-\frac{1}{1+g_ir^2}\right]+\sum_{i=1}^{n_1}\sum_{\substack{j=1\\ j\neq i}}^{n_2}\frac{h_ih_j}{h_i-h_j}\left[\frac{h_i}{1+h_ir}-\frac{h_j}{1+h_jr}\right] \\ &+2\sum_{i=1}^{n_1}\sum_{\substack{j=1\\ j\neq i}}^{n_2}\frac{1}{(g_i+h_j^2)}\left[\frac{g_i^2h_jr+h_j^2g_i}{1+g_ir^2}-\frac{h_j^2g_i}{1+h_jr}\right]+a^2+2bc+c+\sum_{i=1}^{n_1}\frac{2g_i}{1+g_ir^2} . \end{aligned} \tag{20}$$

Now if we put the expression for $V^{\rm eff}(r)$ from (16) in the left-hand side of Eq. (20), and equate each power of r and the coefficients of $1/(1+g_ir^2)$, $1/(1+h_ir)$, and $r/(1+g_ir)$, we get

$$b=l+1\ , \tag{21a}$$

$$c^2+2ad=p_2\ , \tag{21b}$$

$$2cd=p_3\ , \tag{21c}$$

$$d=-\sqrt{p_4}\ , \tag{21d}$$

$$p_1=2ac+d(4n_1+2n_2+2l+4)\ . \tag{21e}$$

In Eq. (21d), the negative sign has been taken to ensure that

$$\exp\left[-\int^x W(t)dt\right]$$

is normalizable.

The constants a, g_i, and h_i are found from the following constraint equations:

$$a+\sum_{i=1}^{n_2} h_i=\frac{\alpha}{2(l+1)}\ , \tag{22a}$$

$$4ag_i-4d+2\frac{g_i^2h_j}{g_i+h_j^2}=0,\quad i=1,\ 2,\ \ldots,\ n_1 \tag{22b}$$

$$4bg_i-4c-4\sum_{j\,(\neq i)}\frac{g_ig_j}{g_i-g_j}+2\sum_j\frac{h_j^2g_i}{g_i+h_j^2}+2g_i=0\ ,\quad i=1,\ 2,\ \ldots,\ n_1 \tag{22c}$$

and

$$2ah_j-2bh_j^2-2c+\frac{2d}{h_j}+\sum_{i\,(\neq j)}\frac{h_j^2h_i}{h_j-h_i}=0\ ,\quad j=1,\ 2,\ \ldots,\ n_2\ . \tag{22d}$$

Finally,

$$E-E_s=-\left[a^2+2bc+(4n_1+2n_2+1)c-\sum_j\frac{2d}{h_j}\right]. \tag{23}$$

Since there are $6+2n_1+n_2$ equations and $4+n_1+n_2$ variables (namely a, b, c, d, g_i, and h_j) to solve, there will be $2+n_1$ restraints on the parameters l, α, p_1, p_2, p_3, and p_4. Hence our early assertion that n_1 cannot be greater than 4. We give below some specific examples for obtaining exact solutions.

Case I. We take $n_1=n_2=0$. There will be two constraints on the parameters. If we take l, α, p_3, and p_4 as arbitrary, then we have

$$a=\frac{\alpha}{2(l+1)}\ ,\qquad p_1=2ac+2bd+2d=-\frac{\alpha p_3}{2(l+1)\sqrt{p_4}}-(2l+4)\sqrt{p_4}\ , \tag{24}$$

and

$$p_2=c^2+2ad=\frac{p_3^2}{4p_4}-\frac{\alpha\sqrt{p_4}}{(l+1)}\ . \tag{25}$$

The energy is given by

$$E=-(a^2+2bc+c)\ . \tag{26}$$

This is the result obtained by Dutra.[44] To show the equivalence, we identify his p with $2l+\frac{3}{2}$, and our α, p_1, p_2, p_3, and p_4 with Dutra's $-2e$, 2λ, ω^2, 2α, and 2β, respectively. Then from Eqs. (24) and (25), we have

$$8e=\left[\frac{\omega^2}{\sqrt{2\beta}}-\frac{\alpha^2}{(2\beta)^{3/2}}\right](2p+1)\ , \tag{27}$$

$$8\lambda=\frac{2\alpha\omega^2}{\beta}-\frac{\alpha^3}{\beta^2}-2\sqrt{2\beta}(2p+5)\ , \tag{28}$$

and

$$\psi=r^{l+1}\exp(ar+cr^2/2+dr^3/3)=r^{(p+1/2)/2}\exp\left[-\left[\frac{\omega^2}{2\sqrt{2\beta}}-\frac{\alpha^2}{4(2\beta)^{3/2}}\right]r-\frac{\alpha}{2\sqrt{2\beta}}r^2-\frac{\sqrt{2\beta}}{3}r^3\right]. \tag{29}$$

These results are identical to those obtained by Dutra.[44]

Case II. We take $n_1=1$ and $n_2=0$. There will be three constraints. If we choose l, α, and p_4 arbitrarily, then p_3, p_1, and p_2 are given by

$$p_3=(l+1)(4l+6)p_4/\alpha\ ,$$

$$p_1=-\frac{\alpha p_3}{2(l+1)\sqrt{p_4}}-\sqrt{p_4}(2l+8)\ ,$$

$$p_2=\frac{p_3^2}{4p_4}-\frac{\alpha\sqrt{p_4}}{(l+1)}\ .$$

Also we have

$$E=-\left[\frac{\alpha^2}{4(l+1)^2}-(2l+7)\frac{p_3}{2\sqrt{p_4}}\right] \tag{30}$$

and

$$\psi=(1+g_1r^2)r^{l+1}\exp\left[\frac{\alpha r}{2(l+1)}-\frac{p_3r^2}{4\sqrt{p_4}}-\frac{\sqrt{p_4}r^3}{3}\right], \tag{31}$$

where

$$g_1=-2(l+1)\sqrt{p_4}/\alpha\ .$$

Case III. We take $n_1=0$ and $n_2=1$. There are two constraints. If we select l, α, p_3, and p_4 as independent parameters, then p_1 and p_2 are given by

$$p_1=2ac+d(2l+6)\ ,$$

and

$$p_2=c^2+2ad\ .$$

Also, we get

$$E=-(a^2+2bc+2c-2d/h_1)\ , \tag{32}$$

and

$$\psi=r^{l+1}(1+h_1r)\exp(ar+cr^2/2+dr^3/3)\ , \tag{33}$$

where b, c, and d are obtained from (21a), (21c), and (21d), and a and h_1 are determined from

$$a+h_1=\frac{\alpha}{2(l+1)}\ , \tag{34}$$

$$2ah_1-2bh_1^2-2c+\frac{2d}{h_1}=0\ . \tag{35}$$

Other solutions with larger values of n_1 and n_2 can similarly be found.

III. SHIFTED 1/N EXPANSION

Imbo, Pagnamenta, and Sukhatme[51] have described the method for obtaining the energy eigenvalues in the shifted $1/N$ expansion formalism. The necessary final expressions for obtaining the eigenenergies for the potential (6) are given below.

In the shifted $1/N$ method, one works with an effective potential, the position of the minimum, r_0, of which in our case is determined from

$$(2l+1)+(2n+1)\left[\frac{\alpha-3p_1r^2-8p_2r^3-15p_3r^4-24p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]^{1/2}=[2r(-\alpha+p_1r^2+2p_2r^3+3p_3r^4+4p_4r^5)]^{1/2}\ , \tag{36}$$

where n_r is the radial quantum number. We also have

$$\bar{k}^2=2r(-\alpha+p_1r^2+2p_2r^3+3p_34^4+4p_4r^5)\ , \tag{37}$$

and the energy is given by

$$E=\frac{\bar{k}^2}{r_0^2}\left[\frac{1}{4}-\frac{1}{2}\left[\frac{\alpha+p_1r^2+p_2r^3+p_3r^4+p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]+\frac{\beta^{(1)}}{\bar{k}^2}+\frac{\beta^{(2)}}{\bar{k}^3}+0\left[\frac{1}{\bar{k}^4}\right]\right]\ . \tag{38}$$

The quantities $\beta^{(1)}$ and $\beta^{(2)}$ appearing in the corrections to the leading order of the energy expansion are

$$\beta^{(1)}=\tfrac{1}{8}(1-a)(3-a)+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4-\frac{1}{\omega}[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\ , \tag{39}$$

$$\begin{aligned}\beta^{(2)}=&(1+2n_r)\tilde{\delta}_2+3(1+2n_r+2n_r^2)\tilde{\delta}_4+5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6\\&-\omega^{-1}[(1+2n_r)\tilde{\varepsilon}_2^2+12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4+2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2+2\tilde{\varepsilon}_1\tilde{\delta}_1\\&+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3+30(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5+6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1+2(11+30n_r+30n_r^2)\tilde{\varepsilon}_3\tilde{\delta}_3\\&+10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5]\\&+\omega^{-2}[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2+24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4\\&+8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\varepsilon_4+12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]\\&-\omega^{-3}[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3+30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4]\ ,\end{aligned} \tag{40}$$

in which

$$\tilde{\varepsilon}_j=\varepsilon_j/\omega^{j/2},\quad \tilde{\delta}_j=\delta_j/\omega^{j/2}\ ,$$

$$\omega=\left[\frac{\alpha-3p_1r^2-8p_2r^3-15p_3r^4-24p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]^{1/2}\ ,$$

$$a=2-(2n_r+1)\omega\ ,$$

$$\delta_1=-\tfrac{2}{3}\delta_2=-(1-a)(3-a)/2\ ,$$

$$\delta_3=-\tfrac{4}{5}\delta_4=2\varepsilon_1=-\tfrac{4}{3}\varepsilon_2=2(2-a)\ ,$$

$$\varepsilon_3=-\frac{1}{2}\left[\frac{\alpha-2p_1r^2-4p_2r^3-5p_3r^4-4p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]\ ,$$

$$\varepsilon_4=\frac{1}{4}\left[\frac{3\alpha-5p_1r^2-10p_2r^3-15p_3r^4-22p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]\ ,$$

$$\delta_5=-\frac{1}{2}\left[\frac{2\alpha-3p_1r^2-6p_2r^3-9p_3r^4-12p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]\ ,$$

$$\delta_6=\frac{1}{4}\left[\frac{5\alpha-7p_1r^2-14p_2r^3-21p_3r^4-28p_4r^5}{\alpha-p_1r^2-2p_2r^3-3p_3r^4-4p_4r^5}\right]\ .$$

IV. RESULTS AND DISCUSSION

Calculations were carried out for the three cases and the results are shown in Tables I–III. For ease of

TABLE I. Comparison of the eigenvalues calculated from the shifted $1/N$ expansion with the exact supersymmetric values for case I.

Set No.	l	α	p_3	p_4	p_1	p_2	E (shifted $1/N$ expansion) $n_r=0$	E (SUSY)
1	0	−1	−0.2	0.1	−1.581 14	0.416 23	−1.154 71	−1.198 68
2	1	−1	−0.2	0.1	−2.055 48	0.258 11	−1.638 74	−1.643 64
3	2	−1	−0.2	0.1	−2.635 23	0.205 41	−2.240 25	−2.241 37
4	3	−1	−0.2	0.1	−3.241 33	0.179 06	−2.861 33	−2.861 67
5	0	−1	−1.0	0.1	−2.846 05	2.816 23	−121.623 25	−4.993 42
6	1	−1	−1.0	0.1	−2.687 94	2.658 11	−7.975 46	−7.968 19
7	2	−1	−1.0	0.1	−3.056 87	2.605 41		−11.095 75
8	3	−1	−1.0	0.1	−3.557 56	2.579 06		−14.245 87
9	0	−1	1.0	0.1	0.316 23	2.816 23	4.512 43	4.493 42
10	1	−1	1.0	0.1	−1.106 80	2.658 11	7.845 17	7.843 19
11	2	−1	1.0	0.1	−2.002 78	2.605 41	11.040 64	11.040 19
12	3	−1	1.0	0.1	−2.766 99	2.579 06	14.214 77	14.214 62
13	0	−5	−1.0	0.1	−9.170 61	4.081 14	−11.143 19	−10.993 42
14	1	−5	−1.0	0.1	−5.850 21	3.290 57	−9.473 67	−9.468 19
15	2	−5	−1.0	0.1	−5.165 05	3.027 05	−11.767 19	−11.762 42
16	3	−5	−1.0	0.1	−5.138 70	2.895 28	−14.624 23	−14.620 87
17	0	−5	2.0	0.1	14.546 48	11.581 14	3.255 20	3.236 83
18	1	−5	2.0	0.1	6.008 33	10.790 57	14.255 18	14.248 89
19	2	−5	2.0	0.1	2.740 64	10.527 05	21.443 29	21.441 50
20	3	−5	2.0	0.1	0.790 57	10.395 28	28.070 52	28.069 87
21	0	−5	2.0	0.5	4.242 64	5.535 53	−1.998 77	−2.007 36
22	1	−5	2.0	0.5	−0.707 11	3.767 77	5.516 39	5.508 57
23	2	−5	2.0	0.5	−3.299 83	3.178 51	9.207 70	9.205 05
24	3	−5	2.0	0.5	−5.303 30	2.883 88	12.338 33	12.337 30

identification, each data set has been assigned a number given in column 1 of each table. In each table, the independent parameters are listed first, followed by dependent parameters. For three of the data sets, no r_0 value could be determined in the shifted $1/N$ method, and for these sets, there is no entry in the energy column. We discuss each case one by one.

Case I (Table I). It will be noticed that for all data sets for which E (shifted $1/N$ expansion) could be calculated, there is a satisfactory agreement with the exact supersymmetric value, except for one. This exception is data set No. 5. Here it turns out that the energy series [Eq.

TABLE II. Comparison of the eigenvalues calculated from the shifted $1/N$ expansion with the exact supersymmetric values for case II. Asterisk denotes large discrepancy with E (SUSY).

Set No.	l	α	p_4	p_1	p_2	p_3	E (shifted $1/N$ expansion) $n_r=0$	E (SUSY)
25	0	−1	0.1	−3.478 51	1.216 23	−0.60	−6.913 43	−6.890 78
26	1	−1	0.1	−4.743 42	10.158 11	−2.00	8.517 20*	−28.523 00
27	2	−1	0.1	−6.008 33	44.205 41	−4.20	38.861 69*	−73.076 39
28	3	−1	0.1	−7.273 24	129.679 06	−7.20	94.294 97*	−148.010 22
29	0	−1	1.0	−11.000 00	10.000 00	−6.00	−21.277 98	−21.250 00
30	1	−1	1.0	−15.000 00	100.500 00	−20.00	37.350 13*	−90.062 50
31	2	−1	1.0	−19.000 00	441.333 33	−42.00	133.519 09*	−231.027 78
32	3	−1	1.0	−23.000 00	1296.250 00	−72.00	309.517 25*	−468.015 62
33	0	−10	0.1	−3.478 51	3.171 28	−0.06	−25.663 67	−25.664 08
34	1	−10	0.1	−4.743 42	1.681 14	−0.20	−9.131 29	−9.096 05
35	2	−10	0.1	−6.008 33	1.495 09	−0.42	−10.056 89	−10.082 64
36	3	−10	0.1	−7.273 24	2.086 57	−0.72	−16.361 60	−16.361 96
37	0	−10	1.0	−11.000 00	10.090 00	−0.60	−27.093 83	−27.100 00
38	1	−10	1.0	−15.000 00	6.000 00	−2.00	−15.148 01	−15.250 00
39	2	−10	1.0	−19.000 00	7.743 33	−4.20	−25.872 40	−25.877 78
40	3	−10	1.0	−23.000 00	15.460 00	−7.20		−48.362 50

(38)] is divergent, the contributions from the $\beta^{(1)}$ and $\beta^{(2)}$ terms being very large. The shape of this potential is shown in Fig. 1. Besides the well at $r=0$, it will be noticed that there is another well.

Case II (Table II). Except for those six cases which are marked with an asterisk, the calculated values by the shifted $1/N$ expansion are in good agreement with the exact values. We then consider the six discrepant cases. The supersymmetric method does not necessarily always give the lowest level for a given l. If this were the source of discrepancy, then the shifted $1/N$ expansion value should lie lower than the supersymmetric (SUSY) value, but the opposite is true in all the six cases. Thus this possibility has to be ruled out. The shape of the potential for data set No. 26 is shown in Fig. 2. There are seen to be two wells of different depths, the second one being the deeper of the two. To find out exactly what is happening, numerical integration of the Schrödinger equation was resorted to for calculating the eigenvalues for data set No. 26. Calculations were carried out only for the l value that occurs in the data set, i.e., $l=1$. The results are shown in Table IV. $n=1$ indicates the lowest level for a given l. It will be noticed that the SUSY calculation gives the lowest level but the shifted $1/N$ calculation corresponds to the $n=8$ level. It appears that this is the first level with $l=1$ in the first well, the lower lying seven oth-

TABLE III. Comparison of the eigenvalues calculated from the shifted $1/N$ expansion with the exact supersymmetric values for case III. Asterisk denotes large discrepancy with E (SUSY).

Set No.	l	α	p_3	p_4	p_1	p_2	E (shifted $1/N$ expansion) $n_r=0$	E (shifted $1/N$ expansion) $n_r=1$	E (SUSY)
41	0	−1	−0.2	0.1	−1.882 16	0.084 80	−2.774 82	−0.142 09*	−0.374 83[a]
42	1	−1	−0.2	0.1	−2.420 36	−0.009 46	−3.871 27	−0.695 14	−0.748 66[a]
43	2	−1	−0.2	0.1	−3.027 14	−0.035 13	−4.830 20	−1.211 44	−1.228 80[a]
44	3	−1	−0.2	0.1	−3.650 46	−0.044 27	−5.721 52	−1.733 35	−1.739 46[a]
45	0	−1	−1.0	0.1	−2.838 26	2.688 18	−8.049 06	−5.016 45	−4.870 44[a]
46	1	−1	−1.0	0.1	−2.712 14	2.536 46	−11.352 99	−7.893 91	−7.783 21[a]
47	2	−1	−1.0	0.1	−3.097 61	2.487 07	−14.664 84	−10.936 59	−10.850 66[a]
48	3	−1	−1.0	0.1	−3.610 66	2.463 19	−17.959 81	−14.011 96	−13.943 79[a]
49	0	−1	1.0	0.1	1.633 03	3.206 08	5.651 32	15.921 21	5.633 29
50	0	−1	1.0	0.1	−3.793 68	2.120 74	0.832 11	8.369 60	8.121 23[a]
51	0	−1	1.0	0.1	0.421 40	2.963 75	4.675 57	14.323 57	4.656 65
52	1	−1	1.0	0.1	−0.050 62	2.995 84	9.270 83	18.888 19	9.268 94
53	1	−1	1.0	0.1	−4.426 10	2.120 75	3.591 37	11.520 93	11.452 75[a]
54	1	−1	1.0	0.1	−1.004 57	2.805 05	8.115 04	17.350 90	8.113 07
55	2	−1	1.0	0.1	−1.221 75	2.888 10	12.439 16	22.013 95	12.438 74
56	2	−1	1.0	0.1	−4.945 91	2.143 27	6.445 21	14.805 97	14.777 66[a]
57	2	−1	1.0	0.1	−1.869 79	2.758 50	11.450 76	20.806 27	11.450 32
58	3	−1	1.0	0.1	−2.238 54	2.811 24	15.427 71	25.085 91	15.427 57
59	3	−1	1.0	0.1	−5.471 04	2.164 74	9.362 74	18.091 49	18.076 97[a]
60	3	−1	1.0	0.1	−2.567 82	2.745 38	14.837 20	24.397 19	14.837 06
61	0	−5	−1.0	0.1	−8.787 27	3.877 98	−11.017 95	−10.022 97*	−10.683 86[a]
62	1	−5	−1.0	0.1	−5.793 59	3.152 75	−12.074 94	−9.298 73	−9.230 78[a]
63	2	−5	−1.0	0.1	−5.162 61	2.900 07	−14.868 22	−11.555 36	−11.480 25[a]
64	3	−5	−1.0	0.1	−5.163 50	2.773 75	−18.010 87	−14.351 64	−14.288 55[a]
65	0	−5	2.0	0.1	18.314 68	12.021 20	4.708 84	30.970 85	4.689 27
66	0	−5	2.0	0.1	0.908 76	10.280 61	−2.369 32	16.965 02*	15.922 10[a]
67	0	−5	2.0	0.1	14.612 94	11.651 03	3.291 94	28.142 18	3.273 47
68	1	−5	2.0	0.1	10.258 08	11.278 79	17.234 51	36.271 02	17.228 38
69	1	−5	2.0	0.1	−2.808 90	9.972 09	7.699 41	22.875 60	22.622 71[a]
70	1	−5	2.0	0.1	6.043 21	10.857 30	14.310 86	32.105 16	14.304 56
71	2	−5	2.0	0.1	6.713 81	10.987 61	25.155 29	42.559 60	25.153 57
72	2	−5	2.0	0.1	−4.474 72	9.868 76	14.430 36	29.116 14	29.025 07[a]
73	2	−5	2.0	0.1	2.767 85	10.593 01	21.519 44	37.957 19	21.517 65
74	3	−5	2.0	0.1	4.421 31	10.821 60	32.159 84	48.957 04	32.159 21
75	3	−5	2.0	0.1	−5.552 82	9.824 19	20.756 27	35.449 19	35.408 19[a]
76	3	−5	2.0	0.1	0.815 28	10.461 00	28.167 69	44.195 42	28.167 05
77	0	−5	2.0	0.5	−2.161 91	3.040 37	−5.607 63	8.485 07*	7.331 44[a]
78	1	−5	2.0	0.5	−5.136 90	2.259 98	0.570 12	11.529 51	11.192 15[a]
79	2	−5	2.0	0.5	−7.144 35	1.963 36	3.479 11	14.508 47	14.373 13[a]
80	3	−5	2.0	0.5	−8.843 45	1.820 92	6.061 15	17.488 15	17.421 84[a]

[a]Indicates an excited state ($n_r=1$).

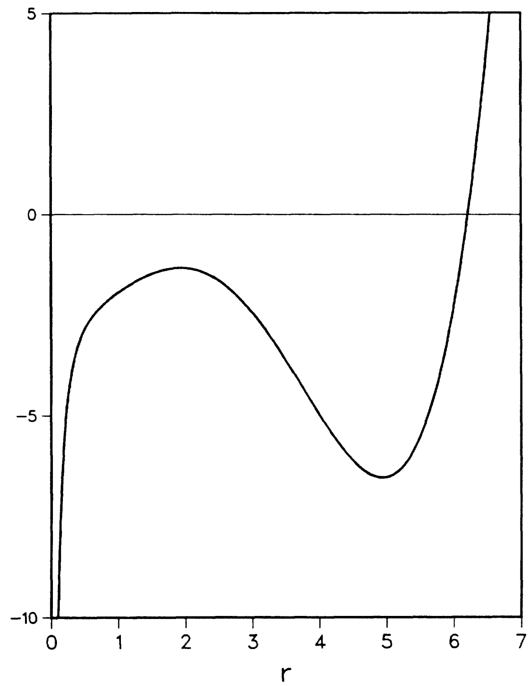

FIG. 1. Shape of the potential for data set No. 5. Both $V(r)$ and r are in atomic units.

er levels all belong to the second well. We searched for a possible second r_0 value satisfying condition (36), but the result was negative. It would appear that the shifted $1/N$ expansion method ignores the existence of the second well. The situation concerning the other five sets, namely, Nos. 27, 28, 30, 31, and 32, is similar. All consist of

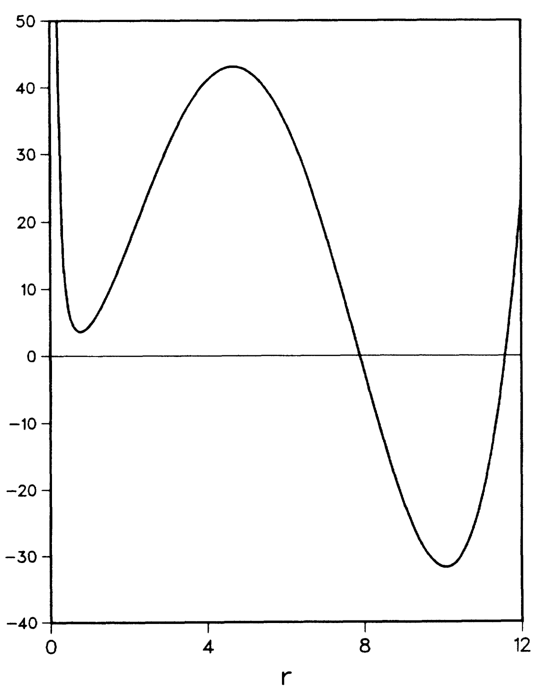

FIG. 2. Shape of the potential for data set No. 26. Both $V(r)$ and r are in atomic units.

TABLE IV. Energy eigenvalues for data set No. 26 potential obtained by numerical integration of the Schrödinger equation. All values are for $l=1$.

n	E
1	−28.253
2	−22.119
3	−15.832
4	−9.6683
5	−3.6361
6	2.2559
7	7.9972
8	8.5160
9	13.575

double-well potentials similar to the one shown in Fig. 2, and the reason of the discrepancy is similar to that for set No. 26.

It is known that for such double minimum potentials, even with numerical methods, it is sometimes difficult to obtain accurate eigenvalues for lower levels.[57–59] The fact that supersymmetric considerations can be used to obtain an exact value, though only for a single level, can be used as a benchmark to assess the accuracy of a numerical integration method and/or to fine tune a computer program which employs such a method.

Case III (Table III). Equations (34) and (35) give rise to a cubic equation in h_1. Consequently, sometimes there is one real root and sometimes there are three. In Table III, column 2, when there is the same l value in three consecutive lines, the three lines correspond to the three real roots for h_1. Whenever $h_1 < 0$, the SUSY wave function [see Eq. (33)] has a node and hence the SUSY eigenvalue is that of the excited state $n_r = 1$. Such cases are marked with a superscript a in the last column of Table III. Hence shifted $1/N$ expansion values were calculated for $n_r = 1$ also and are shown in column 9. A comparison of columns 8 and 9 with 10 shows that in most of the cases the shifted $1/N$ expansion value agrees with the SUSY value. Generally speaking, the agreement of the shifted $1/N$ expansion values with the SUSY values is not as good as in Tables I and II and in a few cases it is poor or very poor. Such cases are marked by an asterisk in Table III and discussed below.

Set No. 41. There is a very large discrepancy between the shifted $1/N$ expansion value ($n_r = 1$) and the SUSY value. The shape of this potential is shown in Fig. 3. Energy levels were also determined by the numerical integration of the Schrödinger equation and the eigenvalues for the first two levels are −2.8011 and −0.3748. Thus we find that the shifted $1/N$ expansion gives reasonable value for the first level, but fails completely for the second. This has to be attributed to the shape of the potential. We have seen earlier that for set 5 also, which has a shape similar to set 41, the shifted $1/N$ expansion failed for the first level. It would be reasonable to infer that for potentials having shapes of the type shown in Figs. 1 and 3, the shifted $1/N$ expansion is liable to fail for one or more levels. Which level would be affected

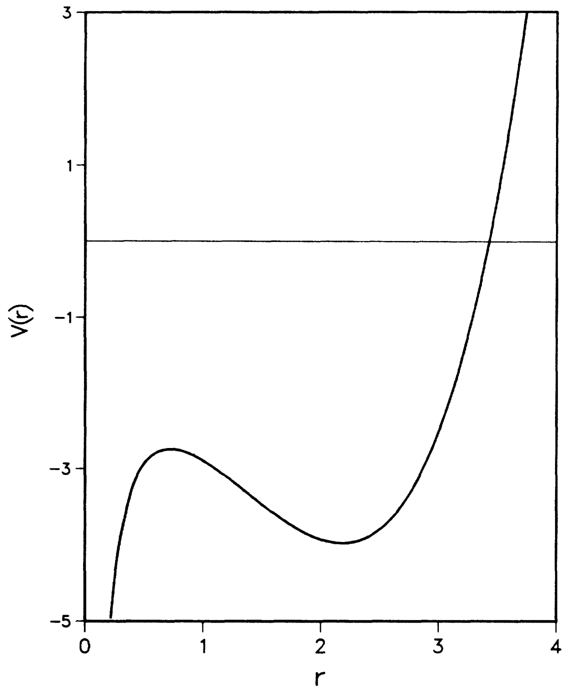

FIG. 3. Shape of the potential for data set No. 41.

would depend on the potential parameters.

Set No. 61. The discrepancy is due to the shape of this potential which is similar to that shown in Fig. 3.

Sets Nos. 66 and 77. The shape of the set No. 66 potential is shown in Fig. 4. The shape of the set No. 77 potential is similar to that of Set No. 66. It appears that for such potential shapes also, the shifted $1/N$ expansion can give inaccurate results.

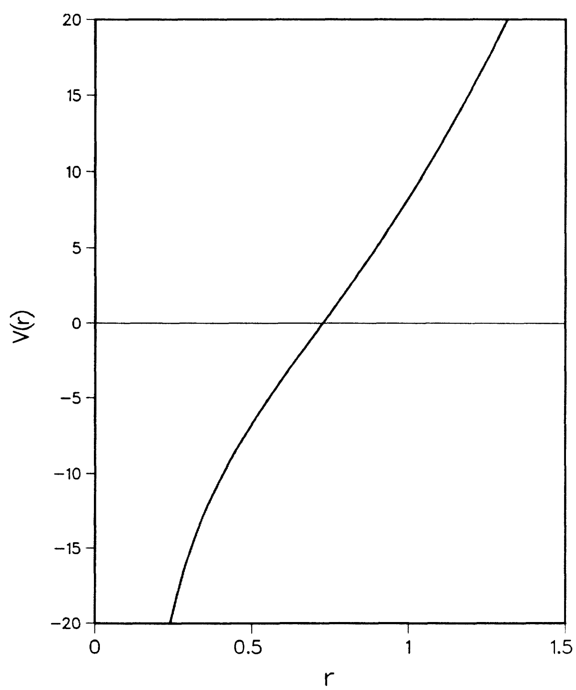

FIG. 4. Shape of the potential for data set No. 66.

V. EXTENSION TO POTENTIALS WITH HIGHER POWERS OF r

In this section we wish to show that the method that we have used in Sec. II can also be used for potentials which involve higher powers of r. The highest power of r has to be even. Suppose we include terms in powers of r^5 and r^6. Then the effective potential can be written as

$$V^{\text{eff}}(r)=\frac{l(l+1)}{r^2}+\frac{\alpha}{r}+p_1r+p_2r^2+p_3r^3+p_4r^4+p_5r^5+p_6r^6 . \tag{41}$$

As an illustration, we shall obtain the exact solution for the simplest case, i.e., $n_1=0$ and $n_2=0$.

We write the superpotential W as

$$W=a+\frac{b}{r}+cr+dr^2+er^3 . \tag{42}$$

Of the eight parameters (including l) in Eq. (41) only five are free. We assume that l, α, p_4, p_5, and p_6 are the independent parameters. Then, following the method of Sec. II, we get the following relations:

$$b=l+1, \quad a=\frac{\alpha}{2(l+1)} ,$$

$$e=-\sqrt{p_6}, \quad d=\frac{p_5}{2e} ,$$

and

$$c=(p_4-p_5^2/4e^2)/2e .$$

The parameters p_1, p_2, and p_3 are derived to be

$$p_1=2ac+2bd+2d ,$$

$$p_2=c^2+2ad+3e+2bc ,$$

and

$$p_3=2ae+2cd .$$

And finally,

$$E=-(a^2+2bc+c) ,$$

and

$$\psi=re^{-r^2/2-r^3/6-r^4/4} .$$

VI. CONCLUSIONS

We have developed a method for obtaining a family of exact solutions for the potential (6). The method can also be used for potentials which involve higher powers of r. A comparison of eigenenergies obtained from the shifted $1/N$ expansion method shows that if the potential has more than one well, the latter method can give poor or erroneous results or it may give incorrect radial quantum number for a level. Thus if one is dealing with such potentials, caution is necessary in using the shifted $1/N$ expansion method.

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