Pseudo-spin-$\frac{1}{2}$ technique for multiphoton processes in a two-level atom

Chang J. Lee
Chemical Analysis Laboratory, Korea Institute of Science and Technology, Seoul 136-791, Korea
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We employ supersymmetry arguments to calculate the exact eigenvalue spectra of a general multimode, multiphoton Hamiltonian for a two-level atom. First, supercharges are built from both the atom and the photon fields. The exact diagonalization of the Hamiltonian is achieved by utilizing spinorial behavior of these supercharges.

I. INTRODUCTION

Quantum theory of radiation and the interaction of radiation with matter show many unusual nonclassical phenomena. The one-photon Jaynes-Cummings model[1,2] (JCM) has long been a standard model for treating many of such phenomena, and the popularity of this simplified model stems partly from the fact that it can be solved exactly. Among the most prominent of these quantum effects is the squeezing of the electromagnetic field, or more generally quantities satisfying su(n) and su(1,1) Lie algebras.[3–5] Squeezing is an example of an inherently multiphoton effect, and deeper squeezing has been predicted in models with multiphoton extension of the JCM.[6] Other multiphoton processes, such as the two-photon version[7] of superradiation, have also been discovered. In view of the role the JCM has played in one-photon processes, it should be valuable to investigate exactly solvable models for various multiphoton processes.

The technique of dressing has been successfully employed to obtain exact eigenvalue spectra for the JCM and certain models describing two-photon processes in a two-level atom[8] (TLA). Usually one speaks of an atom dressed by an electromagnetic field. However, it is useful to treat the atom (or more precisely, the electron) and the photons on an equal dynamical footing as elementary excitations of the corresponding fields. Then, both the atom and the photons are (further) dressed by the interaction. The excitation spectrum of the interacting many-body system can be obtained from the poles of the pertinent Green's function in the frequency domain. Alternatively, one may perform a direct canonical transformation from the original system to a system of weakly interacting (or, hopefully, independent) fictitious particles. This approach has the advantage of being more intuitively appealing. However, unlike the quantum-field theoric methods, where the calculations are performed systematically with diagrams, it is often difficult to find appropriate transformations. Thus, to date there seems to be no general procedure leading to exact canonical dressing for models describing higher-order nonlinear optical processes. Furthermore, the atom-photon system is an interacting boson-fermion system. Consequently, usual canonical transformations involving particles with the same spin statistics cannot be applied.

The purpose of this paper is to extend systematically the exact dressing of the one- and the two-photon Hamiltonians to a model Hamiltonian describing multimode, multiphoton processes in a TLA. As discussed above, the TLA-photon system is considered here as a system of interacting bose- and fermi-type excitations. Therefore, it is natural to employ supersymmetry arguments to treat such a system. In fact, several authors[9–12] have discussed various supersymmetry aspects of the JCM. Later in this paper it will be shown that superalgebra can be extended to the multiphoton problem and thereby the transformation from the original interacting system to that of completely decoupled elementary excitations can be performed systematically. The basic idea is to construct pseudo-spin-$\frac{1}{2}$ operators from the atomic and the electromagnetic field modes, and take advantage of the rotational properties of angular momentum. In Sec. II the Hamiltonian for a two-level system is given in terms of particle creation and annihilation operators. Section III discusses the free-field JCM in terms of supersymmetry quantum mechanics. Then the supersymmetry argument is extended to the exact eigenvalue problem of the JCM and the general multimode, multiphoton processes in a TLA.

II. ELECTRON AND PHOTON FIELDS AND THE TWO-LEVEL SYSTEM

A. Canonical quantization

Let $b_i^\dagger$ (b_i) and $c_j^\dagger$ (c_j) be creation (annihilation) operators for the photon mode i ($i=1,\ldots,k$) and the electron in the atomic state $|j\rangle$ ($j=1,2$), respectively. For a given time they satisfy the canonical quantization rules

$$\begin{aligned}&[b_{i'},b_i^\dagger]=\delta_{i'i},\quad [b_{i'},b_i]=[b_{i'}^\dagger,b_i^\dagger]=0\ ,\\ &\{c_{j'},c_j^\dagger\}=\delta_{j'j},\quad \{c_{j'},c_j\}=\{c_{j'}^\dagger,c_j^\dagger\}=0\ ,\\ &[b_i,c_j]=[b_i,c_j^\dagger]=[b_i^\dagger,c_j]=[b_i^\dagger,c_j^\dagger]=0\ .\end{aligned} \tag{1}$$

Next we introduce the transition operator[13] f and its Hermitian conjugate $f^\dagger$ defined by

$$f=c_1^\dagger c_2,\quad f^\dagger=c_2^\dagger c_1\ , \tag{2}$$

and impose the normalization condition

$$\sum_{j=1}^{2} c_j^\dagger c_j = 1 , \tag{3}$$

because the cases in which both levels are either empty or filled are not of interest here. Then using Eqs. (1)–(3), one can verify the commutation and anticommutation rules

$$\{f,f^\dagger\}=1, \quad \{f,f\}=\{f^\dagger,f^\dagger\}=0 ,$$
$$[b_i,f]=[b_i,f^\dagger]=[b_i^\dagger,f]=[b_i^\dagger,f^\dagger]=0 . \tag{4}$$

It follows that f ($f^\dagger$) can be regarded as fermion operators, and can be realized by 2×2 matrices

$$f=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \quad f^\dagger=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} . \tag{5}$$

B. The two-level system

The Hamiltonian for a two-level system with free electron and photon fields, after a Fourier transform, may be written as

$$H^0=\sum_{i=1}^{k} \omega_i b_i^\dagger b_i + \sum_{j=1}^{2} \varepsilon_j c_j^\dagger c_j . \tag{6}$$

In the above, ω_i and ε_j are energies of the ith mode of the electromagnetic field and the atomic state $|j\rangle$, respectively. The zero-point vibrations do not give rise to any physical significance, and hence are ignored. The zero of the atomic energy is chosen such that $\varepsilon_1+\varepsilon_2=0$, with a difference $\omega_0\equiv\varepsilon_2-\varepsilon_1>0$. Then $\varepsilon_1=-\frac{1}{2}\omega_0$ and $\varepsilon_2=\frac{1}{2}\omega_0$, and the atomic states may be represented by

$$|2\rangle \equiv \begin{bmatrix}1\\ 0\end{bmatrix} \equiv |+\rangle, \quad |1\rangle \equiv \begin{bmatrix}0\\ 1\end{bmatrix} \equiv |-\rangle . \tag{7}$$

Thus the product basis for the two-level system is $|n_1,\ldots,n_k;\pm\rangle \equiv |n_1\rangle\cdots|n_k\rangle\otimes|\pm\rangle$.

The number operator for the fermion operator f is

$$\mathbf{n}_f=f^\dagger f=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} , \tag{8a}$$

and for photons

$$\mathbf{n}_{bi}\equiv\mathbf{n}_i=b_i^\dagger b_i , \tag{8b}$$

where the subscript b is inserted to emphasize spin-statistics. These number operators satisfy the eigenvalue equations

$$\mathbf{n}_{bi}|n_i\rangle=n_i|n_i\rangle ,$$
$$\mathbf{n}_f|-\rangle=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=0\begin{bmatrix}0\\ 1\end{bmatrix} ,$$
$$\mathbf{n}_f|+\rangle=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=1\begin{bmatrix}1\\ 0\end{bmatrix} . \tag{9}$$

Thus we have the correspondence $|-\rangle\leftrightarrow|n_f=0\rangle$ and $|+\rangle\leftrightarrow|n_f=1\rangle$, and all subsequent discussions will be confined to the two-dimensional space spanned by either the basis set $\{|n_1,\ldots,n_k;\pm\rangle\}$ or the boson-fermion number states $\{|n_{b1},\ldots,n_{bk};n_f\rangle\}$.

III. EXACT EIGENVALUE SPECTRA OF m-PHOTON HAMILTONIAN

A. Model Hamiltonians

The Hamiltonian considered here for multimode, multiphoton (m) processes is the sum of H^0 given by Eq. (6) and the interaction between the atom and photons of the form

$$V=\frac{1}{2}\left[\Lambda\prod_{i=1}^{k} b_i^{\nu_i\zeta_i} c_1^\dagger c_2+\Lambda^*\prod_{i=1}^{k} b_i^{-\nu_i\zeta_i} c_2^\dagger c_1\right] , \tag{10}$$

where $\zeta_i=(+)$ or $(-)$, $b_i^+\equiv b_i^\dagger$ and $b_i^-\equiv b_i$, $\sum_{i=1}^{k}\nu_i|\zeta_i|=m$ (ν_i is a positive integer), and Λ is the c-number coupling parameter. After some algebra the Hamiltonian reduces to

$$H^0=\omega_0 f^\dagger f+\sum_{i=1}^{k}\omega_i b_i^\dagger b_i ,$$
$$V=\frac{1}{2}\left[\Lambda\prod_{i=1}^{k} b_i^{\nu_i\zeta_i} f+\Lambda^*\prod_{i=1}^{k} b_i^{-\nu_i\zeta_i} f^\dagger\right]-\frac{\omega_0}{2} , \tag{11}$$

where $\frac{1}{2}\omega_0$ has been included in V for convenience.

The Hamiltonian for the JCM is obtained with $m=k=\nu\zeta=+1$, and λ real,

$$H^0=\omega_0 f^\dagger f+\omega b^\dagger b, \quad V=\frac{\lambda}{2}(f^\dagger b+b^\dagger f)-\frac{\omega_0}{2} . \tag{12}$$

Likewise, three special cases for two-photon processes[7] ($m=2$) are

$$\text{(a)}\ k=1,\ \ \nu=2,\ \ \zeta=+,\ \ H_a=\omega_0 f^\dagger f+\omega b^\dagger b+\frac{\lambda}{2}(b^2 f^\dagger+b^{\dagger 2} f)-\frac{\omega_0}{2} ,$$

$$\text{(b)}\ k=2,\ \ \nu_i=1,\ \ \zeta_i=+,\ \ H_b=\omega_0 f^\dagger f+\sum_i\omega_i b_i^\dagger b_i+\frac{\lambda}{2}(b_1 b_2 f^\dagger+b_1^\dagger b_2^\dagger f)-\frac{\omega_0}{2} ,$$

$$\text{(c)}\ k=2,\ \ \nu_i=1,\ \ \zeta_1=-,\ \ \zeta_2=+,\ \ H_c=\omega_0 f^\dagger f+\sum_i\omega_i b_i^\dagger b_i+\frac{\lambda}{2}(b_1^\dagger b_2 f^\dagger+b_1 b_2^\dagger f)-\frac{\omega_0}{2} ,$$

while Eq. (11) reduces to the Hamiltonian of Sukumar and Buck[14] when $k=1$, $\nu=m$, and $\zeta=+$.

B. JCM and supersymmetry

Before we explore the eigenvalue problem of the general Hamiltonian given by Eq. (11), it is instructive to consider the simplest model, Eq. (12). The transition operators and number operators defined by Eqs. (2) and (8), respectively, are symmetry generators in the sense of Haag *et al.*[15] They are of even symmetry, in that they connect particles with the same spin-statistics. In addition, Eq. (12) contains symmetry generators Q and $Q^\dagger$ containing an equal number of bosonic and fermionic degrees of freedom (one for each),

$$Q^\dagger = f^\dagger b, \quad Q = b^\dagger f \ . \tag{13}$$

Obviously, the effect of $Q^\dagger$ and Q is to transform a boson into a fermion, and vice versa. Thus the state $|n_b, n_f\rangle$ transforms under Q and $Q^\dagger$ as

$$Q|n_b,n_f\rangle \propto |n_b+1, n_f-1\rangle \ ,$$

$$Q^\dagger|n_b,n_f\rangle \propto |n_b-1, n_f+1\rangle \ .$$

Consequently, the basis set for the JCM $\{|n+1;-\rangle, |n;+\rangle\}$ transforms under Q and $Q^\dagger$ as

$$Q|n,+\rangle \propto |n+1,-\rangle \ ,$$

$$Q^\dagger|n+1,-\rangle \propto |n,+\rangle \ .$$

The statistics-changing (and hence odd) generators of symmetry Q and $Q^\dagger$ are generators of *supersymmetry*.

A supersymmetry Hamiltonian is given by the anticommutator between supersymmetry generators Q and $Q^\dagger$ as

$$H_{SS} = \{Q^\dagger, Q\}\omega \ , \tag{14}$$

and it satisfies the commutation rule

$$[Q, H_{SS}] = [Q^\dagger, H_{SS}] = 0 \ . \tag{15}$$

Equations (14) and (15) constitute supersymmetry algebra. The Hamiltonian for the supersymmetry generator Q ($Q^\dagger$) given by Eq. (13) is then

$$H_{SS} = \omega\{f^\dagger b, b^\dagger f\} = \omega(b^\dagger b + f^\dagger f) \ , \tag{16}$$

which is identical to the free-field JCM Hamiltonian at resonance.

In the above supersymmetry quantum-mechanical model there are two kinds of excitations: fermionic and bosonic excitations. The total excitation number operator[16] is the sum of each excitation number operator:

$$\mathbf{N} = \mathbf{n}_b + \mathbf{n}_f = b^\dagger b + f^\dagger f \ . \tag{17}$$

From Eq. (16) we have

$$H_{SS} = \mathbf{N}\omega \ , \tag{18}$$

so $\mathbf{N}$ commutes with H_{SS} and the eigenvalue of this conserved quantity is

$$\langle n_b, n_f|\mathbf{N}|n_b, n_f\rangle = \langle n+1, 0|\mathbf{N}|n+1, 0\rangle = \langle n, 1|\mathbf{N}|n, 1\rangle = n+1 \ , \tag{19}$$

or

$$N = n_b + n_f = n+1 \ . \tag{20}$$

C. The construction of pseudo-spin-$\frac{1}{2}$ operators

The invariance of Q is guaranteed by Eq. (15) and Q is called the supercharge. It is well known that in a two-dimensional isotropic oscillator the angular momentum operator defined by $L = -i(b_1^\dagger b_2 - b_2^\dagger b_1)$ serves as a charge.[17] By analogy, one may wish to relate the supercharge with angular momentum. For the isotropic two-dimensional harmonic oscillator the angular momentum is given in terms of two bosonic modes. However, the supercharge connects a bosonic mode with a fermionic mode, and changes the total spin of a state by $\frac{1}{2}$. Therefore, the supercharge must be spinorial. Indeed, the normalized operators q and $q^\dagger$ as defined by

$$q = (n+1)^{-1/2} Q, \quad q^\dagger = (n+1)^{-1/2} Q^\dagger \ , \tag{21}$$

are quantized according to Fermi-Dirac statistics,

$$\{q, q^\dagger\} = 1, \quad \{q, q\} = \{q^\dagger, q^\dagger\} = 0 \ . \tag{22}$$

Therefore, in the space spanned by the basis set $\{|n,+\rangle, |n+1,-\rangle\}$, $q^\dagger$ and q may be considered as the raising and the lowering operators $l_\pm = l_1 \pm i l_2$ of a pseudo-spin-$\frac{1}{2}$.[18]

The supercharges given by Eq. (13) are in the simplest form. They carry only the interaction between a boson and a fermion. However, more general supercharges can be defined by allowing interactions among bosons. These supercharges may be written as[19]

$$Q = B^\dagger f, \quad Q^\dagger = B f^\dagger \ , \tag{23}$$

where B ($B^\dagger$) is an arbitrary function of boson operators. Here we are concerned with the boson-boson interaction of the form

$$B^\dagger = \prod_{i=1}^{k} b_i^{\nu_i \zeta_i}, \quad B = \prod_{i=1}^{k} b_i^{-\nu_i \zeta_i} \ . \tag{24}$$

Then the corresponding supercharges are given by

$$Q = \prod_{i=1}^{k} b_i^{\nu_i \zeta_i} f, \quad Q^\dagger = \prod_{i=1}^{k} b_i^{-\nu_i \zeta_i} f^\dagger \ , \tag{25}$$

and they satisfy

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \ . \tag{26}$$

Let us now define a new Hermitian operator $\mathbf{N}$ as

$$\mathbf{N} = \{Q, Q^\dagger\} \ . \tag{27}$$

Appendix A gives an explicit expression of $\mathbf{N}$ as a function of $\{\mathbf{n}_{bi}\}$ and $\mathbf{n}_f$. Since H^0 is a function of these number operators, $[\mathbf{N}, H^0] = 0$. V can also be expressed in terms of the supercharges as

$$V=\frac{1}{2}(\Lambda Q+\Lambda^* Q^\dagger)-\frac{\omega_0}{2} \ . \tag{28}$$

Because $[Q,\mathbf{N}]=[Q^\dagger,\mathbf{N}]=0$, the commutation between V and $\mathbf{N}$ reads as

$$[V,\mathbf{N}]=\frac{1}{2}[\Lambda Q+\Lambda^* Q^\dagger-\omega_0,\mathbf{N}]=0 \ . \tag{29}$$

Hence, $\mathbf{N}$ commutes with H and is a conserved quantity. In fact, Eq. (17) is a special case of Eq. (27), with Q and $Q^\dagger$ given by Eq. (13). Similar to Eq. (21) one may define normalized operators q and $q^\dagger$ as

$$q=N^{-1/2}Q, \quad q^\dagger=N^{-1/2}Q^\dagger \ , \tag{30}$$

where N is the eigenvalue of $\mathbf{N}$. They also satisfy Eq. (22) and pseudo-spin-$\frac{1}{2}$ operators may be defined as

$$\begin{aligned} l_1&=(2|\Lambda|)^{-1}(\Lambda^* q^\dagger+\Lambda q) \ ,\\ l_2&=(2i|\Lambda|)^{-1}(\Lambda^* q^\dagger-\Lambda q), \quad i=\sqrt{-1}\\ l_3&=f^\dagger f-\tfrac{1}{2} \ . \end{aligned} \tag{31}$$

The derivation of l_3 and commutation relations with l_1 and l_2 are given in Appendix B.

D. The eigenvalue spectrum of the m-photon Hamiltonian

In the interaction picture defined by the transformation

$$U=\exp(-iH_0 t) \tag{32}$$

with

$$H_0=\sum_{i=1}^{k}\omega_i(b_i^\dagger b_i+\nu_i\zeta_i f^\dagger f) \ , \tag{33}$$

the m-photon Hamiltonian reduces to

$$\begin{aligned} H_I&=\frac{\Delta\omega}{2}[f^\dagger,f]+\frac{\sqrt{N}}{2}(\Lambda^* q^\dagger+\Lambda q)-\frac{\Omega}{2}\\ &=\Delta\omega l_3+|\Lambda|\sqrt{N}l_1-\frac{\Omega}{2} \ . \end{aligned} \tag{34}$$

In the above, $\Delta\omega$ $(=\omega_0-\Omega)$ denotes detuning, with $\Omega=\sum_i \nu_i\zeta_i\omega_i$. From the rotational properties of angular momentum, it immediately follows that Eq. (34) is diagonal in a frame defined by the transformation

$$R(\theta)=\exp(-2i\theta l_2) \ , \tag{35}$$

where $\tan 2\theta=(\Delta\omega)^{-1}|\Lambda|\sqrt{N}$, and the eigenvalue equation becomes

$$\tilde{H}_I|\tilde{m}\rangle\equiv\left[\omega_e\tilde{l}_3-\frac{\Omega}{2}\right]|\tilde{m}\rangle=\left[\tilde{m}\omega_e-\frac{\Omega}{2}\right]|\tilde{m}\rangle \ , \quad \tilde{m}=\pm\tfrac{1}{2} \ , \tag{36}$$

with $\omega_e=(\Delta\omega^2+|\Lambda|^2 N)^{-1/2}$ and the transformed quantity defined by $\tilde{A}\equiv R^\dagger A R$. Note that if we rewrite $\tilde{H}_I$ as

$$\tilde{H}_I=\omega_e\tilde{f}^\dagger\tilde{f}+E_0 \ , \tag{37}$$

with the ground-state energy $E_0=-\frac{1}{2}(\Omega+\omega_e)$, it may be regarded as the Hamiltonian for noninteracting elementary excitations ($\tilde{f}^\dagger\tilde{f}=\tilde{l}_3+\frac{1}{2}=0$ or 1, in number) with the excitation spectrum given by ω_e. The eigenstates of $\tilde{H}_I$ are

$$\begin{aligned} |\tilde{m}_+\rangle&=R(\theta)|n_1,\ldots,n_k;+\rangle\\ &=\cos\theta|n_1,\ldots,n_k;+\rangle\\ &\quad+(\Lambda/|\Lambda|)\sin\theta|n_1',\ldots,n_k';-\rangle \ ,\\ |\tilde{m}_-\rangle&=R(\theta)|n_1',\ldots,n_k';-\rangle\\ &=\cos\theta|n_1',\ldots,n_k';-\rangle\\ &\quad-(\Lambda^*/|\Lambda|)\sin\theta|n_1,\ldots,n_k;+\rangle \ , \end{aligned} \tag{38}$$

where n_i and n_i' are related by $n_i=n_i'-\nu_i\zeta_i$. The derivation of Eq. (38) is given in Appendix C.

Next we show commutation relations between H_0 and l_j $(j=1,2,3)$ given by Eq. (31). It is trivial to show that $[l_3,H_0]=0$, so we concentrate on other commutations. For a given mode (suppressing the index for simplicity) we have

$$[b^\dagger b,b^{\nu\zeta}]=\nu\zeta b^{\nu\zeta} \ . \tag{39}$$

Using (B7) and Eq. (39) with the notation following Eq. (10), one can readily show that

$$\left[b_j^+b_j^-+\nu_j\zeta_j f^+f^-,\ \prod_{i=1}^{k}b_i^{\pm\nu_i\zeta_i}f^{\mp}\right]=0 \ . \tag{40}$$

Hence it follows that

$$[H_0,l_j]=0 \quad (j=1,2,3) \ , \tag{41}$$

and consequently

$$[H_0,R(\theta)]=0 \ . \tag{42}$$

Therefore, H_0 is invariant under the transformation $R(\theta)$. For the JCM, H_0 is equal to H_{SS} and Eq. (42) follows directly from the fact that a supersymmetric Hamiltonian is invariant under a supergauge transformation. In any event, the eigenstates $|\tilde{m}_\pm\rangle$ $(\tilde{m}_\pm=\pm\frac{1}{2})$ of $\tilde{H}_I$ must also be the eigenstates of H_0, and the eigenvalue of the total Hamiltonian H is

$$E=\sum_{i=1}^{k}n_i'\omega_i+\tilde{m}\omega_e-\frac{\Omega}{2} \ . \tag{43}$$

For the JCM, $k=1$, $\zeta\nu=+1$, $n'=n+1$, and $\Omega=\omega$. Thus

$$E_{\mathrm{JCM}}=(n+\tfrac{1}{2})\omega\pm\tfrac{1}{2}\omega_e \ , \tag{44}$$

which is identical to the result in the literature,[20] and the solution of the eigenvalue problem for the two-photon Hamiltonians H_a, H_b, and H_c obtained by the dressed-atom technique and the single-mode, m-photon Hamiltonian[6,14] can also be readily recovered.

IV. CONCLUSIONS

In a graded vector space, both bosons and fermions are treated on an equal footing. In this paper the electrons and photons were regarded as excitations in such a space, and there resulted a unified view for arbitrary multimode, m-photon processes ($m=1,2,\ldots$) of the type $B^\dagger f$ and $Bf^\dagger$ as generalized supercharges. From this point of view a systematic method for the exact diagonalization of such arbitrary m-photon Hamiltonians was developed. The procedure is as follows: firstly, construct pseudo-spin-$\frac{1}{2}$ operators from the spinorial behavior of the supercharges. Secondly, express the Hamiltonian in terms of these pseudo-spin-$\frac{1}{2}$ operators, and then transform the coupled Hamiltonian into that of noninteracting elementary excitations utilizing the algebra of angular momentum. The results for some simple cases treated in the literature are readily recovered.

The supersymmetry arguments were possible due to the existence of the fermion operators f and $f^\dagger$ satisfying Eq. (4). Given as in Eq. (5), they are defining representations of the group SU(2). For M-level atoms ($M>2$) the transition operators $f_{\mu\nu}=c_\mu^\dagger c_\nu$ for the entire atomic manifold satisfy su(M) Lie algebra (with the understanding that the trace vanishes). Thus the Lie superalgebra we employed can no longer be exploited. Nonetheless, because of the group chain structure $\mathrm{SU}(M)\supset\mathrm{SU}(M-1)\cdots\supset\mathrm{SU}(2)$, if we focus on the two states $|\alpha\rangle$ and $|\beta\rangle$ connected by transitions, there exist ($M\times M$) representations for $f_{\alpha\beta}$ that satisfy SU(2) algebra. For each of such subspaces the supersymmetry arguments are still tenable.

The realization of the SU(2) algebra in Sec. III in terms of boson-fermion modes parallels that of the quasi-spin formalism[21] in nuclear physics with two fermions [Eqs. (2)–(5)] and that of Schwinger[22] with two bosons. It has been shown by Buzano *et al.*[11] that the JCM Hamiltonian can be written as $H=a^\mu X_\mu$, where X_μ are the generators of the superunitary group U(1/1). They achieved the diagonalization of H by a superspace rotation with an exponential adjoint representation along with an odd gradation of Λ. Without the grading the rotation is closely related to the rotation through some angle ϕ, $R_1(\phi)=\exp(-2i\phi l_1)$. For the multiphoton Hamiltonian, however, the supergroup structure is not manifest. Still, the interaction representation Hamiltonian [Eq. (34)] is an element of what we may call the pseudo-spin group $\mathrm{SU}^{\mathrm{PS}}(2)$, which underlies the method of diagonalization in this paper. Finally, if one imposes certain restrictions, the Hamiltonian for one-photon processes in a multilevel atom can be solved exactly by use of the resolvent operator method.[23] Multiphoton extension of this latter approach and squeezing for the multimode, multiphoton Hamiltonian treated in this paper are currently under investigation.

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APPENDIX A

Let us first suppose that the electromagnetic field has only one mode. From the identity

$$b^{\dagger\nu}b^{\nu}=b^\dagger b(b^\dagger b-1)\cdots(b^\dagger b-\nu+1)\,,\qquad b^{\nu}b^{\dagger\nu}=(b^\dagger b+1)(b^\dagger b+2)\cdots(b^\dagger b+\nu)\,, \tag{A1}$$

the function $g(\mathbf{n}_b)$ defined by

$$g(\mathbf{n}_b)=b^{\nu\zeta}b^{-\nu\zeta},\quad \zeta=(+)\text{ or }(-) \tag{A2}$$

can be expressed in terms of $\mathbf{n}_b$ as

$$g(\mathbf{n}_b)=\prod_{j=1}^{\nu}\left[\mathbf{n}_b+\frac{1-\zeta(2j-1)}{2}\right]. \tag{A3}$$

Therefore, for k different modes in the electromagnetic field we have

$$g(\{\mathbf{n}_{bi}\})=\prod_{i=1}^{k}b_i^{\nu_i\zeta_i}b_i^{-\nu_i\zeta_i}=\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}\left[\mathbf{n}_{bi}+\frac{1-\zeta_i(2j-1)}{2}\right]. \tag{A4}$$

The Hermitian operator $\mathbf{N}$ defined by Eq. (27) is then

$$\mathbf{N}(\{\mathbf{n}_{bi}\},\mathbf{n}_f)=\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}\left[\mathbf{n}_{bi}+\frac{1-\zeta_i(2j-1)}{2}\right](1-\mathbf{n}_f)+\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}\left[\mathbf{n}_{bi}+\frac{1+\zeta_i(2j-1)}{2}\right]\mathbf{n}_f\,. \tag{A5}$$

APPENDIX B

In the same vein as the discussion following Eq. (22) one may define the raising and the lowering operators of a pseudo-spin-$\frac{1}{2}$ as

$$l_+=(\Lambda^*/|\Lambda|)q^\dagger,\quad l_-=(\Lambda/|\Lambda|)q\,. \tag{B1}$$

The third component may then be obtained from

$$l_3=\tfrac{1}{2}[l_+,l_-]=\frac{1}{2N}(BB^\dagger f^\dagger f-B^\dagger Bff^\dagger)\,, \tag{B2}$$

where B ($B^\dagger$) are given by Eq. (24). From the idempotency $(f^\dagger f)^2=f^\dagger f$, the following identities result:

$$f^\dagger f=2(f^\dagger f-\tfrac{1}{2})f^\dagger f,\quad ff^\dagger=-2(f^\dagger f-\tfrac{1}{2})ff^\dagger\,. \tag{B3}$$

Therefore, (B2) becomes

$$l_3=\frac{1}{N}(BB^\dagger f^\dagger f+B^\dagger Bff^\dagger)(f^\dagger f-\tfrac{1}{2})\,. \tag{B4}$$

The quantity in the first parenthesis is none but $\mathbf{N}$. It does not commute with b ($b^\dagger$), B ($B^\dagger$), or f ($f^\dagger$). However, these individual operators never appear in the Hamiltonians in Sec. III, and all the terms in the Hamiltonians commute with $\mathbf{N}$. Therefore, $\mathbf{N}/N$ may be regarded as an identity operator. (Incidentally, $\mathbf{N}$ is a Casimir operator in the JCM, which according to Schur's lemma is a multiple of the unit operator.) Thus (B4) reduces to

$$l_3=f^\dagger f-\tfrac{1}{2}\,. \tag{B5}$$

Commutation rules with l_1 and l_2 are easily verified:

$$[l_2,l_3]=(2i|\Lambda|\sqrt{N})^{-1}[\Lambda^* Bf^\dagger-\Lambda B^\dagger f, f^\dagger f-\tfrac{1}{2}]$$
$$=i(2|\Lambda|\sqrt{N})^{-1}(\Lambda^* Bf^\dagger+\Lambda B^\dagger f)=il_1 ,$$
$$[l_1,l_3]=(2|\Lambda|\sqrt{N})^{-1}[\Lambda^* Bf^\dagger+\Lambda B^\dagger f, f^\dagger f-\tfrac{1}{2}]$$
$$=-(2|\Lambda|\sqrt{N})^{-1}(\Lambda^* Bf^\dagger-\Lambda B^\dagger f)=-il_2 . \quad \text{(B6)}$$

In the above

$$[f^+f^-,f^\pm]=\pm f^\pm \quad \text{(B7)}$$

with the notation following Eq. (10) is used.

APPENDIX C

Suppose the supercharges given by either Eq. (25) or Eq. (30) connect the states $|\{n_i\},+\rangle$ and $|\{n_i'\},-\rangle$. Then the following equations must hold:

$$q^\dagger|n_1,\ldots,n_k;+\rangle=q|n_1',\ldots,n_k';-\rangle=0 ,$$
$$q|n_1,\ldots,n_k;+\rangle=|n_1',\ldots,n_k';-\rangle ; \quad \text{(C1)}$$
$$q^\dagger|n_1',\ldots,n_k';-\rangle=|n_1,\ldots,n_k;+\rangle .$$

The relation between the photon occupation numbers n_i and n_i' is

$$n_i=n_i'-\nu_i\zeta_i , \quad \text{(C2)}$$

as can be obtained, for example, from

$$q^\dagger|n_1',\ldots,n_k';-\rangle=N^{-1/2}\prod_{i=1}^{k} b_i^{-\nu_i\zeta_i}f^\dagger|n_i'\rangle|0\rangle$$
$$=\prod_{i=1}^{k}|n_i'-\nu_i\zeta_i\rangle|1\rangle$$
$$=|n_1,\ldots,n_k;+\rangle . \quad \text{(C3)}$$

The states $\{|n_1,\ldots,n_k;1\rangle,|n_1',\ldots,n_k';0\rangle\}$ are also eigenstates of $\mathbf{N}$:

$$\mathbf{N}|n_1,\ldots,n_k;1\rangle=\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}\left[n_i+\frac{1+\zeta_i(2j-1)}{2}\right]|n_1,\ldots,n_k;1\rangle=N_+|n_1,\ldots,n_k;1\rangle ,$$
$$\mathbf{N}|n_1',\ldots,n_k';0\rangle=\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}\left[n_i'+\frac{1-\zeta_i(2j-1)}{2}\right]|n_1',\ldots,n_k';0\rangle=N_-|n_1',\ldots,n_k';0\rangle . \quad \text{(C4)}$$

In view of Eq. (18) the two states are superpartner states, and thus must be degenerate. Indeed, if we substitute (C2) into (C4), we have $N_+=N_-$.

In obtaining the state $|\tilde{m}\rangle$, it is convenient to rewrite Eq. (35) as

$$R(\theta)=\mathbb{1}\cos\theta-|\Lambda|^{-1}(\Lambda^* q^\dagger-\Lambda q)\sin\theta . \quad \text{(C5)}$$

Then from (C1) it follows that

$$R(\theta)|n_1,\ldots,n_k;+\rangle=|n_1,\ldots,n_k;+\rangle\cos\theta+(\Lambda/|\Lambda|)q|n_1,\ldots,n_k;+\rangle\sin\theta$$
$$=|n_1,\ldots,n_k;+\rangle\cos\theta+(\Lambda/|\Lambda|)|n_1',\ldots,n_k';-\rangle\sin\theta ,$$
$$R(\theta)|n_1',\ldots,n_k';-\rangle=|n_1',\ldots,n_k';-\rangle\cos\theta-(\Lambda^*/|\Lambda|)q^\dagger|n_1',\ldots,n_k';-\rangle\sin\theta$$
$$=|n_1',\ldots,n_k';-\rangle\cos\theta-(\Lambda^*/|\Lambda|)|n_1,\ldots,n_k;+\rangle\sin\theta . \quad \text{(C6)}$$

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