Quantum-noise suppression in lasers via memory-correlation effects

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We discuss the influence of atomic memory effects on the field fluctuations of a laser in a general context. We derive a Fokker-Planck equation for the field that takes into account the long lifetime of the lasing atoms. Using this equation, we discuss two consequences of atomic memory. First, we find that even in the presence of saturation, memory effects can lead to a reduction of spontaneous-emission noise for short measurement times. Second, we quite generally show that atomic memory effects lead to time-dependent diffusion coefficients.

I. INTRODUCTION

The search for new ways of quantum-noise reduction in lasers has been of particular interest over the last few years. It has been demonstrated, both theoretically and experimentally, that a suppression of fluctuations in the field beyond the standard limits is possible. Examples include the pump-noise-suppressed laser,[1,2] the correlated-spontaneous-emission laser (CEL),[3] the two-photon CEL,[4] and the polarization CEL.[5] Recently, a different scheme for the reduction of spontaneous emission noise has been proposed[6] in which atomic memory effects play a significant role. In this type of laser, atoms with long lifetimes are used as the lasing material. It was found that for short measurement times it is possible to reduce the spontaneous-emission noise below the limit of ordinary lasers. This can be intuitively understood in the following way. In an ordinary laser the decay of an atom due to spontaneous emission happens on a far shorter time scale than the overall evolution of the field. Therefore one can take a spontaneous-emission event as an instantaneous event, which has a δ-like impact on the electromagnetic field. These random "kicks" gradually lead to a diffusion of the phase of the radiation field over the whole angle of 2π. If, however, the atomic lifetime is comparable to the time over which the field is measured, the approximation of δ-like impulses of spontaneous-emission events no longer holds. Instead, one has to account for the fact that an excited atom can only partially contribute to spontaneous-emission noise because it did not have enough time to decay. A detailed analysis showed[6] that the diffusion of the phase in a laser with long-lived atoms is given by

$$\langle \varphi^2(t) \rangle = \frac{\alpha}{2\bar{n}} \left[t + \frac{1}{\Gamma}(e^{-\Gamma t} - 1) \right] . \tag{1}$$

Here α is the linear gain coefficient of the laser and $\bar{n}$ the mean number of photons inside the cavity. From Eq. (1) we see that for times which are long compared to the atomic lifetime Γ^{-1} the phase variance grows linearly with time:

$$\langle \varphi^2(t) \rangle = \frac{\alpha}{2\bar{n}} t \quad (\text{for } t \gg \Gamma^{-1}) . \tag{2}$$

This is the famous Schawlow-Townes result[7] for the phase diffusion in a laser. However, for times which are of the order of Γ^{-1} or even shorter, the uncertainty of the phase is smaller than the one given by Eq. (2). Expanding the exponential in Eq. (1) we find

$$\langle \varphi^2(t) \rangle = \frac{\alpha}{2\bar{n}} \frac{1}{2} \Gamma t^2 \quad (\text{for } t \leq \Gamma^{-1}) . \tag{3}$$

Thus the spontaneous-emission noise is reduced below the usual Schawlow-Townes limit due to atomic memory effects. However, the result in Eq. (1) is based on an analysis which is linear in the electromagnetic field. One could now raise the question of whether the saturation terms in a nonlinear analysis diminish or even eliminate the atomic memory effect. The importance of this question becomes particularly clear on recalling the nonlinear result for the phase diffusion in an ordinary laser (i.e., one with no memory effects). As is well known,[8] the result in Eq. (2) is modified to

$$\langle \varphi^2(t) \rangle = \frac{\alpha + \gamma}{4\bar{n}} t , \tag{4}$$

in which γ is the cavity damping rate of the radiation field. For lasers around threshold the parameters α and γ are nearly identical so that there is no quantitative difference between the two results in that operation regime. In contrast, the physical interpretation of the results could be very different. If one would interprete the phase noise in Eq. (4) to originate in equal parts from spontaneous emission and from the damping of the cavity, the atomic memory effect would be significantly reduced. While the spontaneous-emission noise is directly

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affected by a long lifetime of the lasing atoms, the cavity damping of the electromagnetic field is completely independent of it. Therefore one could argue that atomic memory effects could at most reduce the phase diffusion by a factor of 2. On the other hand, Eq. (2) suggests that the phase diffusion of the field is completely due to spontaneous emission. Thus the effect of atomic memory could make the diffusion coefficient arbitrarily small in the limit of short measurement times. We will show in this paper that the second interpretation is the correct one. We derive a master equation for the reduced density operator of the field which accounts for both, nonlinear saturation terms and atomic memory effects. The master equation is then converted into a Fokker-Planck equation for the quasiprobability distribution of the radiation field. This allows us to directly discuss phase diffusion in a laser with atomic memory. We demonstrate that the diffusion of the phase is indeed completely governed by the interaction between field and atoms. Thus atomic memory effects can make the phase diffusion arbitrarily small for short measurement times. This is the content of Sec. II of this paper.

An additional feature emerging from our analysis of atomic memory effects in Sec. II is that, in contrast with the usual, no-memory effects, situation, the diffusion coefficients in the Fokker-Planck equation turn out to be time dependent. To understand the origin of this time dependence, we look in Sec. III at a simple but physically relevant case where the Langevin equation for the phase is governed by a damped-driven equation with colored noise. Under the assumption of Gaussian phase fluctuation, we obtain a Fokker-Planck equation whose coefficients turn out to be time dependent. We next show that this result is not specific to our example but is indeed a general feature. Thus the origin of time dependence of the diffusion coefficient lies in the colored noise leading us to conclude that memory effects, among other features, will necessarily result in a time-dependent diffusion.

II. ATOMIC MEMORY EFFECTS IN NONLINEAR ORDER

In this section we discuss atomic memory effects in the scope of a nonlinear laser theory. Our laser model consists of three-level atoms which are injected into a laser cavity, as depicted in Fig. 1. The upper two atomic levels constitute the lasing transition which interacts with one mode of the radiation. The lowest atomic level is an inert ground state to which an atomic excitation decays with a rate Γ. Before the atoms are injected into the cavity they are initially prepared in their upper excited state through some excitation mechanism. The Hamiltonian for such system can be written as

$$H=\hbar\omega a^{\dagger}a+\sum_j(\varepsilon_a|a\rangle\langle a|+\varepsilon_b|b\rangle\langle b|+\varepsilon_c|c\rangle\langle c|)_j +\hbar g\sum_j\Theta(t-t_j)V_j\ , \tag{5}$$

with

$$V_j=a^{\dagger}\sigma^j+\sigma^{j\dagger}a\ . \tag{6}$$

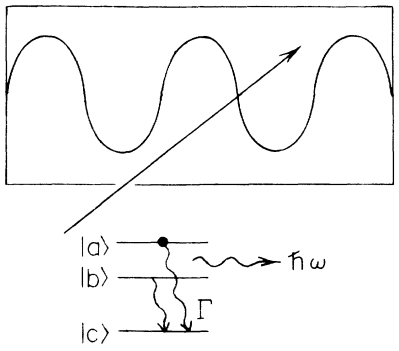

FIG. 1. Laser model. Three-level atoms are initially prepared in the upper atomic level a and are injected into the laser cavity. The two atomic levels a and b constitute the lasing transition while the lower level c is an inert ground state to which an atomic excitation decays.

Here a and $a^{\dagger}$ are the usual creation and annihilation operators for the electromagnetic field while σ^j is the atomic polarization operator $(|a\rangle\langle b|)_j$ for the jth atom. The parameter g specifies the amount of coupling between atoms and field and $\Theta(t)$ is the step function. The evolution of our physical system is then given by the equation of motion for the density operator:

$$\dot{\rho}=\frac{1}{i\hbar}[H,\rho]\ . \tag{7}$$

It is convenient to change from the Schrödinger picture to an interaction picture.[9] We define the new density operator

$$\rho_I(t)=\exp(iH_0t/\hbar)\rho(t)\exp(-iH_0t/\hbar)\ . \tag{8}$$

Here H_0 is the Hamiltonian for the field and atoms without an interaction and is given by the first two terms in Eq. (5). For simplicity we will assume resonance between the electromagnetic field mode of the cavity and the lasing atomic transition, i.e., $\omega=(\varepsilon_a-\varepsilon_b)/\hbar$. Then the equation of motion for the density operator ρ_I becomes particularly simple

$$\dot{\rho}_I(t)=-ig\sum_j\Theta(t-t_j)[V_{Ij},\rho_I(t)]\ . \tag{9}$$

In the following discussion we will drop the index I from the density matrix and the interaction matrix, keeping in mind that all the quantities are specified in the interaction picture. We can now obtain an equation of motion for the reduced density matrix for the field ρ^f by tracing Eq. (9) over all atoms. The result is

$$\dot{\rho}^f(t)=-ig\sum_j\Theta(t-t_j)\mathrm{Tr}_A[V_j,\rho_j^f(t)]+\mathcal{L}\rho^f\ , \tag{10}$$

in which ρ_j^f is the reduced density operator, in which all atoms apart from the jth atom have been traced out. We have also added a term $\mathcal{L}\rho^f$ which accounts for the damping of the field through cavity losses. This term is derived in the standard way by coupling the radiation field to a heat bath and by tracing over the variables of

the reservoir. If the reservoir is assumed to be at zero temperatures, the explicit form of the loss contribution is given by[8]

$$(\dot{\rho}^f)_{\text{loss}}=\mathcal{L}\rho^f=-\frac{\gamma}{2}(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) . \quad (11)$$

The parameter γ is the damping rate of the electromagnetic field inside the cavity.

We next derive an equation of motion for the reduced density operator $\rho_j^f(t)$. This equation has to take into account the decay of an atomic excitation to the inert ground state. Such decay can again be achieved by coupling the atoms to a heat reservoir.[10] We then obtain the following equation of motion for the reduced density operator $\rho_j^f(t)$:

$$\dot{\rho}_j^f(t)=-ig\Theta(t-t_j)[V_j,\rho_j^f(t)] -\frac{\Gamma}{2}(|a\rangle\langle a|\rho_j^f+\rho_j^f|a\rangle\langle a|+|b\rangle\langle b|\rho_j^f+\rho_j^f|b\rangle\langle b|-2|c\rangle\langle a|\rho_j^f|a\rangle\langle c|-2|c\rangle\langle b|\rho_j^f|b\rangle\langle c|)_j . \quad (12)$$

The first term on the right-hand side of Eq. (12) arises from the coupling between the jth atom and the field. The remaining terms are the standard[10] decay terms, which are due to the interaction of the atom with a heat bath. The parameter Γ is the atomic decay rate which we assume to be the same for the two atomic levels a and b.

Two approximations have been made to obtain Eq. (12). First, we have assumed that the atomic decay rate Γ is much larger than the cavity damping rate γ, as is typically the case. We note that such an assumption does not rule out the possibility of measuring memory-correlation effects in a time shorter than the atomic lifetime. We can then neglect the influence of cavity losses during the interaction of the field with a single atom. Therefore the loss contribution from Eq. (11) has been omitted in the Eq. (12). Second, we have assumed that the evolution of an atom is independent of all other atoms, thus neglecting atom-atom correlation effects. Such effects have been shown[11] to be very small in ordinary laser operation and become important only if the laser is operated at very high intensities. For our application it is therefore an excellent approximation to neglect the influence of other atoms in the equation of motion for ρ_j^f. We can simplify this equation even further. As is clear from Eq. (10), only matrix elements of the density matrix ρ_j^f, which involve the upper atomic levels a and b, are necessary to determine the evolution of the field. This is due to the form of interaction between atoms and field [see Eq. (6)] in which only the atomic dipole operator $\sigma=|a\rangle\langle b|$ is coupled to the electromagnetic field. It is therefore possible to use an effective equation for the density operator ρ_j^f, which yields the same matrix elements for the atomic states $|a\rangle$ and $|b\rangle$ as Eq. (12):

$$\dot{\rho}_j^f(t)=-ig\Theta(t-t_j)[V_j,\rho_j^f(t)]-\Gamma\rho_j^f(t) . \quad (13)$$

We can now eliminate the explicit contribution of atomic decay by defining a new operator $\tilde{\rho}_j^f(t)$:

$$\rho_j^f(t)=e^{-\Gamma(t-t_j)}\tilde{\rho}_j^f(t) . \quad (14)$$

Substituting this into Eqs. (10) and (13) we obtain

$$\frac{d}{dt}\rho^f(t)=-ig\sum_j f(t,t_j)\text{Tr}_{A_j}[V_j,\tilde{\rho}_j^f(t)]+\mathcal{L}\rho^f , \quad (15)$$

$$\frac{d}{dt}\tilde{\rho}_j^f(t)=-ig\Theta(t-t_j)[V_j,\tilde{\rho}_j^f(t)] , \quad (16)$$

with

$$f(t,t_j)=\Theta(t-t_j)e^{-\Gamma(t-t_j)} . \quad (17)$$

It is interesting to notice that the expression (16) for the reduced density operator $\tilde{\rho}_j^f$ is the same as for a single, nondecaying atom. Hence all the effects of the atomic decay process have been absorbed into the interaction function $f(t,t_j)$.

We can now start to solve the above set of equations and derive an equation for the electromagnetic field alone. We assume that the laser started its operation at some time in the past so that all transient effects have already decayed. At $t=0$ we than start our observation by measuring the state of the radiation field. It is therefore useful to integrate Eq. (16) from the beginning of the measurement at $t=0$:

$$\tilde{\rho}_j^f(t)=\tilde{\rho}_j^f(0)-ig\int_0^t dt'\Theta(t-t_j)[V_j,\tilde{\rho}_j^f(t')] . \quad (18)$$

For atoms with injection times larger than zero we should let t_j be the lower integration limit. However, the Θ function in Eq. (16) guarantees that both expressions are identical. Simultaneously, it is understood that $\tilde{\rho}_j^f(0)$ is equal to $\tilde{\rho}_j^f(t_j)$ for all $t_j\geq 0$. To avoid confusion, we will therefore denote the initial state of the density operator by $\tilde{\rho}_j^f(t_0)$ and choose the value of t_0 as 0 or t_j, depending on whether the atoms are injected before or after the start of the measurement at $t=0$. Substituting Eq. (18) into Eq. (15) yields

$$\dot{\rho}^f(t)=-ig\sum_j f(t,t_j)\text{Tr}_{A_j}[V_j,\tilde{\rho}_j^f(t_0)] -g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j) \times\text{Tr}_{A_j}[V_j,[V_j,\tilde{\rho}_j^f(t')]]+\mathcal{L}\rho^f . \quad (19)$$

Hence we have replaced the density operator ρ_j^f at time t by the known initial condition plus a term which is of higher orders in the coupling constant g. One could now use Eq. (19) to derive a linear theory of a laser with atom-

ic memory effects in the density-matrix approach. However, our main interest in this paper is to study the influence of saturation terms. Hence an expansion to higher orders in the coupling constant g is necessary. This is obtained by iterating Eq. (19), i.e., by substituting the formal integral of $\tilde{\rho}_j^f$, as given by Eq. (18), into the expression for ρ^f. As is well known from the ordinary quantum theory of the laser,[12] it is sufficient to stop the expansion in the fourth order of g, provided the laser is operated not too high above threshold. The result is

$$\begin{aligned}\dot{\rho}^f(t)=&-ig\sum_j f(t,t_j)\mathrm{Tr}_A[V_j,\tilde{\rho}_j^f(t_0)]-g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathrm{Tr}_A[V_j,[V_j,\tilde{\rho}_j^f(t_0)]]\\&+ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\mathrm{Tr}_A[V_j,[V_j,[V_j,\tilde{\rho}_j^f(t_0)]]]\\&+g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)\mathrm{Tr}_A[V_j,[V_j,[V_j,[V_j,\tilde{\rho}_j^f(t_0)]]]]+\mathcal{L}\rho^f\ . \end{aligned}\tag{20}$$

We now turn to the discussion of the initial value of the density operator ρ_j^f. For atoms which are injected after the start of the measurement, i.e., with $t_j \geq 0$, the electromagnetic field at time t_0 is uncorrelated with the initial atomic preparation. Therefore we can write

$$\tilde{\rho}_j^f(t_0)=\tilde{\rho}_j(t_0)\otimes\rho^f(t_0)\ . \tag{21}$$

Such a factorization of the combined density operator at the time t_0 also holds for atoms with $t_j<0$. This is due to the measurement of the field at $t=0$, which exactly determines the initial state of the electromagnetic field.

We next note that all the terms in Eq. (20), apart from the damping contribution of the cavity, involve the interaction function $f(t,t_j)$. This function has its maximum value at time t_j and decays on a time scale Γ^{-1}, which was assumed to be much shorter than the photon decay time γ^{-1}. One can then make the approximation that the field does not change much during the decay time Γ^{-1} and substitute $\rho^f(t_0)$ by $\rho^f(t)$. In fact, one finds[2] that such an approximation becomes exact if one assumes a Poissonian distribution over the injection times. Such a fluctuation in the pump mechanism, for example, arises naturally when the atoms are pumped through an external light source.

For the discussion of the atomic density operator at time $t=t_0$ we have to distinguish two different cases. For atoms, which enter the cavity after the start of the measurement, the density operator is determined by their initial preparation. Thus only the upper atomic level is occupied and $\tilde{\rho}_j^f(t_0)$ can be written in a matrix notation as

$$\tilde{\rho}_j(t_0)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\quad(t_j\geq 0)\ . \tag{22}$$

In contrast, atoms with injection times smaller than zero have already interacted with the radiation field for some time so that $\tilde{\rho}_j^f(0)$ is not given by Eq. (22). However, it is still possible to find an explicit expression for the density matrix for these atoms. For this we recall that the equation of motion for $\tilde{\rho}_j^f$, as given by Eq. (16), is identical to the one in which a single, nondecaying two-level atom interacts with a radiation field. Such a problem, known as the Jaynes-Cummings model, has been extensively studied.[13] In the case that the atom is initially in the upper atomic state and the radiation field is given by a classical electromagnetic field $\mathcal{E}$, the density operator for the atom alone is found to be[13]

$$\tilde{\rho}_{\mathrm{atom}}(t)=\begin{bmatrix}\cos^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}(t-t_s)] & \dfrac{i\mathcal{E}}{2(\mathcal{E}\mathcal{E}^*)^{1/2}}\sin[2g(\mathcal{E}\mathcal{E}^*)^{1/2}(t-t_s)]\\ \dfrac{-i\mathcal{E}^*}{2(\mathcal{E}\mathcal{E}^*)^{1/2}}\sin[2g(\mathcal{E}\mathcal{E}^*)^{1/2}(t-t_s)] & \sin^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}(t-t_s)]\end{bmatrix}\ . \tag{23}$$

Here the time t_s denotes the start of the interaction between the field and the atom. We can now use Eq. (23) to obtain an expression for $\tilde{\rho}_j(0)$ by setting $t_s=t_j$ and $t=0$. The result is

$$\tilde{\rho}_j(0)=\begin{bmatrix}\cos^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j] & \dfrac{-i\mathcal{E}}{2(\mathcal{E}\mathcal{E}^*)^{1/2}}\sin[2g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j]\\ \dfrac{i\mathcal{E}^*}{2(\mathcal{E}\mathcal{E}^*)^{1/2}}\sin[2g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j] & \sin^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j]\end{bmatrix}\quad(\text{for } t_j<0)\ . \tag{24}$$

The field variable $\mathcal{E}$, which appears in Eq. (24), is to be determined self-consistently from the final equation for the electromagnetic field.

In order to simplify our notation for the two different cases of the initial atomic density operator, we make the definition

$$\tilde{\rho}_j(0)=\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} . \tag{25}$$

The parameters A_j, B_j, C_j, and D_j are given by either Eq. (22) or (24), depending on whether the injection time t_j is before or after $t=0$. We can now substitute Eq. (25) together with Eq. (21) into Eq. (20) and obtain a master equation for the reduced density matrix of the field. The commutators, which appear in Eq. (20), have been evaluated in Appendix A. The final result for the reduced density operator of the field is then found to be

$$\begin{aligned}\dot{\rho}^f(t)=&-ig\sum_j f(t,t_j)[C_j(a\rho^f-\rho^f a)+B_j(a^\dagger\rho^f-\rho^f a^\dagger)]\\
&-g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)[A_j(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a)+D_j(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger)]\\
&+ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)[C_j(aa^\dagger a\rho^f-3a^\dagger a\rho^f a+3a\rho^f aa^\dagger-\rho^f aa^\dagger)\\
&\qquad+B_j(a^\dagger aa^\dagger\rho^f-3aa^\dagger\rho^f a^\dagger+3a^\dagger\rho^f a^\dagger a-\rho^f a^\dagger aa^\dagger)]\\
&+g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)\\
&\qquad\times[aa^\dagger aa^\dagger\rho^f-4a^\dagger aa^\dagger\rho^f a+6aa^\dagger\rho^f aa^\dagger-4a^\dagger\rho^f aa^\dagger a+\rho^f aa^\dagger aa^\dagger]+\mathcal{L}\rho^f .\end{aligned} \tag{26}$$

In the above equation we have suppressed the explicit time dependence of the operators. However, it is important to remember that all the operators are specified at time t. The master equation can now be transformed into an equivalent Fokker-Planck equation by using the Glauber-Sudarshan[14] P representation. This representation identifies the density operator for the radiation field with a corresponding quasiprobability distribution. It is defined by

$$\rho^f(t)=\int d^2\mathcal{E}\,P(\mathcal{E},\mathcal{E}^*,t)|\mathcal{E}\rangle\langle\mathcal{E}| , \tag{27}$$

in which $|\mathcal{E}\rangle$ is a coherent state of the radiation field and $P(\mathcal{E},\mathcal{E}^*,t)$ a quasiprobability distribution. Substituting Eq. (27) into (26) and converting the creation and annihilation operators into partial derivatives of the function P,[8] we obtain the Fokker-Planck equation

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathcal{E},\mathcal{E}^*,t)=&-\frac{\partial}{\partial\mathcal{E}}(d_{\mathcal{E}}P)-\frac{\partial}{\partial\mathcal{E}^*}(d_{\mathcal{E}^*}P)\\
&+\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}(2D_{\mathcal{E}\mathcal{E}^*}P)\\
&+\frac{\partial^2}{\partial\mathcal{E}^2}(D_{\mathcal{E}\mathcal{E}}P)+\frac{\partial^2}{\partial\mathcal{E}^{*2}}(D_{\mathcal{E}^*\mathcal{E}^*}P) .\end{aligned} \tag{28}$$

The drift coefficients in Eq. (28) are found to be

$$\begin{aligned}d_{\mathcal{E}}=&-\frac{\gamma}{2}\mathcal{E}-ig\sum_j f(t,t_j)B_j+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)(A_j-D_j)\mathcal{E}\\
&+ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)[B_j(1+2|\mathcal{E}|^2)-2C_j\mathcal{E}^2]\\
&-g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)(4|\mathcal{E}|^2\mathcal{E}+7\mathcal{E}) ,\end{aligned} \tag{29}$$

together with its complex conjugate. The diffusion coefficients are given by

$$\begin{aligned}2D_{\mathcal{E}\mathcal{E}^*}=&2g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)A_j+3ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)(B_j\mathcal{E}^*-C_j\mathcal{E})\\
&-g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)(10|\mathcal{E}|^2+8)\end{aligned} \tag{30}$$

and

$$\begin{aligned}D_{\mathcal{E}\mathcal{E}}=&ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)B_j\mathcal{E}\\
&-3g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)\mathcal{E}^2 .\end{aligned} \tag{31}$$

In Eqs. (29)–(31), $A_j,\ldots,D_j$ are now to be considered as functions of the coherent state variables $\mathcal{E}$ and $\mathcal{E}^*$. In Appendix B we have evaluated the sums and integrals in these expressions up to the fourth order in the coupling constant g. The final result for the drift coefficient acquires the simple form

$$d_{\mathcal{E}}=\tfrac{1}{2}[(\alpha-\gamma)-\beta|\mathcal{E}|^2]\mathcal{E} . \tag{32}$$

This result is the same as the one for an ordinary laser without atomic memory effects.[12] The parameter α is the linear gain coefficient, defined by

$$\alpha=\frac{2g^2R}{\Gamma^2}\ , \tag{33}$$

while the nonlinear saturation terms are proportional to β, with

$$\beta=\frac{8g^4R}{\Gamma^4}\ . \tag{34}$$

We therefore see that the evolution of the mean electromagnetic field, as determined by the drift coefficient, is time independent and unaffected by atomic memory effects.

In contrast, the diffusion coefficients have the form

$$2D_{\mathcal{E}\mathcal{E}^*}=\alpha g_1(t)-\beta[\tfrac{5}{4}g_2(t)+g_3(t)+3g_4(t)]|\mathcal{E}|^2\ , \tag{35}$$

$$2D_{\mathcal{E}\mathcal{E}}=-\beta[\tfrac{3}{4}g_2(t)+g_4(t)]\mathcal{E}^2\ . \tag{36}$$

The functions $g_l(t)$, $l=1-4$, which appear in Eq. (35) and (36) are defined and evaluated in Appendix B. A sketch of their time dependence is given in Fig. 2. We see that for measurement times long compared to the atomic lifetime Γ^{-1}, the contributions from $g_3(t)$ and $g_4(t)$ vanish while $g_1(t)$ and $g_2(t)$ approach the value 1. In this limit, the resulting expressions for the diffusion coefficients are the ones for an ordinary laser without atomic memory effect, as expected.[8] However, for short times, i.e., $0\le t\simeq\Gamma^{-1}$, the diffusion coefficients differ from these results. Indeed, we find a reduction of spontaneous-emission noise due to the long lifetime of the atoms as we now will show.

Our main point of interest in this analysis is the influence of atomic memory effects on the diffusion of the phase. Therefore we make a change of variables and move into a polar coordinate system by defining $\mathcal{E}=\sqrt{I}e^{i\varphi}$. Here I is the intensity and φ the phase of the electromagnetic field. The Fokker-Planck equation in (28) then acquires the form

$$\frac{\partial}{\partial t}P(I,\varphi,t)=-\frac{\partial}{\partial\varphi}(d_\varphi P)-\frac{\partial}{\partial I}(d_I P)+\frac{\partial^2}{\partial I\partial\varphi}(2D_{I\varphi}P)+\frac{\partial^2}{\partial I^2}(D_{II}P)+\frac{\partial^2}{\partial\varphi^2}(D_{\varphi\varphi}P)\ . \tag{37}$$

The drift and diffusion coefficients in the above equation are given by the coefficients of the original Fokker-Planck equation (28). For example, for the phase φ we find the expressions

$$d_\varphi=\frac{1}{2iI}[\mathcal{E}^*d_{\mathcal{E}}-\mathcal{E}d_{\mathcal{E}^*}-(D_{\mathcal{E}\mathcal{E}}e^{-2i\varphi}-D_{\mathcal{E}^*\mathcal{E}^*}e^{2i\varphi})]\ , \tag{38}$$

$$D_{\varphi\varphi}=\frac{1}{4I}[2D_{\mathcal{E}\mathcal{E}^*}-(D_{\mathcal{E}\mathcal{E}}e^{-2i\varphi}+D_{\mathcal{E}^*\mathcal{E}^*}e^{2i\varphi})]\ . \tag{39}$$

On substituting the results from the Eqs. (32), (35), and (36) into the above expressions, we obtain

$$d_\varphi=0 \tag{40}$$

and

$$D_{\varphi\varphi}=\frac{1}{4I}\{\alpha g_1(t)-\beta[\tfrac{1}{2}g_2(t)+g_3(t)+2g_4(t)]I\}=\frac{1}{4I}(\alpha-\tfrac{1}{2}\beta I)(1-e^{-\Gamma t})\ . \tag{41}$$

In the last step we have used the explicit expressions for the functions $g_l(t)$, which were calculated in the Appendix B. We can now use the coefficients d_φ and $D_{\varphi\varphi}$ to discuss the mean motion and the fluctuations of the phase. We immediately see from the Fokker-Planck equation that

$$\frac{d}{dt}\langle\varphi\rangle=\langle d_\varphi\rangle=0\ . \tag{42}$$

Therefore the phase of the laser is not locked but can freely diffuse over the entire angle of 2π. For the variance we find

$$\frac{d}{dt}\langle\varphi^2\rangle=\langle 2D_{\varphi\varphi}\rangle=\frac{1}{2I_0}(\alpha-\tfrac{1}{2}\beta I_0)(1-e^{-\Gamma t})\ , \tag{43}$$

in which I_0 is the steady-state intensity of the laser. This value can be obtained, for example, by setting the diffusion coefficient $d_{\mathcal{E}}$ in Eq. (32) equal to zero. The result is the familiar expression

$$I_0=\frac{\alpha-\gamma}{\beta}\ . \tag{44}$$

We can now obtain an expression for the contribution of spontaneous-emission noise to the phase diffusion by integrating Eq. (43) from the start of the measurement at $t=0$:

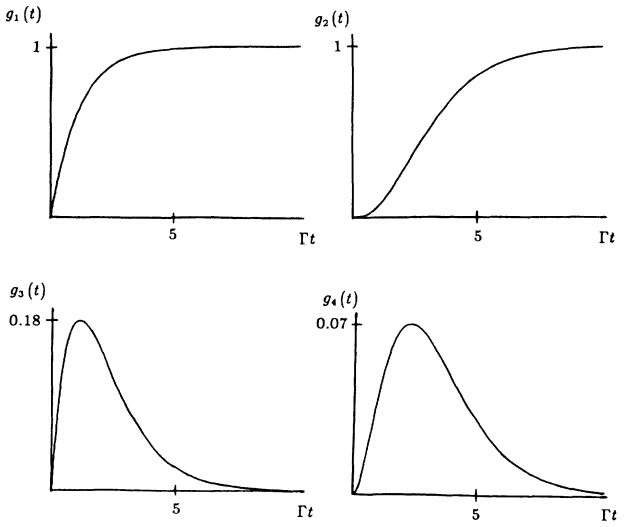

FIG. 2. Plot of the functions $g_l(t)$, $l=1-4$, which appear in the expressions (35) and (36) for the diffusion coefficients of the electromagnetic field.

$$\langle \varphi^2(t)\rangle = \frac{1}{2I_0}(\alpha - \tfrac{1}{2}\beta I_0)\left[t + \frac{1}{\Gamma}(e^{-\Gamma t}-1)\right] . \quad (45)$$

We first note that if we neglect the saturation term βI_0, this result becomes equal to Eq. (1), which we discussed in the Introduction. The effect of saturation can now be seen by substituting Eq. (44) into Eq. (45):

$$\langle \varphi^2(t)\rangle = \frac{1}{4I_0}(\alpha+\gamma)\left[t + \frac{1}{\Gamma}(e^{-\Gamma t}-1)\right] . \quad (46)$$

Thus the time dependence of the phase diffusion, which reveals the atomic memory effects, remains unchanged by the saturation terms. For measurement times which are short compared to the atomic lifetime we again find that the phase variance increases quadratically in time. On the other hand, for long measurement times the increase becomes linear as in the Schawlow-Townes result. Therefore the interpretation of Eq. (4), that the phase noise originates in equal parts from spontaneous emission and the damping of the cavity, proved to be incorrect. The diffusion of the phase is completely driven by spontaneous-emission events and can thus be significantly reduced by atomic memory effects.

III. TIME-DEPENDENT DIFFUSION AS A RESULT OF COLORED NOISE

An unusual feature of the calculations in Sec. II is the explicit time dependence of the diffusion coefficients. From the explicit forms of the coefficients $g_l(t)$ derived in the Appendix B, we recognize that the time dependence of various diffusion coefficients disappears in the limit of large atomic decay rate Γ, that is, for $\Gamma t \gg 1$. Thus in the "no-memory" limit the diffusion coefficients become time independent. In this section we show that it is the presence of noise color that leads to time-dependent diffusion coefficients. This result can be brought out in a relatively simple yet physically relevant example and employing the language of the more conventional formulation of Fokker-Planck equation.

We consider the phase $\varphi(t)$ to satisfy the damped-driven equation

$$\dot{\varphi}(t) = -\lambda\varphi(t) + \epsilon(t) , \quad (47)$$

where λ is the phase-damping constant and the driving term $\epsilon(t)$ is a Gaussian noise with zero mean and exponential correlation function

$$\langle \epsilon(t)\rangle = 0, \quad \langle \epsilon(t)\epsilon(s)\rangle = \frac{D}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right] . \quad (48)$$

We first obtain the probability $P(\varphi,t)$ for the phase assuming Gaussian phase fluctuations and then construct an appropriate Fokker-Planck equation for $P(\varphi,t)$. This example is indeed relevant to the present atomic memory problem when formulated in terms of the Langevin approach, and it was this approach that was used in our earlier discussion of the linear theory.[6]

This probability, by definition, is given by

$$P(\varphi,t) = \langle \delta(\varphi(t)-\varphi)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \langle e^{-y\varphi(t)}\rangle e^{-iy\varphi} . \quad (49)$$

Using the solution

$$\varphi(t) = \varphi_0 e^{-\lambda t} + \int_0^t dt' e^{-\lambda(t-t')}\epsilon(t') \quad (50)$$

of Eq. (47), where φ_0 is the constant of integration, along with Eq. (48) in Eq. (49) and under the assumption of a zero-mean Gaussian phase fluctuation for $\varphi(t)$, we find, after a little algebra indicated in Appendix C, the result

$$P(\varphi,t) = \frac{\exp[-(\varphi-\varphi_0 e^{-\lambda t})^2/2\langle \bar{\varphi}^2(t)\rangle]}{[2\pi\langle \bar{\varphi}^2(t)\rangle]^{1/2}} , \quad (51)$$

where

$$\begin{aligned}\langle \bar{\varphi}^2(t)\rangle &= \int_0^t dt' \int_0^t dt'' e^{-2\lambda t + \lambda(t'+t'')}\langle \epsilon(t')\epsilon(t'')\rangle \\ &= \frac{D\tau e^{-2\lambda t}}{(\lambda\tau+1)(\lambda\tau-1)}\left[1 + e^{2\lambda t} - 2e^{(\lambda\tau-1)(t/\tau)} - \frac{(e^{2\lambda t}-1)}{\lambda\tau}\right] . \quad (52)\end{aligned}$$

The Gaussian nature of the probability $P(\varphi,t)$ is of course a direct consequence of our assumption of the Gaussian nature of phase fluctuations.

It is now easy to show (see Appendix C) that $P(\varphi,t)$ in Eq. (51) satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \lambda\frac{\partial}{\partial\varphi}(\varphi P) + \mathcal{D}\frac{\partial^2 P}{\partial\varphi^2} , \quad (53)$$

with a time-dependent diffusion coefficient

$$\begin{aligned}\mathcal{D}(t) &= \lambda\langle \bar{\varphi}^2(t)\rangle + \frac{1}{2}\frac{d}{dt}\langle \bar{\varphi}^2(t)\rangle \\ &= \frac{D}{(\lambda\tau+1)}[1 - e^{-(\lambda\tau+1)(t/\tau)}] . \quad (54)\end{aligned}$$

In the limit $\lambda \to 0$, that is, for the case of no damping but in the presence of colored noise, we find from Eq. (54) that

$$\mathcal{D}(t)\Big|_{\lambda\to 0} = D(1 - e^{-t/\tau}) . \quad (55)$$

On the other hand, we find from Eq. (54) that in the white noise limit $\tau \to 0$, the diffusion coefficient is

$$\mathcal{D}(t)\Big|_{\tau\to 0} = D , \quad (56)$$

and is time independent. We thus conclude that for a Gaussian fluctuating phase colored noise gives rise to a

time-dependent diffusion coefficient.

One may think that these results may be specific to the above simple case. For this purpose it is instructive to compare the above results with the exact results of Hänggi[15] and others.[16] According to Hänggi, the single-event probability $P(\varphi,t)$ for a stochastic variable φ whose motion is described by the Langevin equation

$$\dot{\varphi}=f(\varphi)+\epsilon(t) ,$$

where the colored additive Gaussian noise obeys $\langle\epsilon(t)\rangle=0$ and $\langle\epsilon(t)\epsilon(s)\rangle=C_2(t,s)$, satisfies the exact master equation

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\varphi}[f(\varphi)P] + \frac{\partial^2}{\partial\varphi^2}\int_0^t ds\, C_2(t,s)\left\langle\delta(\varphi(t)-\varphi)\frac{\delta\varphi(t)}{\delta\epsilon(s)}\right\rangle , \tag{57a}$$

where

$$\frac{\delta\varphi(t)}{\delta\epsilon(s)}=\Theta(t-s)\exp\left[\int_s^t du\left[\frac{\partial f[\varphi(u)]}{\partial\varphi(u)}\right]\right] . \tag{57b}$$

In our problem, $f(\varphi)=-\lambda\varphi$ and $C_2=(D/\tau)\exp(-|t-s|/\tau)$. These yield $\delta\varphi(t)/\delta\epsilon(s)=\Theta(t-s)\exp[-\lambda(t-s)]$, a result independent of φ. Consequently, the above master equation, Eq. (57), reduces to the Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\lambda\frac{\partial}{\partial\varphi}(\varphi P) + \left[\frac{D}{\tau}\int_0^t ds\exp\left[-\frac{|t-s|}{\tau}-\lambda(t-s)\right]\right]\frac{\partial^2 P}{\partial\varphi^2} , \tag{58}$$

with the diffusion coefficient

$$\mathcal{D}(t)=\frac{D}{\tau}\int_0^t ds\exp\left[-\frac{|t-s|}{\tau}-\lambda(t-s)\right] . \tag{59}$$

The diffusion coefficient in Eq. (59) is the same as that in Eq. (54), thereby making the two Fokker-Planck equations, Eqs. (53) and (58), identical. If instead of colored noise, we had white noise so that $C_2(t,s)=2D\delta(t-s)$, then the second term on the right-hand side of Eq. (57a) would go over into $D(\partial^2P/\partial\varphi^2)$, resulting in a Fokker-Planck equation with time-independent diffusion coefficient; a result inferred from our simple example. Thus we conclude that the presence of atomic memory will necessarily result in a time-dependent diffusion.

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APPENDIX A

In this appendix we evaluate the commutators and traces that appear in Eq. (20). Let V be the interaction part of the Hamiltonian as given by Eq. (6), i.e.,

$$V=a^\dagger\sigma+\sigma^\dagger a . \tag{A1}$$

Here σ is the atomic dipole operator $|b\rangle\langle a|$. It is useful to apply a matrix notation for the atomic operators. In this notation V acquires the form

$$V=\begin{bmatrix}0 & a\\ a^\dagger & 0\end{bmatrix} . \tag{A2}$$

Furthermore, let the density operator for the field and the atom be given by [c.f. Eq. (21)]

$$\rho(t_0)=\rho^f(t)\otimes\rho_{\text{atom}}(t_0) , \tag{A3}$$

in which ρ_{atom} is the density operator for the atom alone. This density operator specifies the state of the atom at time t_0 and is taken to be [Eq. (25)]

$$\rho_{\text{atom}}(t_0)=\begin{bmatrix}A & B\\ C & D\end{bmatrix} . \tag{A4}$$

We can then readily calculate

$$[V,\rho(t_0)]=[V,\rho^f\otimes\rho_{\text{atom}}] = \begin{bmatrix}Ca\rho^f-B\rho^f a^\dagger & Da\rho^f-A\rho^f a\\ Aa^\dagger\rho^f-D\rho^f a^\dagger & Ba^\dagger\rho^f-C\rho^f a\end{bmatrix} . \tag{A5}$$

Taking the trace of both sides of Eq. (A5), we obtain

$$\mathrm{Tr}_A[V,\rho(t_0)]=C(a\rho^f-\rho^f a)+B(a^\dagger\rho^f-\rho^f a^\dagger) . \tag{A6}$$

The index A indicates that the trace is taken with respect to the atomic variables. Analogously we find

$$\mathrm{Tr}_A[V,[V,\rho(t_0)]]=A(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a) +D(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) \tag{A7}$$

and

$$\mathrm{Tr}_A[V,[V,[V,\rho(t_0)]] =C(aa^\dagger a\rho^f-3a^\dagger a\rho^f a+3a\rho^f aa^\dagger-\rho^f aa^\dagger a) +B(a^\dagger aa^\dagger\rho^f-3aa^\dagger\rho^f a^\dagger +3a^\dagger\rho^f a^\dagger a-\rho^f a^\dagger aa^\dagger) . \tag{A8}$$

For the trace which involves four commutators we can make a significant simplification. As mentioned before, it is sufficient to restrict our analysis to terms up to fourth order in the coupling constant g. The term proportional to the four commutators in Eq. (20) is already multiplied by a factor of order g^4. Thus the trace only needs to be calculated in lowest order of g. It is easy to see by expansion of Eq. (24) that the atomic density operator has the form

$$\rho_{\text{atom}}(t_0)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+O(g) . \tag{A9}$$

Therefore the term of interest is

$$\mathrm{Tr}_A\left[V,\left[V,\left[V,\left[V,\rho^f\otimes\begin{bmatrix}1&0\\0&0\end{bmatrix}\right]\right]\right]\right]$$
$$=aa^\dagger aa^\dagger\rho^f-4a^\dagger aa^\dagger\rho^f a+6aa^\dagger\rho^f aa^\dagger$$
$$-4a^\dagger\rho^f aa^\dagger a+\rho^f aa^\dagger aa^\dagger\ . \tag{A10}$$

APPENDIX B

In this appendix we evaluate the sums and integrals which appear in the drift and diffusion coefficients of the electromagnetic field. We start our analysis by expanding the coefficients A_j, B_j, C_j, and D_j, as given by Eq. (24) in powers of the coupling constant g. For atoms with injection times t_j smaller than zero we then find

$$A_j=\cos^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j]=1-g^2t_j^2|\mathcal{E}|^2+O(g^4)\ , \tag{B1}$$

$$B_j=-\frac{i}{2}\frac{\mathcal{E}}{(\mathcal{E}\mathcal{E}^*)^{1/2}}\sin[2g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j]$$
$$=-igt_j\mathcal{E}+i\tfrac{2}{3}g^3t_j^3|\mathcal{E}|^2\mathcal{E}+O(g^5)\ , \tag{B2}$$

$$C_j=B_j^*\ , \tag{B3}$$

$$D_j=\sin^2[g(\mathcal{E}\mathcal{E}^*)^{1/2}t_j]=g^2t_j^2|\mathcal{E}|^2+O(g^4)\ . \tag{B4}$$

For atoms with injection times t_j larger than zero we immediately obtain from Eq. (22)

$$A_j=1,\quad B_j=0,\quad C_j=0,\quad D_j=0\ . \tag{B5}$$

We can now substitute these expressions into the different terms of the drift and diffusion coefficients and evaluate their contributions up to fourth order in the coupling constant g.

The drift coefficient for the electromagnetic field is given by [c.f. Eq. (29)]

$$d_{\mathcal{E}}=-\frac{\gamma}{2}\mathcal{E}-ig\sum_j f(t,t_j)B_j+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)(A_j-D_j)\mathcal{E}$$
$$+ig^3\int_0^t dt'\int_0^{t'}dt''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)2(B_j|\mathcal{E}|^2-C_j\mathcal{E}^2)$$
$$-g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}d'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)4|\mathcal{E}|^2\mathcal{E}\ . \tag{B6}$$

Here we have neglected terms of order 1 compared to terms proportional to the intensity $|\mathcal{E}|^2$ of the radiation field. The second and third term in Eq. (B6) are the only ones which make contribution in the second order in g. We find

$$-ig\sum_j f(t,t_j)B_j+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)(A_j-D_j)\mathcal{E}$$
$$=-ig\sum_{\substack{j\\t_j<0}} f(t,t_j)(-igt_j\mathcal{E}+i\tfrac{2}{3}g^3t_j^3|\mathcal{E}|^2\mathcal{E})+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathcal{E}$$
$$+g^2\int_0^t dt'\sum_{\substack{j\\t_j<0}} f(t,t_j)\Theta(t'-t_j)(-2g^2t_j^2|\mathcal{E}|^2\mathcal{E})$$
$$=-g^2\sum_{\substack{j\\t_j<0}} f(t,t_j)t_j\mathcal{E}+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathcal{E}$$
$$+\tfrac{2}{3}g^4\sum_{\substack{j\\t_j<0}} f(t,t_j)t_j^3|\mathcal{E}|^2\mathcal{E}-2g^4\int_0^t dt'\sum_{\substack{j\\t_j<0}} f(t,t_j)\Theta(t'-t_j)t_j^2|\mathcal{E}|^2\mathcal{E}\ . \tag{B7}$$

The two terms of order g^2 can be combined into one expression by noting that

$$-g^2\sum_{\substack{j\\t_j<0}} f(t,t_j)t_j\mathcal{E}+g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathcal{E}=g^2\sum_{\substack{j\\t_j<0}} f(t,t_j)\left[\int_{-\infty}^0 dt'\Theta(t'-t_j)+\int_0^t dt'\Theta(t'-t_j)\right]\mathcal{E}$$
$$+g^2\int_{-\infty}^t dt'\sum_{\substack{j\\t_j>0}} f(t,t_j)\Theta(t'-t_j)\mathcal{E}$$
$$=g^2\int_{-\infty}^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathcal{E}\ . \tag{B8}$$

We next transform the sum over all atoms into an integration over the injection times t_j, i.e., $\sum_j\to R\int_{-\infty}^{\infty}dt_j$, in which R is the mean atomic rate. Substituting the definition (17) for the interaction function $f(t,t_j)$, we obtain our final expression for the terms of order g^2,

$$g^2\int_{-\infty}^{t}dt'\sum_j f(t,t_j)\Theta(t'-t_j)\mathcal{E}=g^2\int_{-\infty}^{t}dt'R\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t-t_j)}\mathcal{E}$$

$$=\frac{g^2R}{\Gamma}\int_{-\infty}^{t}dt'e^{-\Gamma(t-t')}\mathcal{E}=\frac{g^2R}{\Gamma^2}\mathcal{E}$$

$$=\frac{\alpha}{2}\mathcal{E}\ . \tag{B9}$$

Here α is the linear gain coefficient for the laser which is defined by

$$\alpha=\frac{2g^2R}{\Gamma^2}\ . \tag{B10}$$

The terms of order g^4 in the drift coefficient are given by the last two terms in Eq. (B7) and Eq. (B6):

$$\text{Terms of order } g^4=\tfrac{2}{3}g^4\sum_{\substack{j\\ t_j<0}}f(t,t_j)t_j^3|\mathcal{E}|^2\mathcal{E}-2g^4\int_0^t dt'\sum_{\substack{j\\ t_j<0}}f(t,t_j)\Theta(t'-t_j)t_j^2|\mathcal{E}|^2\mathcal{E}$$

$$+4g^4\int_0^t dt'\int_0^{t'}dt''\sum_{\substack{j\\ t_j<0}}f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)t_j|\mathcal{E}|^2\mathcal{E}$$

$$-4g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)|\mathcal{E}|^2\mathcal{E}\ . \tag{B11}$$

If one groups the atoms according to their injection times and uses similar arguments as in Eq. (B8), it is easy to see that all fourth-order terms combine to the simple expression

$$-4g^4\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\int_{-\infty}^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)|\mathcal{E}|^2\mathcal{E}\ . \tag{B12}$$

This can be evaluated by again transforming the sum over atoms into an intergration over the injection times

$$-4g^4\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\int_{-\infty}^{t''}dt'''R\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)e^{-\Gamma(t-t_j)}|\mathcal{E}|^2\mathcal{E}$$

$$=-4\frac{g^4R}{\Gamma}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\int_{-\infty}^{t''}dt'''e^{-\Gamma(t-t''')}|\mathcal{E}|^2\mathcal{E}$$

$$=-4\frac{g^4R}{\Gamma^4}|\mathcal{E}|^2\mathcal{E}$$

$$=-\tfrac{1}{2}\beta|\mathcal{E}|^2\mathcal{E}\ . \tag{B13}$$

The parameter β is the saturation coefficient for the laser and is defined by

$$\beta=\frac{8g^4R}{\Gamma^4}\ . \tag{B14}$$

Combining the results of Eqs. (B9) and (B13) we obtain our final result for the drift coefficient in a laser with atomic memory effects

$$d_{\mathcal{E}}=\tfrac{1}{2}(\alpha-\gamma)\mathcal{E}-\tfrac{1}{2}\beta|\mathcal{E}|^2\mathcal{E}\ . \tag{B15}$$

We now turn to the evaluation of the diffusion coefficients. From Eqs. (30) and (31) we obtain, after substituting the expansions of the parameters A_j, B_j, C_j, and D_j,

$$2D_{\mathcal{E}\mathcal{E}^*}=2g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)-2g^4\int_0^t dt'\sum_{\substack{j\\ t_j<0}}f(t,t_j)\Theta(t'-t_j)t_j^2|\mathcal{E}|^2$$

$$+6g^4\int_0^t dt'\int_0^{t'}dt''\sum_{\substack{j\\ t_j<0}}f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)t_j|\mathcal{E}|^2$$

$$-10g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)|\mathcal{E}|^2 \tag{B16}$$

and

$$2D_{\mathcal{E}\mathcal{E}}=2g^4\int_0^t dt'\int_0^{t'}dt''\sum_{\substack{j\\ t_j<0}} f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)t_j\mathcal{E}^2$$
$$-6g^4\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j)\mathcal{E}^2 . \quad \text{(B17)}$$

We next define four time-dependent functions by

$$g_1(t)=\frac{2g^2}{\alpha}\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j) , \quad \text{(B18)}$$

$$g_2(t)=\frac{8g^4}{\beta}\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\sum_j f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)\Theta(t'''-t_j) , \quad \text{(B19)}$$

$$g_3(t)=\frac{2g^4}{\beta}\int_0^t dt'\sum_{\substack{j\\ t_j<0}} f(t,t_j)\Theta(t'-t_j)t_j^2 , \quad \text{(B20)}$$

$$g_4(t)=-2\frac{g^4}{\beta}\int_0^t dt'\int_0^{t'}dt''\sum_{\substack{j\\ t_j<0}} f(t,t_j)\Theta(t'-t_j)\Theta(t''-t_j)t_j . \quad \text{(B21)}$$

Then the diffusion coefficients take on the simple forms

$$2D_{\mathcal{E}\mathcal{E}^*}=\alpha g_1(t)-\beta[\tfrac{5}{4}g_2(t)+g_3(t)+3g_4(t)]|\mathcal{E}|^2 \quad \text{(B22)}$$

and

$$2D_{\mathcal{E}\mathcal{E}}=-\beta[\tfrac{3}{4}g_2(t)+g_4(t)]\mathcal{E}^2 . \quad \text{(B23)}$$

It remains to evaluate the explicit form of the functions $g_l(t)$. For this we transform the sums over atoms into corresponding integrals over the injection times and perform the integration. A straightforward calculation yields

$$g_1(t)=1-e^{-\Gamma t} , \quad \text{(B24)}$$

$$g_2(t)=1-e^{-\Gamma t}(1+\Gamma t+\tfrac{1}{2}\Gamma^2t^2) , \quad \text{(B25)}$$

$$g_3(t)=\tfrac{1}{2}\Gamma te^{-\Gamma t} , \quad \text{(B26)}$$

$$g_4(t)=\tfrac{1}{8}\Gamma^2t^2e^{-\Gamma t} . \quad \text{(B27)}$$

APPENDIX C

In this appendix we derive the single-event probability of the stochastic process, Eq. (47), starting from the definition in Eq. (49):

$$P(\varphi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dy\langle e^{iy\varphi(t)}\rangle e^{-iy\varphi} . \quad \text{(C1)}$$

When we substitute the solution for φ, Eq. (50), into Eq. (C1), we arrive at

$$P(\varphi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dy\langle e^{iy\bar{\varphi}(t)}\rangle\exp[-iy(\varphi-\varphi_0e^{-\lambda t})] , \quad \text{(C2)}$$

where

$$\bar{\varphi}(t)=\int_0^t dt'\exp[-\lambda(t-t')]\epsilon(t') . \quad \text{(C3)}$$

On recalling Eq. (48), we have

$$\langle\bar{\varphi}(t)\rangle=\int_0^t dt'e^{-\lambda(t-t')}\langle\epsilon(t')\rangle=0 ,$$

and if we assume Gaussian phase fluctuation of $\bar{\varphi}(t)$, we find that

$$\langle e^{iy\bar{\varphi}(t)}\rangle=\exp\left[-\frac{y^2}{2}\langle\bar{\varphi}^2(t)\rangle\right] . \quad \text{(C4)}$$

Substituting Eq. (C4) into Eq. (C2), we arrive at

$$P(\varphi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dy\exp\left[-\frac{y^2}{2}\langle\bar{\varphi}^2(t)\rangle - iy(\varphi-\varphi_0e^{-\lambda t})\right] ,$$

which on completing the square in the exponent of the integrand yields, after some minor algebra,

$$P(\varphi,t)=\frac{1}{[2\pi\langle\bar{\varphi}^2(t)\rangle]^{1/2}}\exp\left[-\frac{(\varphi-\varphi_0e^{-\lambda t})^2}{2\langle\bar{\varphi}^2(t)\rangle}\right] . \quad \text{(C5)}$$

We are now left with calculating the average $\langle\bar{\varphi}^2(t)\rangle$. With the help of Eq. (48), we find that

$$\langle\bar{\varphi}^2(t)\rangle=\frac{D}{\tau}e^{-2\lambda t}\int_0^t dt'\int_0^t dt''e^{\lambda(t'+t'')-[(|t'-t''|)/\tau]} ,$$

which on evaluating the integrals gives

$$\langle\bar{\varphi}^2(t)\rangle=\frac{D\tau e^{-2\lambda t}}{(\lambda\tau+1)(\lambda\tau-1)}\left[1-2e^{(\lambda\tau-1)(t/\tau)} + e^{2\lambda t}-\frac{(e^{2\lambda t}-1)}{\lambda\tau}\right] . \quad \text{(C6)}$$

In the limit of white noise, $\tau\to 0$, this reduces to

$$\langle\bar{\varphi}^2(t)\rangle=\frac{D}{\lambda}(1-e^{-2\lambda t}) . \quad \text{(C7)}$$

To obtain the Fokker-Planck equation corresponding to the distribution in Eq. (C5), we calculate the expression

$$I \equiv \lambda \frac{\partial}{\partial \varphi}(\varphi P) + \mathcal{D} \frac{\partial^2 P}{\partial \varphi^2} , \tag{C8}$$

with

$$\mathcal{D}(t) = \lambda \langle \bar{\varphi}^2(t) \rangle + \frac{1}{2} \frac{d}{dt} \langle \bar{\varphi}^2(t) \rangle . \tag{C9}$$

When we substitute Eq. (C5) into Eq. (C8), we obtain

$$I = -\frac{1}{2} \frac{d/dt \langle \bar{\varphi}^2(t) \rangle P}{\langle \bar{\varphi}^2(t) \rangle} - \frac{\lambda \varphi_0 e^{-\lambda t}(\varphi - \varphi_0 e^{-\lambda t}) P}{\langle \bar{\varphi}^2(t) \rangle} + \frac{1}{2} \frac{(\varphi - \varphi_0 e^{-\lambda t})^2 d/dt \langle \bar{\varphi}^2(t) \rangle}{\langle \bar{\varphi}^2(t) \rangle^2} P . \tag{C10}$$

If we evaluate $\partial P/\partial t$, we find that it is precisely the same as the right-hand side of Eq. (C10). This establishes the Fokker-Planck equation (53) for the probability distribution in Eq. (51).

We conclude by noting that in the limit $\tau = 0$, Eq. (C9) with the help of Eq. (C7) reduces to the time-independent diffusion

$$\mathcal{D}(t) = D(1 - e^{-2\lambda t}) + D e^{-2\lambda t} = D .$$

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