Spontaneous emission by a system of N two-level atoms in terms of the SU(2)-group representations

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We consider cooperative spontaneous emission by an assembly of N two-level atoms immersed in a high-Q single-mode cavity, when s ($s \ll N$) atoms are initially excited. The atoms are noninteracting with each other except via the action of the electromagnetic field. The eigenvalues of the system and the time development of the mean energy of the atomic subsystem are presented in the form of power series in the parameter $\epsilon=(N-s/2+1/2)^{-1}$ using the SU(2)-group-representation technique.

I. INTRODUCTION

The exactly solvable Jaynes-Cummings (JC) model of a two-level atom coupled to a single-mode quantized radiation field in a lossless cavity is the fundamental model in quantum optics. Despite its simplicity it is able to demonstrate a number of interesting phenomena such as collapses and revivals,[1] sub-Poissonian photon statistics,[2] and squeezing.[3] The simplest generalization of the JC model, leading to collective effects in light-matter interaction, is the so-called Dicke model of a system of identical two-level atoms coupled to a single quantized field mode and located within a wavelength of each other.

The Hamiltonian for the latter model in the rotating-wave approximation reads ($\hbar=1$):

$$H=H_0+Vg \ ,$$
$$H_0=\omega_f a^\dagger a+\omega\sum_{j=1}^{N} S_3^{(j)} \ , \tag{1}$$
$$Vg=\sum_{j=1}^{N}(S_+^{(j)}a+S_-^{(j)}a^\dagger) \ .$$

In Eq. (1) $a^\dagger$ (a) is the photon creation (annihilation) operator and $[a,a^\dagger]=1$. The jth atom is described by the well-known Pauli matrices $S_k^{(j)}$ ($k=3,+,-$). ω_f denotes the frequency of the field mode while ω is the atomic transition frequency. The coupling coefficient g is the same for all the atoms. It is implicit that the transition dipoles are aligned with the mode polarization. In what follows we assume exact resonance and choose the scale in such a way that $\omega_f=\omega=1$.

In general, excluding the cases where $s=1,\ldots,8$, it is impossible for the model in question to obtain exact solutions in a closed form. It has been numerically shown by Walls and Barakat[4] that when only a small number s of atoms from the system of N atoms is initially excited ($s \ll N$), and photons are absent at $t=0$, the spectrum of the Hamiltonian (1) is almost equidistant. This case was discussed later in Refs. 5 and 6. The approximate solutions for the expectation values of the Heisenberg operators in terms of the elliptic functions were presented. In the present paper we consider the same situation, when s initially excited atoms radiate spontaneously in the presence of a large number of $N-s$ unexcited atoms. We take the initial state of the system as symmetrical with respect to the permutations of the atoms. We present a new solution to the problem. Our method consists of construction of the perturbation theory with a small parameter ϵ:

$$\epsilon=\left[N-\frac{s}{2}+\tfrac{1}{2}\right]^{-1} \ . \tag{2}$$

The results obtained in this way are valid for an arbitrary time t.

The method we proposed has the following attractive features when compared to the elliptic solutions.[5,6] First, it offers the possibility of obtaining corrections to the equidistant spectrum. Second, it leads to the very simple forms of the solutions. Our method involves calculations in the representations of the group SU(2). We present here the approximate expressions for the eigenvalues and eigenvectors of the Hamiltonian (1) as well as for the time development of the expectation value of the atomic inversion. We show that our analytical results remain in excellent agreement with those numerically obtained by Walls and Barakat.[4]

II. RESULTS

Let us recall that the excitation number operator $\mathcal{N}$:

$$\mathcal{N}=a^\dagger a+\sum_{j=1}^{N} S_3^{(j)}$$

is an integral of motion. Hence we can only work in the subspace labeled by its eigenvalue. As a consequence of the initial condition the time evolution of the system is restricted to the $(s+1)$-dimensional subspace spanned by the basis vectors

$$|m\rangle=|s-m\rangle_a\otimes|m\rangle_f, \quad 0\le m\le s \ . \tag{3}$$

Here, $|m\rangle_f$ denotes the Fock state of the field while $|s-m\rangle_a$ is the normalized symmetric Dicke state of the atomic subsystem with $s-m$ atoms excited. The initial condition corresponds to $m=0$.

The free part H_0 of the Hamiltonian (1) leads to an unimportant phase factor in the transition amplitudes. In turn, the interaction term V can be written in the basis (3) in a tridiagonal matrix form with the following nonvanishing matrix elements:

$$\langle m-1|Vg|m\rangle=\langle m|Vg|m-1\rangle =\sqrt{m(s-m+1)(N-s+m)} . \quad (4)$$

Choosing the small parameter ϵ according to Eq. (2) one can expand the right-hand side of Eq. (4) in a power series in ϵ. Hence we have

$$Vg=\frac{1}{\sqrt{\epsilon}}(H^{(0)}+\epsilon H^{(1)}+\epsilon^2 H^{(2)}+\cdots) , \quad (5)$$

where the nonvanishing matrix elements of the first three terms in Eq. (5) are

$$\langle m-1|H^{(0)}|m\rangle=\sqrt{m(s-m+1)} , \quad (6a)$$

$$\langle m-1|H^{(1)}|m\rangle=\frac{1}{2}\left[m-\frac{s+1}{2}\right]\sqrt{m(s-m+1)} , \quad (6b)$$

$$\langle m-1|H^{(2)}|m\rangle=-\frac{1}{8}\left[m-\frac{s+1}{2}\right]^2\sqrt{m(s-m+1)} , \quad (6c)$$

and, obviously, $\langle m|H^{(k)}|m-1\rangle=\langle m-1|H^{(k)}|m\rangle$.

It is easily seen from Eq. (6a) that the operator $H^{(0)}$ is equal to the generator $2S_x=S_+ +S_-$ of the $(s+1)$-dimensional irreducible representation of the group SU(2). This gives immediately the eigenvalues and eigenvectors of $H^{(0)}$. In our energy scale we get

$$H^{(0)}|p\rangle^{(0)}=\lambda_p^{(0)}|p\rangle^{(0)},\quad \lambda_p^{(0)}=s-2p ,\quad 0\le p\le s , \quad (7)$$

$$|p\rangle^{(0)}=(\alpha_{op},\ldots,\alpha_{kp},\ldots\alpha_{sp})^T ,$$

where α_{kp} is defined by the formula[7]

$$\alpha_{kp}=\left[\frac{k!p!}{2^s(s-k)!(s-p)!}\right]^{1/2}\sum_{j=0}^{\min(p,k)}\frac{(-2)^j(s-j)!}{j!(k-j)!(p-j)!} . \quad (8)$$

Using the properties of the matrix elements of the group SU(2) (Ref. 7) one can calculate higher-order corrections to the eigenvalues and eigenvectors. Due to our choice of the form of the parameter ϵ, the first-order approximation terms for the eigenvalues vanish, i.e., $\lambda_p^{(1)}=0$. In turn, the eigenvalues and eigenvectors in the second- and first-order approximations are, respectively,

$$\begin{aligned}
&\lambda_p=\lambda_p^{(0)}+\epsilon^2\lambda_p^{(2)} ,\\
&\lambda_p^{(2)}=-\frac{(s-2p)}{16}\left[5p(s-p)-\frac{(s-1)(s-2)}{2}\right] ,\\
&|p\rangle=|p\rangle^{(0)}+\epsilon\sum_{m=0}^{s}|m\rangle\langle m|p\rangle^{(1)} ,\\
&\langle m|p\rangle^{(1)}=-\frac{1}{4}\left[p-\frac{s+1}{2}\right]\sqrt{p(s-p+1)}\alpha_{m,p-1}\\
&\qquad+\frac{1}{4}\left[p-\frac{s-1}{2}\right]\sqrt{(p+1)(s-p)}\alpha_{m,p+1} .
\end{aligned} \quad (9)$$

The time evolution of the atomic inversion can be calculated through the formula

$$E_{\rm at}(t)=\langle 0|e^{iHt}S_3e^{-iHt}|0\rangle , \quad (10)$$

where S_3 is the collective operator equal to $\sum_{j=1}^{N}S_3^{(j)}$.

Expanding the initial vector $|0\rangle$ in the basis of the eigenvectors $|p\rangle$ of the Hamiltonian (1), in the zeroth-order approximation we find that

$$E_{\rm at}^{(0)}(t)=\frac{s}{2}\cos 2\Omega t,\quad \Omega=g\left[N-\frac{s}{2}+\frac{1}{2}\right]^{1/2} . \quad (11)$$

The above quantity has a very simple form, similar to that for the JC model. The collectivity of the system leads, in this approximation, to the change of the time scale only. In turn, in the first-order approximation we arrive at

$$E_{\rm at}^{(1)}(t)=\frac{s}{2}\cos 2\Omega t+\frac{\epsilon}{16}s(s-1)(1-\cos 4\Omega t) . \quad (12)$$

In general, except the case $s=1$, we deal with the additional Rabi frequency 4Ω of the oscillations of the system. The first-order approximation seems to be sufficient for $E_{\rm at}(t)$; if $s\ll N$, we can omit the higher-order terms in ϵ since ϵ is then small as well. The special case takes place for $s=1$ (irrespective of N); all the approximations tend to the same result:

$$E_{\rm at}(t)=\tfrac{1}{2}\cos(2g\sqrt{N}t) . \quad (13)$$

We now consider the radiation rate $I(t)$ defined by

$$I(t)=-dE_{\rm at}(t)/dt . \quad (14)$$

On insertion of Eq. (12) into Eq. (14) we get

$$I^{(1)}(t)=s\Omega\sin 2\Omega t-\frac{\epsilon\Omega}{4}s(s-1)\sin 4\Omega t . \quad (15)$$

In particular, for $s=1$ immediately from Eq. (15) or from Eqs. (14) and (13) we arrive at

$$I^{(1)}(t)_{s=1}=g\sqrt{N}\sin(2g\sqrt{N}t) , \quad (16)$$

in full agreement with the result of Seke.[8] For short times

$$I_{\rm short}^{(1)}(t)_{s=1}=2g^2Nt \quad (17)$$

is N times the rate for one atom only,[9] i.e., for the spontaneous emission from the JC model. Cummings and

Dorri[10] also considered spontaneous emission from a system for N two-level atoms with one atom initially excited albeit for nonsymmetrical excitation, i.e., when the excited atom was distinguished from the others. In this case we deal with the so-called effect of radiation trapping, later dubbed by Cummings[11] "radiation suppression." For short times the radiation rate is then identical with that for one atom only.

In the general case, from Eq. (15) for short times one finds that

$$I^{(1)}_{\text{short}}(t)=2g^2ts(N-s+1)\ . \tag{18}$$

It is easily checked that the above function has its maximum for $s=(N+1)/2$. The radiation rate is then proportional to $(N+1)^2$, namely,

$$I^{(1)}_{\text{short}}(t)=\tfrac{1}{2}g^2t(N+1)^2\ . \tag{19}$$

The system is thus in its superradiant state. Certainly, for very large N the quantity (19) becomes proportional to N^2. So, the collective behavior of the system considered here is identical with the superradiant behavior of an assembly of atoms in a low-Q cavity.[12–14]

III. DISCUSSION

The perturbation solutions for the time evolution of the atomic inversion (or, equivalently, of the radiation rate) and for the eigenvalues presented here hold for $s \ll N$. Walls and Barakat[4] have derived computer solutions for the same quantities. Although they considered the cases for which the inequality $s \ll N$ is hardly satisfied, it is tempting to compare the present analytical results with those given in Ref. 4 in order to verify the quality of our approximate solutions. From the point of view of the approximations used in this paper, the best case discussed by Walls and Barakat[4] is $N=48$, $s=24$. The appropriate eigenvalues are listed in Table I. The agreement between $\Lambda^{(2)}_p=(\lambda^{(0)}_p+\epsilon^2\lambda^{(2)}_p)/\sqrt{\epsilon}$ and $\Lambda^{(*)}_p$ presented by Walls and Barakat[4] is intriguingly excellent.

In Table II the same quantities are compared for $N=s=24$. At first glance, our solutions should seem to be inapplicable to this case. However, inspection of Table II shows that the agreement between $\Lambda^{(2)}_p$ and $\Lambda^{(*)}_p$ is amazingly good except for the eigenvalues with the small absolute magnitudes ($p=9,10,11,13,14,15$).

It was noticed by Walls and Barakat[4] that the spectrum of the eigenvalues tends to be equidistant as s decreases. In fact, the spectrum of the zeroth-order eigenvalues $\Lambda^{(0)}_p=\lambda^{(0)}_p/\sqrt{\epsilon}$ is always equidistant irrespective of the value of s/N. The eigenvalues $\Lambda^{(2)}_p$ contain the term nonlinear in p which destroys this feature of the spectrum. If, however, s/N is sufficiently small, the nonlinear contributions from p become negligible and the eigenvalues $\Lambda^{(2)}_p$ tend to those of $\Lambda^{(0)}_p$.

In order to obtain the time evolution of $E^{(1)}_{\text{at}}(t)$ we had to use the eigenvectors and eigenvalues in the first- and zeroth-order approximations, respectively. In consequence, the time behavior of $E^{(1)}_{\text{at}}(t)$ and $I^{(1)}(t)$ is truly periodic behavior irrespective of the magnitude of the ratio s/N. However, the time dependence of the quantities $E_{\text{at}}(t)$ and $I(t)$, calculated within an accuracy of ϵ^2, may be aperiodic if s/N is not small enough. This is because the spectrum of $\Lambda^{(2)}_p$ is not then equidistant. In such a case one should expect beatings between the terms with different frequencies, resulting in modulation of $E_{\text{at}}(t)$ and $I(t)$. It also is in qualitative agreement with the numerical predictions.[4]

Let us compare now the solution (12) for $E^{(1)}_{\text{at}}(t)$ with that elliptic obtained by Kumar and Mehta.[6] In particular, expanding Eq. (12) in a power series in t, within an accuracy of t^4 we get

$$E^{(1)}_{\text{at}}(t)=\frac{s}{2}-s(N-s+1)g^2t^2 + \frac{s}{3}[N^2-3(s-1)N+\tfrac{5}{4}(s-1)^2]g^4t^4\ . \tag{20}$$

From the result presented by Kumar and Mehta[6] we arrive at

TABLE I. Eigenvalues for $N=48$ and $s=24$. The remaining 12 eigenvalues $\Lambda^{(k)}_p$ ($k=0,2,*$) for $13\le p\le 24$ are readily obtained through $\Lambda^{(k)}_p=-\Lambda^{(k)}_{24-p}$. $\Lambda^{(0)}_p=(1/\sqrt{\epsilon})\lambda^{(0)}_p$, calculated from Eq. (7). $\Lambda^{(2)}_p=(1/\sqrt{\epsilon})(\lambda^{(0)}_p+\epsilon^2\lambda^{(2)}_p)$ calculated from Eq. (9). $\Lambda^{(*)}_p$, computer solutions from Ref. 4.

p	$\Lambda^{(0)}_p$	$\Lambda^{(2)}_p$	$\Lambda^{(*)}_p$
0	145.00	146.72	146.64
1	132.91	133.77	133.76
2	120.83	121.02	121.04
3	108.75	108.43	108.47
4	96.66	96.00	96.03
5	84.58	83.70	83.72
6	72.50	71.52	71.52
7	60.41	59.44	59.43
8	48.33	47.45	47.43
9	36.25	35.53	35.51
10	24.17	23.66	23.64
11	12.08	11.82	11.81
12	0	0	0

TABLE II. Eigenvalues for $N=s=24$. $\Lambda^{(k)}_p=-\Lambda^{(k)}_{24-p}$ for $13\le p\le 24$. Notation for $\Lambda^{(2)}_p$ and $\Lambda^{(*)}_p$ is the same as in Table I.

p	$\Lambda^{(2)}_p$	$\Lambda^{(*)}_p$
0	93.44	91.28
1	82.07	81.57
2	71.64	72.17
3	62.06	63.09
4	53.24	54.33
5	45.10	45.92
6	37.56	37.88
7	30.52	30.23
8	23.91	23.00
9	17.63	16.25
10	11.61	10.05
11	5.76	4.58
12	0	0

$$E_{\rm at}(t)=\frac{s}{2}-s\left[N-s+1+\frac{1}{4N^2}\right]g^2t^2 + \frac{s}{3}\left[N^2-3(s-\tfrac{1}{2})N+2s^2+\frac{1-3s}{2} + \frac{s}{N}\frac{1-4s}{4}\right]g^4t^4 . \quad (21)$$

On the other hand, the direct solution of the Heisenberg equations of motion leads to the following form of $E_{\rm at}(t)$:

$$E_{\rm at}(t)=\frac{s}{2}-s(N-s+1)g^2t^2 + \frac{s}{3}[N^2-3(s-1)N+2(s-1)^2]g^4t^4 . \quad (22)$$

Inspection of the above three solutions shows that our result agrees better with the direct solution (22); the difference is only in the term proportional to t^4.

To conclude briefly, our approximate solutions seem to be suitable for the description of the dynamics of the spontaneous emission from a system of N two-level atoms with s atoms initially excited even for the moderate values of the ratio s/N; the agreement with the real behavior is particularly good for short times, which is easily seen from Eqs. (20) and (22). For the sufficiently small values of s/N the dynamics of the system is almost exactly described for all times by Eqs. (12) and (15) and has the truly periodic behavior.

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APPENDIX

As mentioned, the case $s=2$ (N arbitrary) is exactly solvable. This solution was presented by us in Ref. 15. Shumovsky, Fam Le Kien, and Aliskenderov[16] have solved the particular case $s=N=2$. Having known this result, it is interesting to compare it with our approximate solution (15), which for $s=N=2$ reads

$$I^{(1)}_{s=N=2}(t)=2\Omega\sin2\Omega t-\tfrac{1}{3}\Omega\sin4\Omega t,\quad \Omega=\sqrt{3/2}g . \quad (A1)$$

The exact solution has the form[16]

$$I_{s=N=2}(t)=\tfrac{16}{9}\Omega\sin2\Omega t-\tfrac{2}{9}\Omega\sin4\Omega t . \quad (A2)$$

It is readily seen that the amplitudes of the oscillations are a little greater in our approximate case.

The maximal value of ϵ is $\epsilon_{\max}=1$. This value of ϵ corresponds to the "special case" $s=1$ (N arbitrary) discussed by us earlier in Sec. II. The next in turn highest possible value of ϵ is $\epsilon=\frac{2}{3}$, which just corresponds to $s=N=2$. In this case, obviously, we have to improve the solution (15) taking into account the next approximation in ϵ, i.e., ϵ^2. For this purpose we apply the second-order approximations for the eigenvectors and eigenvalues. In fact, however, the second-order approximation eigenvalues $\lambda_p^{(2)}$ ($p=0,1,2$) are equal to zero for $s=2$, which is easily seen from Eq. (9). Hence the time evolution of the system for $s=2$ remains periodic.

In general, the eigenvectors in the second-order approximation are found to be

$$|p\rangle=|p\rangle^{(0)}+\epsilon|p\rangle^{(1)}+\epsilon^2|p\rangle^{(2)} ,$$
$$|p\rangle^{(2)}=A_p|p\rangle^{(0)}+B_p|p+2\rangle^{(0)}+C_p|p-2\rangle^{(0)} , \quad (A3)$$
$$A_p=-\tfrac{1}{128}[p(s-2p+1)^2(s-p+1) + (p+1)(s-2p-1)^2(s-p)] ,$$
$$B_p=\tfrac{1}{128}[(s-2p)^2-5(s-2p)+5] \times\sqrt{(p+1)(p+2)(s-p-1)(s-p)} , \quad (A4)$$
$$C_p=\tfrac{1}{128}[(s-2p)^2+5(s-2p)+5] \times\sqrt{p(p-1)(s-p+1)(s-p+2)} .$$

From Eq. (10) at (9) and (A3), we find for $s=2$ (N arbitrary)

$$E^{(2)}_{\rm at}(t)=\frac{\epsilon}{8}+\tfrac{3}{16}\epsilon^2+\left[1-\frac{\epsilon^2}{4}\right]\cos2\Omega t - \frac{\epsilon}{8}\left[1-\frac{\epsilon}{2}\right]\cos4\Omega t ,$$
$$\epsilon=1/(N-\tfrac{1}{2}),\quad \Omega=\sqrt{(N-1/2)g} . \quad (A5)$$

On inserting the atomic energy (A5) into (14), we get for $N=2$

$$I^{(2)}_{s=N=2}(t)=\tfrac{16}{9}\Omega\sin2\Omega t-\tfrac{2}{9}\Omega\sin4\Omega t,\quad \Omega=\sqrt{3/2}g , \quad (A6)$$

in full agreement with the exact solution (A2).

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