Influence of pump-phase fluctuations on the squeezing in a degenerate parametric oscillator

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The fluctuations in the phase of the pump in a degenerate optical parametric oscillator are modeled by a classical phase-diffusion process, and their influence on squeezing is studied. To this end, appropriate quadrature variables, relative to the instantaneous phase of the pump, are introduced. The squeezing is calculated both inside and outside the cavity. In both cases it is found to be degraded by the phase fluctuations, which may be understood in terms of the time lag in the response of the system to an instantaneous change in the pump phase. This effect might account for a small part of the residual noise present in current experiments. Correlation functions for the field of the oscillator for finite pump linewidth are derived and discussed.

I. INTRODUCTION

Parametric processes are of great interest in quantum optics because they make it possible to generate light beams that exhibit very strong, often nonclassical, correlations. Such correlations are of interest in themselves, as evidence for the quantum nature of light;[1] but they also offer the hope that they might lead to improvements in the sensitivity of a variety of measurement techniques[2] where the noise common to both beams could be subtracted from the final output. This sensitivity beyond the standard shot-noise limit (of which conventional "squeezing"[3] is the best-known example) has been demonstrated experimentally by several groups using different arrangements;[4] the use of parametric oscillators to generate squeezed light, in particular, has been reviewed by Wu *et al.*[5]

Essential to the parametric process is a strong pump beam, and the important question of how the fluctuations in the pump affect the output of the amplifier has been addressed before[6-8] though only, it appears, in the context of a traveling-wave amplifier. Although pulsed squeezed light has recently been generated in a traveling-wave configuration,[9] the experiments which have produced the largest noise reduction[4] have made use of a cavity to build up the light intensity. The influence of pump fluctuations in such an optical parametric oscillator (OPO) does not, however, appear to have been considered in the literature; this is the object of the present paper.

In general, two kinds of questions may be asked regarding this problem. In the first place, one naturally wants to know to what extent the fluctuations of the pump reduce the nonclassical correlations (e.g., squeezing) in the light generated by the OPO. Second, for applications where the coherence properties of the generated light are important, one is led to ask in which way the finite linewidth of the pump, arising from fluctuations, affects the spectrum of the generated light. The two questions are of course related, since the degree of squeezing of the output beam can be defined in terms of its correlation functions.[10,11] Both shall be answered here, at least as far as second-order coherence is concerned.

We shall restrict ourselves here to phase fluctuations of the pump, which is typically an intense laser beam with well-stabilized amplitude but whose phase drifts in a way that may be approximated by a Wiener-Levy diffusion process. This phase diffusion clearly causes the squeezing of the output light to have only a transient nature when it is defined relative to a hypothetical, fixed reference phase; since, in time, the phase of the pump, which determines which quadrature is amplified or deamplified, will sweep all over a $(0,2\pi)$ interval. As a result, we shall not find any squeezing under stationary conditions—i.e., over a sufficiently long time interval—if we define our quadratures relative to such a fixed, external phase. But, of course, this is not the only possible definition, nor is this, operationally, the way squeezing is detected; rather than making the squeezed light interfere with an external, independent source, it is typically[5] made to interfere with a beam whose phase fluctuations are correlated to those of the pump, so that, if the phase of the pump is $\theta(t)$, squeezing is observed for quadratures defined relative to a time-dependent phase $\theta(t)/2$ (ultimately, a random variable in our model). When squeezing is defined in this way, a stationary process results leading to a well-defined quantum-noise reduction, as we shall see below.

This paper is organized as follows: In Sec. II, the basic equations are introduced and the statements in the preceding paragraph are given mathematical shape. In Sec. III the intracavity squeezing is discussed, and in Sec. IV the correlations for the field are derived and, from them, the spectrum of squeezing of the output field. A summary of our conclusions may be found in Sec. V.

II. FIELD-EVOLUTION EQUATIONS

Our basic model is an extension of that of Wódkiewicz and Zubairy[6] to a cavity situation. Thus we have the

Heisenberg equations of motion for the annihilation and creation operators for the intracavity field mode

$$\frac{da}{dt}=\kappa e^{-i\theta}a^\dagger-\frac{\gamma}{2}a+F(t)\ , \tag{1a}$$

$$\frac{da^\dagger}{dt}=\kappa e^{i\theta}a-\frac{\gamma}{2}a^\dagger+F^\dagger(t)\ , \tag{1b}$$

where κ is the parametric gain, $\theta(t)$ is the phase of the pump field (a classical random variable, see below), γ is the intensity loss rate of the cavity ($1/\gamma$ is the cavity "photon lifetime"), and $F(t)$ is a Langevin operator to account for the cavity losses (see, e.g., Ref. 10):

$$\langle F(t)F(t')\rangle=\langle F^\dagger(t)F^\dagger(t')\rangle=0\ , \tag{2a}$$

$$\langle F(t)F^\dagger(t')\rangle=\gamma\delta(t-t')\ . \tag{2b}$$

The phase $\theta(t)$ of the pump is taken to undergo a Wiener-Levy diffusion process, with diffusion constant D:

$$\langle\theta(t)\rangle=0\ , \tag{3a}$$

$$\langle\dot\theta(t)\dot\theta(t')\rangle=2D\delta(t-t')\ , \tag{3b}$$

from which follows

$$\langle e^{-i\theta(t)}\rangle=e^{-i\theta(0)}e^{-Dt}\ . \tag{4}$$

This leads to a Lorentzian spectrum for the pump, whose linewidth (half-width at half maximum) is D (in radians per second). Note that the angle brackets in (2) denote a quantum-mechanical expectation value whereas in (3) and (4) they denote an average over the classical stochastic process which describes the phase diffusion. In the equations to follow, a single set of angle brackets will denote only the quantum-mechanical expectation value; when the different realizations of $\theta(t)$ are also averaged over, double angle brackets will be used, as in $\langle\langle a(t)\rangle\rangle$.

Consider first the quantum expectation value $\langle a(t)\rangle$ for a given realization of $\theta(t)$. From Eqs. (1) and (2),

$$\frac{d}{dt}\langle a(t)\rangle=\kappa e^{-i\theta(t)}\langle a(t)\rangle^*-\frac{\gamma}{2}\langle a(t)\rangle\ . \tag{5}$$

If the cavity field builds up from vacuum, $\langle a(0)\rangle=0$, and Eq. (5) is solved by $\langle a(t)\rangle=0$ for all t.

Suppose we wanted to investigate the squeezing in the intracavity field for quadratures defined relative to a fixed reference phase ϕ_0. We would define operators

$$a_1=(ae^{i\phi_0}+a^\dagger e^{-i\phi_0})/2\ , \tag{6a}$$

$$a_2=(ae^{i\phi_0}-a^\dagger e^{-i\phi_0})/2i \tag{6b}$$

and inquire about their variances

$$\langle\Delta a_{1,2}^2\rangle=\tfrac{1}{4}(\langle aa^\dagger+a^\dagger a\rangle\pm2\,\mathrm{Re}\langle a^2e^{2i\phi_0}\rangle) \tag{7}$$

[we have used $\langle a(t)\rangle=0$]. We can use (1) to write equations of motion for $\langle a^2\rangle$ [ignore for now the constant reference phase ϕ_0, and remember that we are not yet taking an average over $\theta(t)$]. We find the following closed set of equations:

$$\frac{dx}{dt}=-\gamma x+\kappa y+G_1\ , \tag{8a}$$

$$\frac{dy}{dt}=-\gamma y+2\kappa(x+z)-i\dot\theta y+G_2e^{-i\theta}\ , \tag{8b}$$

$$\frac{dz}{dt}=-\gamma z+\kappa y-2i\dot\theta z+G_1^*e^{-2i\theta}\ , \tag{8c}$$

where

$$x\equiv\langle a^2\rangle\ , \tag{9a}$$

$$y\equiv\langle aa^\dagger+a^\dagger a\rangle e^{-i\theta}\ , \tag{9b}$$

$$z\equiv\langle a^{\dagger2}\rangle e^{-2i\theta}\ , \tag{9c}$$

and

$$G_1\equiv\langle aF+Fa\rangle\ , \tag{10a}$$

$$G_2\equiv\langle Fa^\dagger+F^\dagger a+aF^\dagger+a^\dagger F\rangle\ . \tag{10b}$$

The quantum-noise contributions of G_1 and G_2 may be evaluated as follows. Write

$$a(t)=a(t-\epsilon)+\int_{t-\epsilon}^{t}\dot a(t')dt' \tag{11}$$

then substitute this expression into (10a) and (10b), with $\dot a(t')$ given by Eq. (1a) evaluated at the time t'. The expectation values of the type $\langle F(t)a(t-\epsilon)\rangle$ vanish, since the noise F at time t and the field a at an earlier time are uncorrelated, and $\langle F(t)\rangle=0$. The products of the type

$$\int_{t-\epsilon}^{t}\langle F(t)a^\dagger(t')\rangle dt'$$

vanish also, since $\langle F(t)a^\dagger(t')\rangle$ is zero except at the point $t=t'$ where it is finite, as will be seen presently. Only the products

$$\int_{t-\epsilon}^{t}\langle F(t)F^\dagger(t')\rangle dt'=\gamma\int_{t-\epsilon}^{t}\delta(t-t')dt'=\gamma/2 \tag{12}$$

survive, because of the δ function. These products occur only in the terms $\langle Fa^\dagger\rangle$ and $\langle aF^\dagger\rangle$ in (10b), with the result

$$G_1\equiv0\ ,$$

$$G_2\equiv\frac{\gamma}{2}+\frac{\gamma}{2}=\gamma\ . \tag{13}$$

Introducing the vector $\mathbf{x}=(x,y,z)^t$ (t here stands for transpose) and the matrices

$$M_0=\begin{bmatrix}-\gamma & \kappa & 0\\ 2\kappa & -\gamma & 2\kappa\\ 0 & \kappa & -\gamma\end{bmatrix}\ , \tag{14a}$$

$$M_1=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}\ , \tag{14b}$$

we find that the system (8) may then be written, in compact notation,

$$\frac{d\mathbf{x}}{dt}=[M_0+i\dot\theta(t)M_1]\mathbf{x}+\mathbf{A}e^{-i\theta(t)} \tag{15}$$

where the vector $\mathbf{A}$ is

$$\mathbf{A}=\begin{bmatrix}0\\ \gamma\\ 0\end{bmatrix}. \tag{16}$$

Equation (15) is a stochastic differential equation. If we are interested in the average of $\mathbf{x}$ over an ensemble of realizations of the fluctuating phase $\theta(t)$, we find that we need only solve the following equation:

$$\frac{d}{dt}\langle\mathbf{x}\rangle=(M_0-DM_1^2)\langle\mathbf{x}\rangle+\mathbf{A}e^{-Dt}e^{-i\theta_0}, \tag{17}$$

a result which is proved in the Appendix. Recall that the vector $\langle\mathbf{x}\rangle$ is

$$\langle\mathbf{x}\rangle=\begin{bmatrix}\langle\langle a^2\rangle\rangle\\ \langle\langle a^\dagger a+aa^\dagger\rangle e^{-i\theta}\rangle\\ \langle\langle a^{\dagger 2}\rangle e^{-2i\theta}\rangle\end{bmatrix}. \tag{18}$$

Hence the solution of (17) would give us one of the quantities which we need to calculate the squeezing, namely $\langle\langle a^2\rangle\rangle$. [See Eq. (7).] More precisely, we would use it to calculate $\langle\langle\Delta a_{1,2}^2\rangle\rangle$, the variances averaged over an ensemble of possible realizations of the pump phase $\theta(t)$.

It is not surprising to find that $\langle\langle a^2(t)\rangle\rangle$ goes to zero as $t\to\infty$, because of the diffusion of the pump phase. Indeed, all the eigenvalues of $M_0-DM_1^2$ have a negative real part, and the inhomogeneous term in (17) decays as $\exp(-Dt)$. Thus the general solution to (17) can be written as a superposition of a term that decays as $\exp(-Dt)$ plus three other decaying exponentials. This means that after a long time has passed, because of the diffusion of the phase $\theta(t)$, the generated light can have any phase with equal probability, and so the ensemble average of $a^2(t)$ is equal to zero. Then, by Eq. (7), $\langle\langle\Delta a_{1,2}^2\rangle\rangle$ becomes equal to $(1+2\langle\langle n\rangle\rangle)/4$ (where n is the photon number), so that no squeezing is seen.

This conclusion was to be expected from the general considerations presented in the Introduction. Pictorially, we can think of the field inside the cavity as an "error ellipse" (see, for example, Fig. 1 of Ref. 8 and the discussion therein) whose orientation relative to the axes depends on the pump phase: in time, due to the pump-phase diffusion, the ellipse will rotate randomly about the origin, yielding, on the average, a large error circle for the intracavity field. As long as $D\ll 2\kappa,\gamma,\gamma-2\kappa$, we may expect the solution to Eq. (17) to exhibit, first, a growth of coherence (that is, of the expectation value $\langle\langle a^2\rangle\rangle$), over a time scale $1/(\gamma-2\kappa)$, as the "vacuum error circle" becomes squeezed into an ellipse; then, over a time scale $1/D$, a slow decay of $\langle\langle a^2\rangle\rangle$, as the ellipse's orientation is randomized.

Figure 1, which shows the solution of the system (17) for $\langle\langle a^2\rangle\rangle$, plotted for $D=0.01\gamma$ and $2\kappa=0.9\gamma$, bears this expectation out. On the other hand, it is interesting to look at the actual amount of intracavity squeezing, i.e., $\langle\langle\Delta a_2^2(t)\rangle\rangle$, for this case. This is shown as the dashed line in Fig. 1, which has been calculated from the solutions to (17) and (24), below. One finds that the squeezing never becomes very large and actually disappears long before $\langle\langle a^2\rangle\rangle$ goes to zero. This is clearly because the reduction of $\langle\Delta a_2^2\rangle$ depends crucially on a very precise

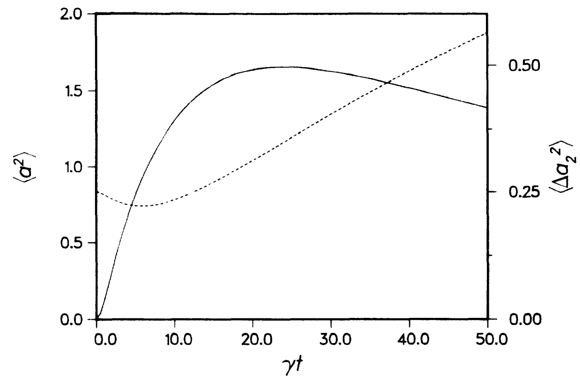

FIG. 1. Growth in time, from vacuum, of the expectation value $\langle\langle a^2\rangle\rangle$ (solid line, axis on the left) and the intracavity squeezing $\langle\langle\Delta a_2^2(t)\rangle\rangle$ relative to a fixed reference phase (dashed line, axis on right), for the parameters $D=0.01\gamma$, $2\kappa=0.9\gamma$.

cancellation between the first and the second term in Eq. (7), and the diffusion of the pump phase, i.e., the random rotation of the squeezed error ellipse, spoils this cancellation very quickly.

While this conclusion holds for squeezing relative to any *fixed* phase ϕ_0, the physically meaningful quadratures are those referred to the instantaneous phase $\theta(t)/2$, as discussed in the Introduction, since, in the end, in the experiments the squeezed light is made to interfere with a beam whose phase fluctuations are correlated to those of the pump in precisely this way (see, e.g., Ref. 5). The interference term between the "local oscillator," which goes as $\exp[-i\theta(t)/2]$, and the "signal," which goes as a, will be a linear combination of $a\exp[i\theta(t)/2]$ and $a^\dagger\exp[-i\theta(t)/2]$. Therefore the "quadratures" of interest are

$$a_1'=\tfrac{1}{2}(ae^{i\theta(t)/2}+a^\dagger e^{-i\theta(t)/2}), \tag{19a}$$

$$a_2'=\frac{1}{2i}(ae^{i\theta(t)/2}-a^\dagger e^{-i\theta(t)/2}). \tag{19b}$$

We may then define the intracavity squeezing in terms of

$$\langle\langle(\Delta a_{1,2}')^2\rangle\rangle=\tfrac{1}{4}(\langle a^\dagger a+aa^\dagger\rangle\pm 2\,\mathrm{Re}\langle\langle e^{i\theta(t)}a^2\rangle\rangle) \tag{20}$$

whereas the spectrum of squeezing outside the cavity must be defined in terms of the correlation functions

$$C_{aa}(\tau)=\langle\langle e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)\rangle\rangle, \tag{21a}$$

$$C_{a^\dagger a}(\tau)=\langle\langle e^{-i\theta(t+\tau)/2}e^{i\theta(t)/2}a^\dagger(t+\tau)a(t)\rangle\rangle \tag{21b}$$

as[10,11]

$$S_\pm(\omega)=4\gamma\int_0^\infty\cos\omega\tau(C_{a^\dagger a}(\tau)\pm C_{aa}(\tau))d\tau \tag{22}$$

in this simple case where, as we shall see below, the Fourier transform of C_{aa} is real (symmetric spectrum). The quantities (20) and (22) are calculated next.

III. INTRACAVITY SQUEEZING

In this section the intracavity squeezing, according to the generalized expression (20), will be calculated. This quantity is in general of little more than academic interest, since it would be hard to measure it directly. Still, it is interesting to compare this case to the theories with no pump-phase diffusion (e.g., Refs. 10 and 12).

We may obtain an equation of motion for the quantity $e^{i\theta}a^2$ very simply from (8). Indeed, introducing the vector $\mathbf{x}'=e^{i\theta}\mathbf{x}$ [with x,y,z given in (9)] we obtain

$$\frac{d\mathbf{x}'}{dt}=[M_0+i\dot{\theta}(t)M_1']\mathbf{x}'+\mathbf{A} \tag{23}$$

with the same M_0 and $\mathbf{A}$ as in (15), whereas M_1' is now

$$M_1'=\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix} .$$

When the result proved in the Appendix [Eqs. (A1)–(A5)] is used again to derive an equation for $\langle \mathbf{x}'\rangle$, we find

$$\frac{d}{dt}\langle \mathbf{x}'\rangle=M'\langle \mathbf{x}'\rangle+\mathbf{A} \tag{24}$$

where now

$$\mathbf{x}'=\begin{bmatrix}\langle e^{i\theta}a^2\rangle\\ \langle aa^\dagger+a^\dagger a\rangle\\ \langle e^{-i\theta}a^{\dagger 2}\rangle\end{bmatrix} , \tag{25a}$$

$$M_1'=\begin{bmatrix}-\gamma-D & \kappa & 0\\ 2\kappa & -\gamma & 2\kappa\\ 0 & \kappa & -\gamma-D\end{bmatrix} . \tag{25b}$$

It is easy to solve for the full time dependence, but we really need be concerned only with the steady state (SS) which the system approaches as $t\to\infty$. The solution to (24) with the left-hand side equal to zero is

$$\langle\langle e^{i\theta}a^2\rangle\rangle_{\rm SS}=\langle\langle e^{-i\theta}a^{\dagger 2}\rangle\rangle_{\rm SS}=\frac{\gamma\kappa}{\gamma^2-4\kappa^2+\gamma D} , \tag{26a}$$

$$\langle\langle aa^\dagger+a^\dagger a\rangle\rangle_{\rm SS}=\frac{\gamma(\gamma+D)}{\gamma^2-4\kappa^2+\gamma D} , \tag{26b}$$

and the quadrature which shows noise reduction, a_2', yields, according to Eq. (20),

$$\langle\langle \Delta a_2'^2\rangle\rangle=\frac{1}{4}\,\frac{1+D/\gamma-2\kappa/\gamma}{1+D/\gamma-(2\kappa/\gamma)^2} . \tag{27}$$

We recall here that in the theory without pump-phase diffusion the threshold for self-oscillation (parametric gain equals loss) is the point $2\kappa=\gamma$, and as this limit is approached from below $\langle \Delta a_1^2\rangle\to\infty$ while $\langle \Delta a_2^2\rangle\to\frac{1}{8}$, corresponding to a noise reduction by a factor of 2. This limit is easily obtained from (27) when $D=0$ [$\langle\langle a_1'^2\rangle\rangle$ is given by an expression identical to (27) except with the opposite sign for κ]. When $D\neq 0$, instead, Eq. (27) depends on $2\kappa/\gamma$ as shown in Fig. 2.

Several features of Fig. 2 are of interest. First, the limit $D=0$ (no phase diffusion) is seen to be a singular one. For any other value of D, the noise in the quadrature a_2' diverges as the threshold of oscillation is approached from below (just as the noise in a_1' does, both when $D=0$ and when $D\neq 0$). This is easily understood from the model of a squeezed ellipse undergoing random rotations. As threshold is approached, the noise along one of the axes approaches infinity. Any amount of phase diffusion, by tilting the ellipse slightly, couples an amount of noise into the other quadrature equal to the projection of this "unsqueezed" noise by the tilt angle, and this projection approaches infinity as threshold is approached, for any finite tilt angle, i.e., for any finite phase diffusion. This is so even though the quadratures a_1' and a_2' are constantly being redefined by the instantaneous pump phase, because there is a time lag in the response of the system (the cavity field) to an instantaneous change in the pump phase. This time lag, in fact, is particularly important near threshold, where the fluctuations in the unsqueezed quadrature become essentially undamped; it is this effect which is responsible for the reduction of the squeezing observed in Fig. 2.

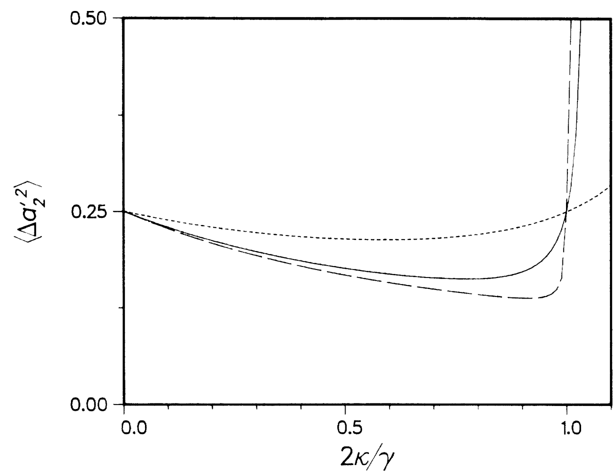

FIG. 2. The steady-state squeezing inside the cavity, defined relative to the pump's phase, as a function of $2\kappa/\gamma$ (parametric gain/losses) for different values of D/γ. Dashed line: $D=\gamma$; solid line: $D=0.1\gamma$; dash-dotted line: $D=0.01\gamma$. Note the increase in the threshold for oscillation with D/γ and the absence of squeezing at $2\kappa=\gamma$ for all $D\neq 0$.

Figure 2 also shows that the threshold of oscillation itself is raised somewhat when $D\neq 0$. At the point $2\kappa=\gamma$ no noise reduction occurs: for any $D\neq 0$, $\langle\langle \Delta a_2'^2\rangle\rangle=\frac{1}{4}$ (its vacuum value) when $2\kappa=\gamma$.

The maximum noise reduction (minimum of $\langle\langle \Delta a_2'^2\rangle\rangle$) occurs somewhat below threshold. In fact, it is easy to calculate that $\langle\langle \Delta a_2'^2\rangle\rangle$ is minimum when

$$\frac{2\kappa}{\gamma}=1+\frac{D}{\gamma}-\left[\frac{D}{\gamma}+\left[\frac{D}{\gamma}\right]^2\right]^{1/2} \simeq 1-\sqrt{D/\gamma} \tag{28}$$

where the approximation holds for $D\ll\gamma$. In this limit, Eq. (28) gives, when substituted back in (27),

$$\langle\langle \Delta a_2'^2\rangle\rangle_{\rm min}\simeq\tfrac{1}{8}(1+\sqrt{D/\gamma}) . \tag{29}$$

Equation (29) shows the effect of phase fluctuations of the pump on the average noise reduction for the intracavity field in steady state, for the OPO below threshold, in the limit when $D \ll \gamma$. (For larger values of D, refer to Fig. 2.)

IV. THE OUTPUT FIELD

A. Noise reduction in the output field

The realization[13] that the output field could exhibit a different quantum-noise reduction than the intracavity field led[10] to the concept of a "spectrum of squeezing" for the output field, where the noise in a small frequency interval (detector bandwidth) was seen to be reduced through a correlation between pairs of modes[14] symmetrically placed about the cavity resonance. To calculate such a spectrum of squeezing [given for our system by Eq. (22)] we need to compute the correlation functions of the mixed output field plus reference beam with relative phase $\theta(t)/2$, given by Eqs. (21).

Once again, we can use the equations of motion (1) to obtain equations for $C_{a^\dagger a}$, C_{aa}. Consider, for instance, C_{aa}. We have, taking derivatives with respect to τ,

$$\frac{\partial}{\partial\tau}[e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)]=\frac{i}{2}\frac{\partial\theta(t+\tau)}{\partial\tau}e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)+e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}\frac{\partial a(t+\tau)}{\partial\tau}a(t)\ . \tag{30}$$

Use now (1) in (30) and take the quantum expectation value of the resulting expression. As long as $\tau>0$,

$$\langle F(t+\tau)a(t)\rangle=0 \tag{31}$$

so we have

$$\frac{\partial}{\partial\tau}\langle e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)\rangle=\frac{1}{2}\left[i\frac{\partial\theta}{\partial\tau}-\gamma\right]\langle e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)\rangle+\kappa\langle e^{-i\theta(t+\tau)/2}e^{i\theta(t)/2}a^\dagger(t+\tau)a(t)\rangle\ . \tag{32a}$$

Similarly

$$\frac{\partial}{\partial\tau}\langle e^{-i\theta(t+\tau)/2}e^{i\theta(t)/2}a^\dagger(t+\tau)a(t)\rangle=\frac{1}{2}\left[-i\frac{\partial\theta}{\partial\tau}-\gamma\right]\langle e^{-i\theta(t+\tau)/2}e^{i\theta(t)/2}a^\dagger(t+\tau)a(t)\rangle$$
$$+\kappa\langle e^{i\theta(t+\tau)/2}e^{i\theta(t)/2}a(t+\tau)a(t)\rangle\ . \tag{32b}$$

When the ensemble average over the realizations of $\theta(t)$ is taken, as in the Appendix, this leads to a system

$$\frac{\partial C_{aa}}{\partial\tau}=-\left[\frac{\gamma}{2}+\frac{D}{4}\right]C_{aa}+\kappa C_{a^\dagger a}\ , \tag{33a}$$

$$\frac{\partial C_{a^\dagger a}}{\partial\tau}=-\left[\frac{\gamma}{2}+\frac{D}{4}\right]C_{a^\dagger a}+\kappa C_{aa}\ , \tag{33b}$$

which does not depend explicitly on t. This may be easily solved to express C_{aa} and $C_{a^\dagger a}$ as functions of τ and of their values at $\tau=0^+$. Noting that, from the definition (21), these values are trivially related to the components of the vector (25a), we may ensure the time independence (stationarity) of C_{aa} and $C_{a^\dagger a}$ by choosing for their values at $\tau=0$ the steady-state values given by Eqs. (26):

$$C_{aa}(0)=\langle\langle e^{i\theta}a^2\rangle\rangle_{SS}=\frac{\gamma\kappa}{\gamma^2-4\kappa^2+\gamma D}\ , \tag{34a}$$

$$C_{a^\dagger a}(0)=\langle\langle a^\dagger a\rangle\rangle_{SS}=\tfrac{1}{2}(\langle\langle a^\dagger a+aa^\dagger\rangle\rangle_{SS}-1)$$
$$=\frac{1}{2}\left[\frac{\gamma(\gamma+D)}{\gamma^2-4\kappa^2+\gamma D}-1\right]\ . \tag{34b}$$

Then the solution to (33) is ($\tau>0$)

$$C_{a^\dagger a}(\tau)=\frac{1}{4}\frac{2\kappa}{\gamma^2-4\kappa^2+\gamma D}[(\gamma+2\kappa)e^{\kappa\tau}-(\gamma-2\kappa)e^{-\kappa\tau}]\times e^{-(\gamma/2+D/4)\tau}\ , \tag{35a}$$

$$C_{aa}(\tau)=\frac{1}{4}\frac{2\kappa}{\gamma^2-4\kappa^2+\gamma D}[(\gamma+2\kappa)e^{\kappa\tau}+(\gamma-2\kappa)e^{-\kappa\tau}]\times e^{-(\gamma/2+D/4)\tau}\ . \tag{35b}$$

This result may be compared to the correlation functions without phase diffusion presented, for instance, in Ref. 15.

From Eqs. (35) and the definition (22) it is easy now to calculate the spectrum of squeezing $S_-(\omega)$:

$$S_-(\omega)=-4\frac{(1-2\kappa/\gamma)2\kappa/\gamma}{[1+D/\gamma-(2\kappa/\gamma)^2](1+D/2\gamma+2\kappa/\gamma)}\times\frac{1}{1+4\omega^2/(\gamma+D/2+2\kappa)^2} \tag{36}$$

for the quadrature which shows noise reduction [i.e., the minus sign has been chosen in Eq. (20)]. For the other quadrature, the sign of κ would be reversed everywhere in Eq. (36).

Equation (36) shows that the spectrum of squeezing is still Lorentzian, as it is for $D=0$, only somewhat broadened. For a given value of D, the maximum noise reduction still occurs at line center, where

$$S_-(0)=-4\frac{(1-2\kappa/\gamma)2\kappa/\gamma}{[1+D/\gamma-(2\kappa/\gamma)^2](1+D/2\gamma+2\kappa/\gamma)} \tag{37}$$

only now $S_-(0)$ is not maximum when $2\kappa=\gamma$, as it would be if $D=0$; indeed we find (just as for the intracavity field) no squeezing at all when $2\kappa=\gamma$ [i.e., $S_-(0)=0$]. Thus, as for the intracavity field, the limit $D=0$ is a singular one.

Figure 3 shows the total noise reduction at $\omega=0$, given by $1+S_-(0)$ (where 1 is the shot-noise level) plotted as a function of $2\kappa/\gamma$ for different values of D/γ. Again we find the noise in both quadratures to diverge as the new threshold, $2\kappa/\gamma=(1+D/\gamma)^{1/2}$, is approached from below. This time, however, one can show that maximum squeezing (minimum noise) occurs for

$$\left[\frac{2\kappa}{\gamma}\right]_{\min}\simeq 1-\left[\frac{D}{\gamma}\right]^{1/3}+\frac{1}{2}\left[\frac{D}{\gamma}\right]^{2/3}+\frac{1}{6}\left[\frac{D}{\gamma}\right] \tag{38}$$

when $D/\gamma \ll 1$. This is a different value (and a different dependence on D/γ) from (28), which is the value of $2\kappa/\gamma$ which maximizes the squeezing in the intracavity field. The terms of order D/γ are necessary if (38) is to be substituted in (37) to evaluate the maximum noise reduction:

$$[1+S_-(0)]_{\min}\simeq\frac{3}{4}\left[\frac{D}{\gamma}\right]^{2/3}. \tag{39}$$

Equation (39) expresses the degradation of the squeezing in the output field due to a finite pump linewidth D. In the ideal case, $D=0$, perfect squeezing, corresponding to $S_-(0)=-1$, is achieved at resonance when the threshold $2\kappa=\gamma$ is approached.

Note again the different dependence on D for the intracavity field [Eq. (29)] and the output field, Eq. (39). Of course, what one calls squeezing inside and outside the cavity are actually rather different physical quantities to begin with. A discussion of their relationship for general quantum optical systems has recently been presented in Ref. 16.

How serious a limitation is (39), in practice? The experiments in Ref. 5 reported a pump linewidth of about 0.1 MHz. Taking the value $\gamma=10$ MHz as typical, we find $D/\gamma=10^{-2}$ and so, from Eq. (39), a minimum value of $1+S_-(0)\simeq 0.04$, a factor of 3 or so below the inferred value of $1+S_-(0)$ for the experiment. Thus pump-phase fluctuations might amount to a significant fraction of the residual noise in the squeezed quadrature.

B. Minimum-uncertainty states

An interesting place to look for the effects of pump-phase diffusion is in the relation $(1+S_+)(1+S_-)=1$, obeyed by a minimum-uncertainty state, and, according to the theory, by the OPO below threshold when $D=0$. This relation has been verified in the experiments reported in Ref. 5 with good accuracy. Our Fig. 4 plots $1+S_+$ versus $1+S_-$, as given by Eq. (36), for different values of D, with the hyperbola $1+S_+=1/(1+S_-)$, corresponding to the $D=0$ case (minimum-uncertainty state), also drawn for reference. This may be immediately compared to Fig. 8 of Ref. 5. For $D/\gamma=0.01$ the deviations from a minimum-uncertainty state appear rather small in this plot. If instead the product $(1+S_+)(1+S_-)$ is plotted, as we have done in Fig. 5, deviations from the ideal value of 1 are seen to be large even for $D/\gamma=0.01$, especially in the region near threshold where squeezing is maximum.

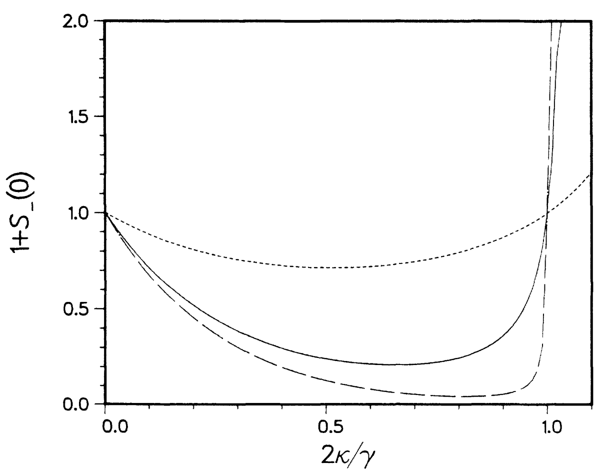

FIG. 3. The on-resonance noise spectrum outside the cavity in the squeezed quadrature, $S_-(0)$, as a function of $2\kappa/\gamma$ for different values of D/γ. Dashed line: $D=\gamma$; solid line: $D=0.1\gamma$; dash-dotted line: $D=0.01\gamma$.

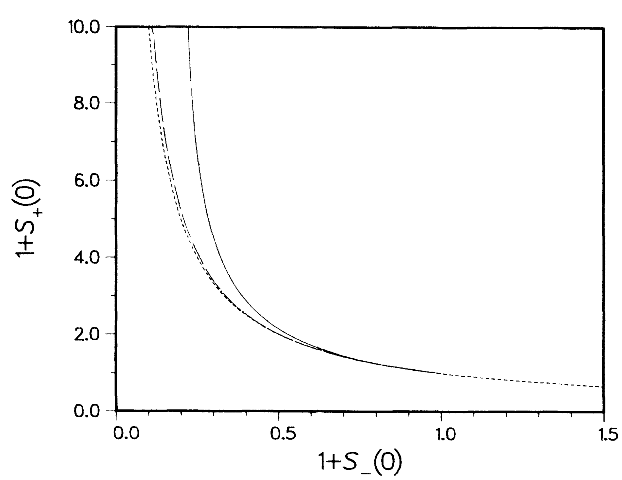

FIG. 4. The noise in one quadrature, $1+S_+(0)$, vs the other one, $1+S_-(0)$, as $2\kappa/\gamma$ is scanned from 0 to 1, for different values of D. A minimum uncertainty state would lie on the dashed hyperbola, obtained for $D=0$. The dash-dotted line corresponds to $D=0.01\gamma$ and the solid line to $D=0.1\gamma$.

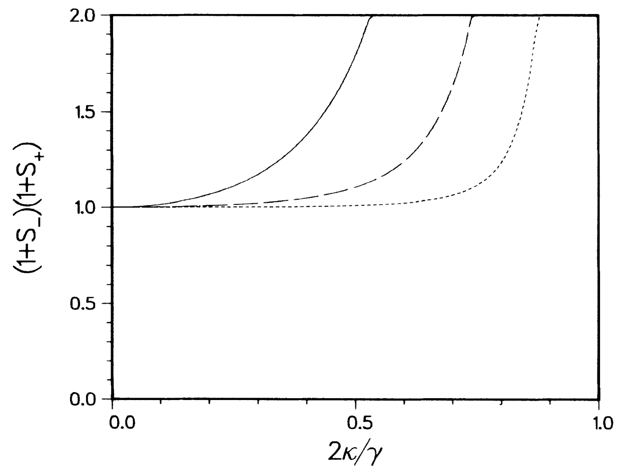

FIG. 5. The product $[1+S_+(0)][1+S_-(0)]$, which is equal to 1 for a minimum-uncertainty state, as a function of $2\kappa/\gamma$, for different values of D. Solid line: $D=0.1\gamma$. Dash-dotted line: $D=0.01\gamma$. Dashed line: $D=0.001\gamma$.

C. Spectrum of the output field

The correlation functions C_{aa} and $C_{a^\dagger a}$ of Eqs. (21) are not really what one would need to calculate the spectrum of the output field alone; rather, they are appropriate to the study of the interference with the local oscillator of phase $\theta(t)/2$. For the ordinary (first-order) spectrum of the output field of the degenerate OPO, we need to evaluate instead

$$C'_{a^\dagger a}(\tau)\equiv\langle\langle a^\dagger(t+\tau)a(t)\rangle\rangle \ . \tag{40a}$$

As before, we can derive a set of coupled equations for $C'_{a^\dagger a}$ and

$$C'_{aa}(\tau)\equiv\langle\langle e^{i\theta(t+\tau)}a(t+\tau)a(t)\rangle\rangle \ . \tag{40b}$$

These are

$$\frac{d}{d\tau}C'_{a^\dagger a}=-\frac{\gamma}{2}C'_{a^\dagger a}+\kappa C'_{aa} \ , \tag{41a}$$

$$\frac{d}{d\tau}C'_{aa}=-\left[\frac{\gamma}{2}+D\right]C'_{aa}+\kappa C'_{a^\dagger a} \ . \tag{41b}$$

The initial conditions for $\tau=0$ should be taken from the steady states calculated in Eqs. (26). The result is

$$\begin{aligned}\langle\langle a^\dagger(t+\tau)a(t)\rangle\rangle&=\frac{2\kappa^2}{\gamma^2-4\kappa^2+\gamma D}\frac{1}{\lambda_1-\lambda_2}(-\lambda_2 e^{\lambda_1\tau}+\lambda_1 e^{\lambda_2\tau})\\&\simeq\frac{1}{4}\frac{2\kappa}{\gamma^2-4\kappa^2+\gamma D}[(\gamma+D+2\kappa)e^{\kappa\tau}-(\gamma+D-2\kappa)e^{-\kappa\tau}]e^{-(\gamma+D)\tau/2}\end{aligned} \tag{42}$$

where λ_1 and λ_2 are the eigenvalues of the system (41):

$$\begin{aligned}\lambda_{1,2}&=\tfrac{1}{2}[-\gamma-D\pm(4\kappa^2+D^2)^{1/2}]\\&\simeq\tfrac{1}{2}(-\gamma-D\pm2\kappa)\end{aligned} \tag{43}$$

and the approximations in Eqs. (42) and (43) hold for $D\ll\kappa$ (first order in D/κ). Equation (42) may be compared with the corresponding result for $D=0$ given, for instance, in Ref. 15. The spectrum obtained from (42) is a superposition of two Lorentzians, of widths $2|\lambda_1|$ and $2|\lambda_2|$, or, to first order in D, $\gamma+D\mp2\kappa$ [full width at half maximum (FWHM)]. (Recall that $2D$ is the FWHM of the pump.) It may be written as

$$\begin{aligned}\tilde{C}_{a^\dagger a}(\omega)&=\frac{1}{\pi}\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left[\frac{1}{\lambda_1^2+\omega^2}-\frac{1}{\lambda_2^2+\omega^2}\right]\\&=-\frac{1}{\pi}\frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)}{(\lambda_1^2+\omega^2)(\lambda_2^2+\omega^2)}\end{aligned} \tag{44}$$

after adding the two Lorentzians and normalizing to unit total area. From the expressions (43) one can see that the pump-phase diffusion is especially important close to threshold, where $2\kappa\to\gamma$ and the first Lorentzian on the first line of Eq. (44) would become a δ function if $D=0$. (This corresponds to the totally undamped fluctuations of the "unsqueezed" quadrature of the field.) The finite value of D yields a spectrum of finite width as $2\kappa\to\gamma$. It is easy to see that in this limit, namely when $2\kappa-\gamma\ll D\ll2\kappa$, the spectrum is dominated by this first Lorentzian which has approximately half the width of the pump, i.e., $2|\lambda_1|\simeq D$. The subtraction of the other Lorentzian, of width $\sim2\gamma$, results in a very slight narrowing of the total spectrum, i.e., a total width slightly less than D.

V. CONCLUSIONS

We have calculated the influence of classical pump-phase fluctuations on the characteristic properties of the light generated by an optical parametric oscillator. A major qualitative difference with the case $D=0$ is that we find that the noise in both quadratures diverges as the OPO is brought close to the threshold of oscillation. Among our quantitative results, we have established how the pump's linewidth enters the correlation functions of the output light [as $\exp(-D/2)$, see Eq. (42)] and the corresponding broadening of the spectrum of squeezing

[Eq. (36)]. We have also shown how the optimum value of $2\kappa/\gamma$ for squeezing depends on D [Eq. (38), for small D/γ, and Fig. 3] and how the noise reduction is degraded as a function of D as well. (It is interesting to note that maximum squeezing for the intracavity field does not, in general, correspond to the maximum for the output field, and that the dependence on D is also different for both.) Finally, we have shown how seriously the phase diffusion of the pump may prevent the realization of a minimum-uncertainty state. Essential to our analysis has been the realization that squeezing is observed through the interference of the squeezed light with a local oscillator with correlated phase fluctuations, so that a correlation function like $C_{a^\dagger a}$ [Eq. (21)], rather than $C'_{a^\dagger a}$ [Eq. (40)], is the relevant quantity for a squeezing experiment in which phase diffusion of the pump is not negligible.

APPENDIX

Suppose we have a stochastic differential equation of the form

$$\frac{dx}{dt}=[M_0+i\dot{\theta}(t)M_1]x+v(\theta) \tag{A1}$$

where x and v are, in general, vectors [as in, e.g., Eq. (15), where $v(\theta)\equiv \mathbf{A}e^{-i\theta}$], although we shall drop the vector indices here to simplify the notation. M_0 and M_1 are constant matrices which in general do not commute, and $\theta(t)$ is a stochastic process, whose derivative $\dot{\theta}(t)$ is δ correlated and Gaussian:

$$\langle \dot{\theta}(t)\dot{\theta}(t')\rangle=2D\delta(t-t') , \tag{A2}$$

$$\langle \dot{\theta}(t_1)\cdots\dot{\theta}(t_{2n-1})\rangle=0 , \tag{A3}$$

$$\langle \dot{\theta}(t_1)\cdots\dot{\theta}(t_{2n})\rangle=\frac{1}{n!}\sum_{P\in S_{2n}} D^n\prod_{j=1}^{n}\delta(t_{P(2j)}-t_{P(2j-1)}) . \tag{A4}$$

In Eq. (A3), P is any permutation of the integers $1,2,\ldots,2n$, and the sum is over all such permutations [the symmetric group S_{2n} of order $(2n)!$]. We want to show that the ensemble average of $x(t)$ obeys then the equation

$$\frac{d}{dt}\langle x\rangle=(M_0-DM_1^2)\langle x\rangle+\langle v(\theta)\rangle . \tag{A5}$$

We shall follow the derivation in Ref. 17, which applies here with minor extensions. First, introduce the matrix $\exp(M_0t)$, defined by the usual power series and whose inverse is clearly $\exp(-M_0t)$. Then introduce as a new variable

$$y(t)=e^{-M_0t}x(t) \tag{A6}$$

so that

$$\langle x(t)\rangle=e^{M_0t}\langle y(t)\rangle . \tag{A7}$$

The stochastic equation obeyed by $y(t)$ is

$$\frac{dy}{dt}=i\dot{\theta}M(t)y+e^{-M_0t}v(\theta) \tag{A8}$$

introducing a new matrix

$$M(t)=e^{-M_0t}M_1e^{M_0t} \tag{A9}$$

which may not commute with itself at different times, as a result of which the formal solution of (A8) has to be written in time-ordered fashion,

$$\begin{aligned} y(t)={}&y(0)+\sum_{n=1}^{\infty}(i)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\dot{\theta}(t_1)\cdots\dot{\theta}(t_n)M(t_1)\cdots M(t_n)y(0)\\ &+\sum_{n=1}^{\infty}(i)^{n-1}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\dot{\theta}(t_1)\cdots\dot{\theta}(t_{n-1})M(t_1)\cdots M(t_{n-1})e^{-M_0t_n}v(\theta(t_n)) \end{aligned} \tag{A10}$$

(with the convention $t_0\equiv t$). Equation (A10) is just the Dyson series obtained by iteration of Eq. (A8).

Consider now the ensemble average value of Eq. (A10). By (A3), we find that in the first sum in Eq. (A10) only the terms with n even survive. For the second sum we need to consider the average

$$\langle \dot{\theta}(t_1)\cdots\dot{\theta}(t_{n-1})v(\theta(t_n))\rangle \tag{A11}$$

with $t_1\geq t_2\geq\cdots\geq t_{n-1}\geq t_n$. Let

$$\theta(t_n)=\theta(t_n-\epsilon)+\int_{t_n-\epsilon}^{t_n}\dot{\theta}(t_{n+1})dt_{n+1} \tag{A12}$$

for arbitrarily small ϵ. Note that, then, $\theta(t_n-\epsilon)$ is decorrelated from all the $\dot{\theta}(t_j)$ appearing in Eq. (A11). We can approximate

$$v(\theta(t_n))\simeq v(\theta(t_n-\epsilon))+\frac{dv}{d\theta}\int_{t_n-\epsilon}^{t_n}\dot{\theta}(t_{n+1})dt_{n+1} \tag{A13}$$

(the rms size of the last term is finite and of order $\epsilon D|dv/d\theta|$, thus the expansion is justified for sufficiently small ϵ). Now substitute (A13) into (A11):

$$\langle \dot\theta(t_1)\cdots\dot\theta(t_{n-1})v(\theta(t_n))\rangle \simeq \langle \dot\theta(t_1)\cdots\dot\theta(t_{n-1})\rangle\langle v(\theta(t_n-\epsilon))\rangle + \frac{dv}{d\theta}\int_{t_n-\epsilon}^{t_n}\langle \dot\theta(t_1)\cdots\dot\theta(t_{n-1})\dot\theta(t_{n+1})\rangle dt_{n+1} \,. \tag{A14}$$

If $\langle v(\theta(t))\rangle$ is a continuous function of t, the limit $\epsilon\to 0$ may be taken in the first term of (A14) without problem. Consider then the second term. When it is substituted into Eq. (A10) one obtains an $(n+1)$-dimensional integral whose integrand, by Eqs. (A3) and (A4), is identically zero if n is odd, whereas if n is even, say $n=2m$, it equals a sum of products of $m\,\delta$ functions as in Eq. (A4). Note, however, that the variable t_n does not appear as the argument of any δ functions, since there is no $\dot\theta(t_n)$ in Eq. (A14). The integral over t_n is then, in all the terms in the sum, of the form

$$\int_0^{t_{n-1}} dt_n \int_{t_n-\epsilon}^{t_n+\epsilon} \delta(t_{n+1}-t_j)dt_{n+1} \tag{A15}$$

where $j\neq n$. Then, because of the time ordering, $t_j \ge t_{n-1} \ge t_n$, we have

$$\int_{t_n-\epsilon}^{t_n+\epsilon} \delta(t_{n+1}-t_j)dt_{n+1} = \begin{cases} 0 & \text{if } t_n\neq t_j \\ 1 & \text{if } t_n = t_j \,. \end{cases} \tag{A16}$$

Thus, at only one point ($t_n=t_j$) is the function of t_n which appears in the integrand of (A15) different from zero, and at that point it is finite; therefore, the integral over t_n vanishes.

We have shown, therefore, that the second term in (A14) does not contribute to $\langle y\rangle$ for any value of $\epsilon\neq 0$. We can then in the limit $\epsilon\to 0$ replace $\langle \dot\theta(t_1)\cdots\dot\theta(t_{n-1})v(\theta(t_n))\rangle$ by $\langle \dot\theta(t_1)\cdots\dot\theta(t_{n-1})\rangle\langle v(\theta(t_n))\rangle$ in the integrand of (A10) when calculating the average value of $y(t)$, which may then be written as

$$\begin{aligned} \langle y(t)\rangle = {}& y(0)+\int_0^t e^{-M_0 t_1}\langle v(\theta(t_1))\rangle dt_1 \\ &+\sum_{n=1}^{\infty}(-1)^n\int_0^t dt_1\cdots\int_0^{t_{2n-1}} dt_{2n}\langle \dot\theta(t_1)\cdots\dot\theta(t_{2n})\rangle M(t_1)..M(t_{2n}) \\ &\qquad\times\left[y(0)+\int_0^{t_{2n}} dt_{2n+1}e^{-M_0 t_{2n+1}}\langle v(\theta(t_{2n+1}))\rangle\right] . \end{aligned} \tag{A17}$$

We now have to deal with

$$I_n(t)\equiv \int_0^t dt_1\cdots\int_0^{t_{2n-1}} dt_{2n}\frac{1}{n!}\sum_{P\in S_{2n}} D^n\prod_{j=1}^{n}\delta(t_{P(2j)}-t_{P(2j-1)})M(t_1)\cdots M(t_{2n}) \tag{A18}$$

according to (A4). This has been done by Fox in Ref. 17 in detail, and it is not necessary to go over the details of the proof here. We can reason by induction as follows. Of the $(2n)!$ permutations (of the subindices $1,\ldots,2n$) over which the sum in (A18) runs, the time ordering of the integrals causes all the ones which do not contain the factor $\delta(t_1-t_2)$ to vanish. (See Ref. 17 for proof of this assertion.) There are, as Fox shows, $2n[(2n-2)!]$ terms in (A18) which do contain the factor $\delta(t_1-t_2)$. They can, of course, be written as $\delta(t_1-t_2)$ times a product of $n-1$ δ's involving a permutation of the remaining $2n-2$ indices. Each distinct permutation of S_{2n-2}, however, appears in $2n$ identical terms in (A18): because $\delta(t_1-t_2)$ can be placed in any position relative to the remaining $n-1$ factors (a total of n possible positions, giving n identical terms) and because for each position there are in (A18) a term $\delta(t_1-t_2)$ and a term $\delta(t_2-t_1)$ which lead, of course, to the same result twice. Thus we can write (A18) as

$$\begin{aligned} I_n(t) = {}& \int_0^t dt_1\int_0^{t_1} dt_2\delta(t_1-t_2) \\ &\times\int_0^{t_2} dt_3\cdots\int_0^{t_{2n-1}} dt_{2n}\frac{1}{n!}2n\sum_{P\in S_{2n-1}} D^n\prod_{j=1}^{n-1}\delta(t_{P(2j)+2}-t_{P(2j-1)+2})M(t_1)\cdots M(t_{2n}) \\ = {}& D\int_0^t dt_1 M^2(t_1)\int_0^{t_1} dt_3\cdots\int_0^{t_{2n-1}} dt_{2n}\frac{D^{n-1}}{n!}\sum_{P\in S_{2n-1}}\delta(t_{P(2j)+2}-t_{P(2j-1)+2})M(t_3)\cdots M(t_{2n}) \end{aligned} \tag{A19}$$

carrying out the integration over t_2 (which brings a factor $\frac{1}{2}$, since t_1 is one of the limits of the integral). We see that, with a simple relabeling of the variables,

$$t_1\to t_1' \,,$$

$$t_3\to t_1, t_4\to t_2,\ldots,t_{2n}\to t_{2n-2}$$

we can write

$$I_n(t)=D\int_0^t dt_1' M^2(t_1')\int_0^{t_1'} dt_1 \cdots \int_0^{t_{2n-3}} dt_{2n-2}\frac{1}{(n-1)!}\sum_{P\in S_{2n-2}} D^{n-1}\prod_{j=1}^{n-1}\delta(t_{P(2j)}-t_{P(2j-1)})M(t_1)\cdots M(t_{2n-2})$$
$$=D\int_0^t dt_1' M^2(t_1')I_{n-1}(t_1') \quad \text{(A20)}$$

which is just of the form necessary to reason by induction, to conclude

$$I_n(t)=D^n\int_0^t dt_1'\int_0^{t_1'} dt_2'\cdots\int_0^{t_{n-1}'} dt_{n-1}' M^2(t_1')M^2(t_2')\cdots M^2(t_n') . \quad \text{(A21)}$$

Then

$$\langle y(t)\rangle = y(0)+\int_0^t e^{-M_0 t_1}\langle v(\theta(t_1))\rangle dt_1$$
$$+\sum_{n=1}^{\infty}(-1)^n D^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n M^2(t_1)\cdots M^2(t_n)\left[y(0)+\int_0^{t_n} dt_{n+1}e^{-M_0 t_{n+1}}\langle v(\theta(t_{n+1}))\rangle\right] . \quad \text{(A22)}$$

This may be directly compared to Eq. (A10), which was the formal solution of (A8), to see immediately the equation that $\langle y(t)\rangle$ must satisfy:

$$\frac{d}{dt}\langle y(t)\rangle = -DM^2(t)\langle y(t)\rangle + e^{-M_0 t}\langle v(\theta)\rangle \quad \text{(A23)}$$

or, using (A7) to go back to the original variable $\langle x\rangle$, along with (A9),

$$\frac{d}{dt}\langle x(t)\rangle = (M_0 - DM_1^2)\langle x(t)\rangle + \langle v(\theta)\rangle , \quad \text{(A5')}$$

which is the result we have repeatedly used in the text. To prove Eq. (17), in particular, we need to calculate the average of $\langle v(\theta)\rangle$ when $v(\theta)$ is proportional to $\exp[-i\theta(t)]$. This is a fairly standard result which may also be obtained as a special case of (A5') itself. Indeed, define

$$x(t)\equiv e^{-i\theta(t)} ; \quad \text{(A24)}$$

it obeys the equation

$$\frac{dx}{dt} = -i\dot\theta(t)x , \quad \text{(A25)}$$

which is of the form (A1) with $M_0=0$, $v=0$, $M_1=-1$. Therefore by (A25)

$$\frac{d}{dt}\langle x(t)\rangle \equiv \frac{d}{dt}\langle e^{-i\theta(t)}\rangle = -D\langle e^{-i\theta(t)}\rangle . \quad \text{(A26)}$$

Hence

$$\langle e^{-i\theta(t)}\rangle = e^{-i\theta(0)}e^{-Dt} , \quad \text{(A27)}$$

which is the result used in Eq. (17) in the paper, for the inhomogeneous term of that equation.

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[1]See, for a recent experiment, R. Gosh and L. Mandel, Phys. Rev. Lett. **59**, 1903 (1987), and references therein.
[2]C. M. Caves, Phys. Rev. D **23**, 1693 (1981); H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983); A. S. Lane, M. D. Reid, and D. F. Walls, Phys. Rev. Lett. **60**, 1940 (1988); B. Yurke, P. Grangier, and R. E. Slusher, J. Opt. Soc. Am. B **4**, 1677 (1987).
[3]See the special issues of J. Mod. Opt. **34** (6/7), (1987) and J. Opt. Soc. Am. B **4** (10), (1987).
[4]M. Xiao, L.-A. Wu, and H. J. Kimble, Phys. Rev. Lett. **41**, 278 (1987); A. Heidmann, R. J. Horowicz, S. Reynaud, E. Giacobino, C. Fabre, and G. Camy, Phys. Rev. Lett. **59**, 2555 (1987).
[5]L.-A. Wu, M. Xiao, and H. J. Kimble, J. Opt. Soc. Am. B **4**, 1465 (1987).
[6]K. Wódkiewicz and M. S. Zubairy, Phys. Rev. A **27**, 2003 (1983).
[7]M. Hillery and M. S. Zubairy, Phys. Rev. A **29**, 1275 (1984).
[8]C. M. Caves and D. D. Crouch, J. Opt. Soc. Am. B **4**, 1535 (1987).
[9]R. E. Slusher, P. Grangier, A. La Porta, B. Yurke, and M. J. Potasek, Phys. Rev. Lett. **59**, 2566 (1987).
[10]M. J. Collet and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).
[11]B. Yurke, Phys. Rev. A **32**, 300 (1985); H. J. Carmichael, J. Opt. Soc. Am. B **4**, 1588 (1987).
[12]G. Milburn and D. F. Walls, Opt. Commun. **39**, 410 (1981).
[13]B. Yurke, Phys. Rev. A **29**, 408 (1984).
[14]C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985).
[15]M. J. Collet, R. Loudon, and C. W. Gardiner, J. Mod. Opt. **34**, 881 (1987).
[16]J. Gea-Banacloche, N. Lu, L. M. Pedrotti, S. Prasad, M. O. Scully, and K. Wódkiewicz, Phys. Rev. A **41**, 369 (1990).
[17]R. F. Fox, J. Math. Phys. **13**, 1196 (1972).