Dark-soliton dynamics and shock waves induced by the stimulated Raman effect in optical fibers

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It is demonstrated that the dynamics of small-amplitude dark solitons in the presence of the stimulated Raman effect in optical fibers may be described by the well-known Korteweg–de Vries–Burgers equation. This approach allows us to explain analytically the temporal self-shift of dark solutions due to self-induced Raman scattering and to predict oscillating shock waves in optical fibers.

I. INTRODUCTION

As is well known, optical pulses may propagate in single-mode fibers without broadening in the form of bright or dark solitons for which the nonlinear refractive index exactly compensates the group-velocity dispersion (GVD).[1] Soliton propagation of bright pulses has been verified in a number of elegant experiments performed in the negative GVD region of the fiber spectrum, i.e., for the wavelengths $\lambda_0 > 1.3$ μm in standard monomode silica fibers (see, e.g., the pioneer paper by Mollenauer, Stollen and Gordon).[2] Because of the difficulty of generating appropriate dark pulses, experimental investigations of dark solitons, predicted for the positive GVD by Hasegawa and Tappert (see Ref. 1), have been started only recently.[3–5] In the experiments, dark solitons were produced and observed on a broad bright pulse with a rapid intensity dip stipulated by a driving pulse or utilizing a specially shaped antisymmetric input pulse of finite extent. But as was demonstrated by numerical[6] and analytical[7] studies, the finite duration of the background pulse isn't the principal limitation for dark-soliton propagation, and the generated dark pulses possess properties similar to exact dark solitons for the cw background. Recently, Weiner *et al.*[8] reported also the discovery of temporal and spectral self-shifts of dark solitons propagating in single-mode fibers. The results have been discussed using numerical solutions of the modified nonlinear Schrödinger (NLS) equation that includes the Raman contribution to the nonlinear index.

The stimulated Raman effect produces a frequency shift of bright optical solitons in single-mode optical fibers.[9–14] The effect was first observed by Mitschke and Mollenauer[9] and was explained analytically by Gordon.[10] When incorporated into the NLS equation this effect appears (in the lowest approximation) as a high-order term of the form $u(|u|^2)_t$ (u being the dimensionless electric-field amplitude) and produces a frequency shift in proportion to the fourth power of the soliton amplitude and to the distance of propagation. As a result, the stimulated Raman effect leads to fission of bright-soliton bound states.[14]

Bright optical solitons are two-parametric, and the Raman effect does not result in changing amplitudes of the solitons, but changes only their frequencies. Unlike bright solitons, dark solitons are one-parametric, and the influence of self-induced Raman scattering on their parameters is more destructive.[8] This paper aims to describe analytically the dynamics of dark solitons under the Raman effect demonstrating that the NLS equation with the Raman contribution to the nonlinear refractive index may be transformed in the small-amplitude limit into the well-known Korteweg–de Vries–Burgers equation. This approach allows us to explain analytically the temporal self-shift of dark solitons due to the Raman self-pumping. Moreover, we predict the steady-state dynamics of optical pulses in the form of shock waves induced by the Raman effect.

II. CONNECTION BETWEEN SMALL-AMPLITUDE DARK SOLITONS AND SOLITONS OF THE KORTEWEG-DE VRIES EQUATION

The propagation of nonlinear pulses in optical fibers with dispersion is well described by the conventional NLS equation, which has the following scaled form (see, e.g., Ref. 1):

$$i\frac{\partial u}{\partial x} - \sigma\frac{\partial^2 u}{\partial t^2} + 2|u|^2 u = 0 , \tag{1}$$

where $u(x,t)$ is the complex electric-field amplitude envelope in a reference frame moving with the pulse, σ is the sign of the GVD of the pulse. The solutions of the equation divide into two different classes depending on the sign of σ. In the case $\sigma = -1$ (negative GVD) the equation possesses stable bright-soliton solutions.[1] At $\sigma = +1$ (positive GVD) the cw solution $|u| = u_0 = \text{const}$ is stable and, as a result, Eq. (1) has soliton solutions in the form of localized dark pulses propagating on the nonlinear cw background. The one-soliton dark pulse has the form[15]

$$u(x,t) = u_0\frac{(\lambda - i\nu)^2 + \exp z}{1 + \exp z}e^{2iu_0^2 x} , \tag{2}$$

where

$$z = 2\nu u_0(t - t_0 - 2\lambda u_0 x), \quad \lambda^2 = 1 - \nu^2 , \tag{3}$$

and it corresponds to the boundary conditions $|u|\to u_0$ at $t\to\pm\infty$; the solution [(2) and (3)] has the single parameter v (unlike a bright soliton), which characterizes the soliton intensity. For simplicity, we will consider $t_0=0$.

We distinguish two limiting cases of the dark soliton. The fundamental dark soliton [Eqs. (2) and (3) at $\lambda=0$],

$$u(x,t)=u_0\tanh(u_0t)e^{2iu_0^2x}\ , \tag{4}$$

is the antisymmetric function of the time with phase shift π and zero intensity at its center. Another limiting case $v^2\ll1$ corresponds to a small-amplitude dark soliton, when the solution [(2) and (3)] may be presented in the form

$$\begin{aligned}u(x,t)&\approx[u_0-\tfrac{1}{2}u_0v^2\mathrm{sech}^2(z/2)]e^{2iu_0^2x+i\phi(x,t)}\ ,\\ \phi(x,t)&=-2v/(1+e^z)\ ,\end{aligned} \tag{5a}$$

where

$$z=2vu_0(t\mp2u_0x\pm u_0v^2x)\ , \tag{5b}$$

and its velocity is close to the velocity of linear excitations of the background. We will demonstrate that the dynamics of the latter solitons may be described by the well-known Korteweg-de Vries (KdV) equation. The connection between the NLS equation at $\sigma=+1$ and the KdV equation allows us to study in a simple manner a number of features related to the dark-soliton dynamics, e.g., creation of dark solitons by an arbitrary input pulse without a threshold,[16] the temporal self-shift of dark solitons,[8] intermode attraction of dark solitons in two-mode or birefringent optical fibers,[17] etc.

To discuss the dark-soliton dynamics in the small-amplitude limit, we look for a solution of Eq. (1) in the form

$$u(x,t)=[u_0+a(x,t)]e^{2iu_0^2x+i\phi(x,t)}\ . \tag{6}$$

Substituting Eq. (6) into Eq. (1), we obtain two equations:

$$\left[\frac{\partial a}{\partial x}-u_0\frac{\partial^2\phi}{\partial t^2}\right]-\left[2\frac{\partial a}{\partial t}\frac{\partial\phi}{\partial t}+a\frac{\partial^2\phi}{\partial t^2}\right]=0\ , \tag{7a}$$

$$u_0\left[\frac{\partial\phi}{\partial x}-4u_0a\right]+\frac{\partial^2a}{\partial t^2}+a\frac{\partial\phi}{\partial x}-(u_0+a)\left[\frac{\partial\phi}{\partial t}\right]^2-6u_0a^2=0\ . \tag{7b}$$

In the linear limit and for $\partial^2a/\partial t^2\ll u_0^2a$, the linear excitations of the cw background may be described by the wave equation

$$\frac{\partial^2\phi}{\partial x^2}-4u_0^2\frac{\partial^2\phi}{\partial t^2}=0$$

supporting a wave motion with two velocities $\pm|C|$ (in the t space), where

$$C^2=4u_0^2\ . \tag{8}$$

Let us use new variables that allow us to divide the directions and to consider the nonlinear dynamics for both of the velocities separately:

$$\tau=\epsilon(t-Cx),\quad y=\epsilon^3x\ , \tag{9}$$

where ϵ is an arbitrary small parameter connected with the soliton amplitude [cf. Eq. (5)]. Additionally, we will find the solutions for the functions $a(x,t)$ and $\phi(x,t)$ in the form of the formal perturbative series (asymptotic series) in the same small-parameter ϵ as the following:

$$a=\epsilon^2a_0+\epsilon^4a_1+\cdots,\quad \phi=\epsilon\phi_0+\epsilon^3\phi_1+\cdots\ . \tag{10}$$

Substituting (10) and (9) into Eq. (7), we obtain the set of equations produced by coefficient at the different powers of ϵ,

$$C\frac{\partial a_0}{\partial\tau}+u_0\frac{\partial^2\phi_0}{\partial\tau^2}=0\ , \tag{11a}$$

$$\left[C\frac{\partial a_1}{\partial\tau}+u_0\frac{\partial^2\phi_1}{\partial\tau^2}\right]+2\frac{\partial a_0}{\partial t}\frac{\partial\phi_0}{\partial\tau}-\frac{\partial a_0}{\partial y}+a_0\frac{\partial^2\phi_0}{\partial\tau^2}=0,\ldots \tag{11b}$$

and

$$u_0\left[C\frac{\partial\phi_0}{\partial\tau}+4u_0a_0\right]=0\ , \tag{11c}$$

$$u_0\left[C\frac{\partial\phi_1}{\partial\tau}+4u_0a_1\right]-u_0\frac{\partial\phi_0}{\partial y}-\frac{\partial^2a_0}{\partial\tau^2}+Ca_0\frac{\partial\phi_0}{\partial\tau}+u_0\left[\frac{\partial\phi_0}{\partial\tau}\right]^2+6u_0a_0^2=0,\ldots\ . \tag{11d}$$

Equations (11a) and (11c) lead to the relation

$$\frac{\partial\phi_0}{\partial\tau}=-\frac{Ca_0}{u_0}\ . \tag{12}$$

Using the expression

$$\left[C\frac{\partial a_1}{\partial\tau}+u_0\frac{\partial^2\phi_1}{\partial\tau^2}\right]$$

defined by Eqs. (11c) and (12), we may obtain from Eq. (11d) the KdV equation

$$2C\frac{\partial a_0}{\partial y}+24u_0a_0\frac{\partial a_0}{\partial\tau}-\frac{\partial^3a_0}{\partial\tau^3}=0\ , \tag{13}$$

which has the soliton solution in the form

$$a_0(\tau,y)=-(\kappa^2/2u_0)\mathrm{sech}^2\left[\kappa\left[\tau+\frac{2\kappa^2y}{C}\right]\right]\ , \tag{14}$$

κ being the KdV-soliton amplitude. Direct comparison the solution (10), (12), (9), and (14) with (15) leads to the simple relation between the dark-soliton parameters and the formal perturbative parameter ϵ (in principle, we may put $\epsilon=1$):

$$vu_0\equiv\epsilon\kappa\ , \tag{15}$$

which demonstrates that the dynamics of small-amplitude dark solitons (5) may be exactly described by the KdV equation (13).

III. DARK SOLITONS UNDER THE RAMAN EFFECT

Let us use the same approach to describe analytically the perturbation-induced dynamics of dark solitons. We will describe the experimentally observed temporal self-shift of dark solitons produced by the Raman self-pumping.[8]

The Raman contribution to the nonlinear refractive index may be described by the modified NLS equation in which the nonlinearity is presented by the term

$$n_2|u|^2 \rightarrow n_2 \left[(1-\tilde{\alpha})|u|^2 + \tilde{\alpha} \int_{-\infty}^{t} f(t-t')|u(t')|^2 dt' \right] ,$$

where $\tilde{\alpha}$ is the fraction of the total (low frequency) nonlinearity with a delayed response, and $f(t-t')$ is the Raman response function.[18] The response function is the Fourier transform of the complex third-order susceptibility, of which the imaginary part is the Raman gain $n''_{2R}(\Omega)$ (see Ref. 18):

$$f(t-t') = \frac{2}{\pi} \int_0^{\infty} n''_{2R}(\Omega) \sin[\Omega(t-t')]\, d\Omega .$$

Because the Raman response function of fused silica is extremely short, an approximate for the response function has been successfully used to model the Raman contribution in the NLS equation by a local term.[10,14] In this approach the function $u(t-s)$ (here $s=t-t'$) can be expanded in a Taylor series about t to yield the perturbed NLS equation

$$i\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial t^2} + 2|u|^2 u = \alpha u \frac{\partial}{\partial t}(|u|^2) , \tag{16}$$

α being in proportion to the Raman gain parameter $\tilde{\alpha}$.

Using the approach presented above, we may calculate the contributions of the term $\alpha u \partial/\partial t(|u|^2)$ in amplitude equations of Section II. In particular, this term leads to the contribution $-2\alpha u_0^2(\partial a/\partial t)$ on the right-hand side of Eq. (7b). The resulting first-order equation for the amplitude a_0 has the form

$$2C\frac{\partial a_0}{\partial y} + 24u_0 a_0 \frac{\partial a_0}{\partial \tau} - \frac{\partial^3 a_0}{\partial \tau^3} = 2\left[\frac{\alpha}{\epsilon}\right] u_0^2 \frac{\partial^2 a_0}{\partial \tau^2} . \tag{17}$$

Equation (17) is the Korteweg–de Vries–Burgers equation arising in different branches of physics, mostly in hydrodynamics (see, e.g., Ref. 19 and references therein). The dynamics of the KdV soliton in the presence of the small perturbation $2(\alpha/\epsilon)u_0^2 \frac{\partial^2 a_0}{\partial \tau^2}$ may be investigated by the perturbation theory for solitons.[19] According to the approach, the evolution of the KdV soliton amplitude κ is described by the equation (see, e.g., Ref. 19):

$$\frac{d\kappa}{dy} = -\frac{2C}{15}(\alpha/\epsilon)\kappa^3 .$$

Taking into account the expression (15) and the connection between y and x defined in Eq. (9), we obtain the resulting equation in the form

$$\frac{d\nu}{dx} = -\frac{2\alpha}{15} C u_0^2 \nu^3 , \tag{18}$$

which yields the evolution of the dark-soliton amplitude:

$$\nu^2 = \frac{\nu_0^2}{1+\text{sgn}C(x/x_0)} , \quad x_0^{-1} = \tfrac{8}{15}\alpha u_0^3 , \tag{19}$$

where $\text{sgn}C = +1$ for $C>0$ and $\text{sgn}C = -1$ for $C<0$.

The result (19) means that the evolution of the dark-soliton parameters is connected with a direction of the soliton motion. In such a situation, two dark solitons with opposite velocities but equal amplitudes, which are produced by a localized input pulse on a cw (or finite-extent) background,[16] will change their velocities and amplitudes by different ways. The dark soliton propagating to the right ($C>0$, $\text{sgn}C=+1$) decreases its amplitude and increases its velocity (in the t space). But the dark soliton propagating to the left ($C<0$, $\text{sgn}C=-1$) increases its amplitude and decreases its velocity (see Fig. 1). As a result, dark solitons moving to the right may disappear due to the Raman effect on distances of order of x_0, x_0 being defined by Eq. (19).

Let us also discuss the result (19) from the viewpoint of the experimental data obtained in Ref. 8. The starting pulse for those experiments and numerical calculations was a tangent hyperbolic dark pulse on a Gaussian background pulse,

$$u(0,t) = A \tanh t \exp(-t^2/T^2) . \tag{20}$$

This input pulse, an antisymmetric function of time with zero intensity and an abrupt π phase shift at its center, closely resembles the fundamental dark soliton (4). In the case of the cw background [or $T \gg 1$ in Eq. (20)] the problem of the dark-soliton generation by the input pulse

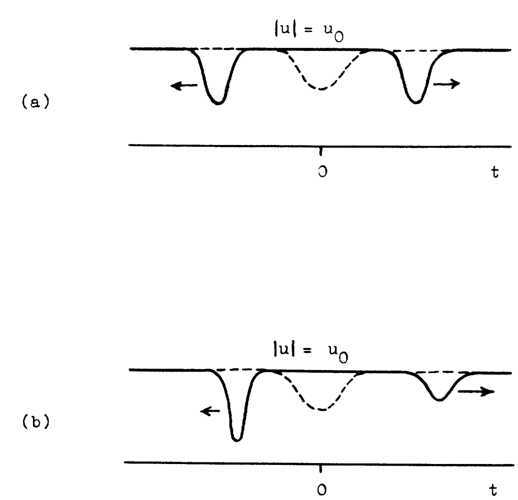

FIG. 1. Decay of an input pulse (dashed line) into two dark solitons (solid line) without the Raman effect (a), and in the presence of the self-induced Raman scattering (b).

(20) may be solved analytically for the NLS equation (see Ref. 20). When $A = N - \beta$, where $N \geq 1$ is an integer and $0 \leq \beta < 1$ is an arbitrary number, the central pulse will always evolve into the fundamental soliton (4) with $u_0 = N - \beta$, which we call the primary soliton. In addition, there are $(N-1)$ soliton pairs, i.e., $2(N-1)$ secondary gray solitons generated under the same constant background $u_0 = N - \beta$ with the parameters $\nu_n = 1 - n/(N-\beta), n = 1, 2, \ldots, N-1$, and propagating with the opposite velocities (in the t space) $\pm 2u_0(1-\nu_n^2)^{1/2}$ from the central fundamental (primary) dark soliton (see figures in Ref. 20 where numerical simulations for the case were made). The number of these secondary dark-soliton pairs is defined by the background intensity. In the experiments by Weiner *et al.*[5,8] one or two pairs were observed and they may be considered as small-amplitude ones.

In the presence of the Raman self-pumping [$\alpha \neq 0$ in Eq. (16)] the generation of dark solitons from the input pulse (20) is modified because the perturbation changes the soliton parameters. For the small-amplitude (gray) solitons the evolution of the soliton amplitude is described by Eq. (19). According to Eq. (19), the solitons propagating to the right decrease their amplitudes; at the same time, the solitons propagating to the left increase their amplitudes. After some distances $x \gtrsim x_0$ [see Eq. (19)] a pair of the gray solitons generated with opposite velocities will be transformed into a single high-contrast (left) gray soliton because its mate (right) soliton will disappear. The central dark pulse that is transformed into the primary dark soliton shifts also to later times and is of lower contrast.[8] The initial evolution of the fundamental dark soliton cannot be described by Eq. (19) because it was obtained in the small-amplitude limit, but its shift is less [see Eq. (18) where the shift speed is proportional to the soliton velocity]. As a result, the output optical pulse must display significant asymmetry due to the temporal self-shift and amplitude decreasing of dark solitons. The effect is in substantial agreement with the experimental and numerical results,[8] where it was observed that the shifts are more evident for high-power input pulses. The latter result may be also explained in the framework of our approach, because the distance describing the effect of the self-shift of dark solitons x_0 is inversely proportional to the third power of the background intensity u_0 [see Eq. (19)].

IV. SHOCK WAVES INDUCED BY THE RAMAN EFFECT

As is well known, the Korteweg–de Vries–Burgers equation (17) has, instead of unstable solitons, another type of a steady-state moving solution in the form of a shock wave with an oscillating structure (see, e.g., Ref. 21). Looking for a solution $a_0(\tau - Wy)$ and imposing the condition that $a_0(-\infty)$ is zero (i.e., $|u| \to u_0$), one obtains the equation [we put $\epsilon = 1$ in Eq. (17)]

$$-2CWv + 12v^2 - v'' = 2\alpha u_0^2 v' , \quad (21)$$

where $v \equiv u_0 a_0$, and the prime stands for the differentiation in respect to the new variable $\xi = \tau - Wy$. This equation describes the motion of a unit-mass particle in the effective potential $U_{\text{eff}}(v) = CWv^2 - 4v^3$ in the presence of the friction force $\gamma v', \gamma \equiv 2\alpha u_0^2$. Straightforward analysis demonstrates the only regular solution to Eq. (21) is the above-mentioned oscillating shock wave. The amplitude of the shock wave u_1 is proportional to the velocity $W, u_1 = \frac{1}{3}|W\,\text{sgn}C|$. The shape of the wave depends on the relation between the wave parameters. In the linear approximation (near the tails of the wave) we may investigate its shape exactly. Substituting $v = \frac{1}{6}CW + f$, $f \ll 1$, into the Korteweg–de Vries–Burgers equation, we obtain the solution $f \sim \exp(p\xi)$, where

$$p = -\frac{\gamma}{2} \pm \left[\frac{\gamma^2}{4} + 2CW\right]^{1/2} .$$

As a result, the shock wave is monotonic one in the dissipation-dominant case, when $\gamma^2 > \gamma_{\text{cr}}^2 \equiv 8|CW|$. In the opposite case, when $\gamma^2 < \gamma_{\text{cr}}^2$, i.e., for

$$\alpha^2 < \alpha_{\text{cr}}^2 = 4|W|/u_0^3 , \quad (22)$$

the shock wave has an oscillating profile shown in Fig. 2. In the case $\gamma^2 \ll \gamma_{\text{cr}}^2$ the oscillatory shock wave may be considered as a bound state of a succession of the KdV solitons (in our case dark solitons) (see Ref. 21). The total number of the solitons (maximum points of the shape) may be estimated in the framework of the perturbation theory for the KdV soliton system[21]: $N \sim \sqrt{|CW|}/\gamma \approx (|W|/\alpha^2 u_0^3)^{1/2} \sim \alpha_{\text{cr}}/\alpha \gg 1$. The length of the oscillations is of the same order (see Ref. 21).

It is evident that more general solutions of the NLS equation with the Raman term in the form of a shock wave or a kink soliton may be looked for as follows: $u(x,t) = F(\xi)\exp[i\phi(x,\xi)], \xi = t - Wx$, with the asymptotics

$$u \to u_0 \exp(2iu_0^2 x), \quad t \to -\infty ,$$

$$u \to u_1 \exp(2iu_1^2 x), \quad t \to +\infty ,$$

where the value $|u_0 - u_1|$ is a function of the velocity W. To prove the assumption, one needs to use numerical simulations.

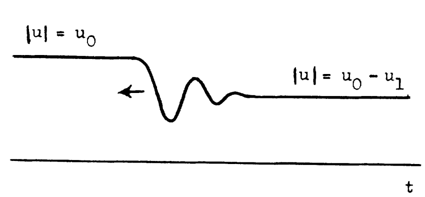

FIG. 2. Shape of the shock wave induced by the Raman effect for the case $CW < 0$; $u_1 = \frac{1}{3}|W\,\text{sgn}C|$.

V. CONCLUSIONS

In conclusion, we demonstrated that the dynamics of small-amplitude optical excitations of the cw background pulse in the presence of the self-induced Raman scattering may be effectively studied in the framework of the Korteweg–de Vries–Burgers equation. The approach allowed us to explain by a simple way the temporal self-shift of dark solitons due to the Raman self-pumping. The dark solitons are one-parametric ones and the influence of the Raman effect on their dynamics is more destructive than for the case of bright solitons. We demonstrated also that under the self-induced Raman scattering there is another type of solitonlike optical pulses in the region of the positive GVD of the fibers. These solitons are the shock waves propagating without changing their shapes and velocities.

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