Polarization-correlated-emission schemes

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We investigate different kinds of polarization-correlated-emission schemes, including a maser and a laser concept. In these systems, coherently prepared atoms drive the electromagnetic field. We demonstrate that the atomic coherence can lead to a reduction of noise fluctuations within a linear theory. In the homogeneously broadened case the amplitude noise is strongly affected by the atomic coherence, leading to noise quenching in the laser and even squeezing in the maser. On the other hand, the phase fluctuations are unaffected by the coherence between the lasing levels. We then study the effect of inhomogeneous broadening of the atomic transition frequency. We find the interesting result that the noise reduction due to the atomic coherence is not destroyed but redistributed between the phase and amplitude of the electromagnetic field.

I. INTRODUCTION

The reduction of the quantum noise in various atomic systems has been the subject of extensive work over the last decade. For example, the generation and utilization of squeezed states, in which the noise fluctuations of one of the quadratures of the electromagnetic field is below the vacuum noise level, has attracted a great deal of interest, both experimentally[1] and theoretically.[2] Furthermore, a big effort has been devoted to the reduction of spontaneous emission noise in the correlated emission laser[3] (CEL), where the quantum noise can be substantially reduced below the Schawlow-Townes limit.[4]

Most of the previous CEL schemes involve correlations between photon pairs, having different frequencies (quantum beat laser),[5a] different polarizations (Hanle laser),[5b] different wave vectors (holographic laser),[5c] or a cascade emission (two-photon CEL).[5d] Recently a new type of correlated spontaneous emission device has been proposed, the polarization CEL,[6] which can produce noise reduced light in a single photon transition. In this type of laser the atoms are excited into a coherent superposition between the two atomic levels which drive the electromagnetic field. It has been shown that such a laser model exhibits a reduction of noise fluctuations in the amplitude and phase of the radiated field. In this paper we want to elaborate on this idea and develop different kinds of polarization-correlated-emission concepts. We show that there is an important, qualitative difference between the maser and laser concept. These analyses are carried out in a linear theory of the electromagnetic field, thus we restrict ourselves to low intensities. It has been shown[6] that in such an operation regime the noise-reducing coherence terms are strongly affected by the statistical properties of the pump mechanism. In fact, an increase of the pump fluctuations reduces the effect of atomic coherence on the noise characteristic of the radiated field. Therefore in this paper we restrict ourselves to the case of a noise-free pump source to analyze the maximal influence of the atomic coherence on the radiated electromagnetic field.

We discuss two different concepts of single photon correlated emission. In the case of the polarization-correlated-emission maser [Fig. 1(a)] we consider two-level atoms which are injected into a microwave cavity in a regular way. This could be feasible in micromaser experiments in which an atomic beam of Rydberg atoms drives the field of a microwave cavity.[7] In our model the atoms enter the cavity in a coherent superposition of upper and lower atomic level and start to interact with one mode of the radiation field for a well-defined interaction time τ. We assume that during the whole interaction time the atoms do not lose their excitation or coherence through any decay process.

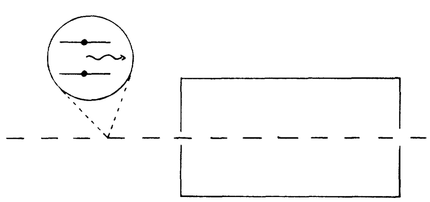
In the case of the polarization-correlated-emission laser [Fig. 1(b)] we consider three-level atoms in which the upper two levels constitute the lasing transition. Again, the atoms are regularly injected into the cavity in a coherent superposition of the two excited levels and start to interact with a resonant mode of the radiation field. Instead of removing the atoms from the cavity after an interaction time τ as in the maser case, the interaction is now limited by a decay of the atomic excitation to a lower-lying, inert ground state. In both the maser and laser case we find that the atomic coherence leads to a significant noise reduction in the amplitude of the radiation field.

In Sec. II, we perform a Langevin analysis of the polarization-correlated-emission maser and laser. In a linear analysis in the electromagnetic field (i.e., second order in the coupling constant), we find that the phase of the field locks with respect to the phase of the injected atomic dipoles. Furthermore, we derive an expression for the steady-state value of the field amplitude under the condition that $\alpha(\rho_{aa}-\rho_{bb})<\gamma$. An analysis of the noise properties of the polarization-correlated-emission maser then shows that the phase diffusion coefficient is the same as for an incoherently pumped maser. In contrast, the amplitude fluctuations are strongly affected by the atomic coherence, leading to a total suppression of the spontaneous emission noise in the case of $\rho_{bb}=\frac{1}{2}$ and even squeezing when $\rho_{bb}>\frac{1}{2}$. In the laser case, we again find that the phase diffusion is not altered with respect to an incoherent pumped case. The atomic coherence only affects the amplitude nose of the electromagnetic field. However, the noise reduction is smaller than in the maser case. We find no squeezing in the laser case and the maximum we can achieve is total noise quieting.

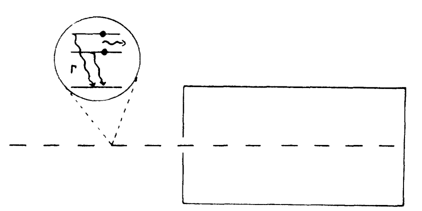
In Sec. III, we follow an alternative approach to the polarization-correlated-emission schemes by performing a density operator analysis. We again restrict ourselves to a linear theory and derive a master equation for the reduced density operator for the radiation field. We then convert this equation into a corresponding Fokker-Planck equation. This enables a direct discussion of the amplitude and phase diffusion coefficients and we fully recover the results from Sec. II.

Both of the above analyses assume that the atoms are homogeneously broadened, with a center frequency which is resonant with the cavity frequency. In Sec. IV, these constraints are relaxed and we study the effects of inhomogeneous broadening. We find that for a broadband distribution over the atomic transition frequencies the noise reduction is now partially shifted from the amplitude to the phase.

Finally, in Sec. V we summarize our results.



(a)



(b)

FIG. 1. Physical models for the polarization-correlated-emission devices. (a) Maser case: Two-level atoms are prepared in a coherent superposition between their lasing levels and are passed through a radiation cavity. Each atom interacts with the field for a given time interval τ before it leaves the cavity. (b) Laser case: The lasing material consists of three-level atoms of which the upper two levels constitute the lasing transition. The lower-lying level is an inert ground state to which an atomic excitation can decay with a rate Γ. The atoms are again prepared in a coherent superposition between the upper two atomic levels and are then injected into the cavity. However, the atoms are not removed from the cavity but interact with the field until they decay to the ground state.

II. LANGEVIN ANALYSIS

We start the analysis of the polarization-correlated-emission maser and laser with the Hamiltonian for our physical models. As described before, we consider coherently prepared atoms which are regularly injected into a radiation cavity (see Fig. 1). The regular atomic injection corresponds to a noise-free pump source[8] so that we do not have to consider any fluctuations due to the pumping mechanism. After entering the cavity the atoms start to interact with one mode of the radiation field. In the maser case the atoms are removed from the cavity after a time interval τ, while in the laser case the atoms leave the interaction through a decay process to a lower-lying ground state. The basic Hamiltonian for both systems can be written as

$$H=\hbar\omega a^\dagger a+\sum_j(\varepsilon_a|a\rangle\langle a|+\varepsilon_b|b\rangle\langle b|)_j+g\hbar\sum_j f(t,t_j)V_j \;, \quad (1)$$

with

$$V_j=a^\dagger\sigma^j+(\sigma^j)^\dagger a \;. \quad (2)$$

Here ε_a and ε_b are the energies of the upper and lower atomic levels of the lasing transition. The parameter g denotes the coupling strength between the electromagnetic field and the atoms. The function $f(t,t_j)$ in Eq. (1) specifies the particular interaction of each individual atom and accounts for the difference between our maser and laser model. We will discuss these two cases separately.

A. Maser

In this case, the jth atom enters the cavity at time t_j and is removed at a later time $t_j+\tau$. The time τ is the time of flight through the cavity, which we assume to be the same for all atoms. Therefore the interaction function f in Eq. (1) is given by the notch function

$$f(t,t_j)=N(t,t_j)=\begin{cases}1 & \text{if } t_j\le t\le t_j+\tau \\ 0 & \text{otherwise}\end{cases} \;. \quad (3)$$

We can now derive the Heisenberg equations of motion for the field and the atomic dipole operator

$$\dot a=-i\omega a-\frac{\gamma}{2}a-ig\sum_j N(t,t_j)\sigma^j+F_\gamma \;, \quad (4)$$

$$\dot\sigma^j=-i\omega\sigma^j+igN(t,t_j)\sigma_z^j a \;, \quad (5)$$

in which σ_z^j is the operator $\sigma_z^j=(|a\rangle\langle a|-|b\rangle\langle b|)_j$. For simplicity we have assumed exact resonance between the radiation field and the atomic $a \rightarrow b$ transition, i.e., $(\varepsilon_a - \varepsilon_b)/\hbar = \omega$. In Eq. (4) we have also accounted for the cavity-induced losses. The parameter γ is the cavity damping rate and F_γ the corresponding Langevin noise force.[9] Note that in the maser case we can neglect any atomic decay during the passing time through the microwave cavity.

We next eliminate the quickly time varying terms in Eqs. (4) and (5) by changing into a rotating frame. For this we define the new operators

$$\tilde{a} = e^{i\omega t} a, \quad \tilde{\sigma}^j = e^{i\omega t} \sigma^j . \tag{6}$$

The corresponding equations of motion for $\tilde{a}$ and $\tilde{\sigma}^j$ are easily found to be

$$\frac{d}{dt}\tilde{a} = -\frac{\gamma}{2}\tilde{a} - ig \sum_j N(t,t_j)\tilde{\sigma}^j + \tilde{F}_\gamma , \tag{7}$$

$$\frac{d}{dt}\tilde{\sigma} = igN(t,t_j)\sigma_z^j \tilde{a} . \tag{8}$$

For convenience we will drop the tilde in the following discussion, keeping in mind that all operators are specified in a rotating frame. We can now formally integrate Eq. (8) and substitute the result into Eq. (7). We then find

$$\dot{a} = -\frac{\gamma}{2}a - ig \sum_j N(t,t_j)\sigma^j(t_j) + g^2 \int_{-\infty}^{t} dt' \sum_j N(t,t_j)N(t',t_j)\sigma_z^j(t')a(t') + F_\gamma . \tag{9}$$

The second term on the right-hand side of Eq. (9) contributes to the drift of the electromagnetic field as well as to its noise. In order to separate these contributions we add and subtract the average value of the second term in Eq. (9). For this we note that the initial value of the atomic dipole operator is given by the prepared atomic coherence

$$\langle \sigma^j(t_j) \rangle = \mathrm{Tr}[\sigma^j \rho(t_j)] = \rho_{ab} . \tag{10}$$

Furthermore, let us assume that the radiation field is slowly varying on the time scale of the atomic evolution so that we can approximate $a(t')$ by $a(t)$ inside the integral. This means that we are considering a high-q cavity in which the photons have a long lifetime. Finally we restrict ourselves to an analysis up to second order in the coupling constant g which corresponds to a linear theory in the electromagnetic field. We can then substitute $\sigma_z^j(t')$ in Eq. (9) by its expectation value at t_j, i.e., $\langle \sigma_z^j(t_j) \rangle = \rho_{aa} - \rho_{bb}$. With these assumptions Eq. (9) becomes

$$\dot{a} = -\frac{\gamma}{2}a - ig\rho_{ab} \sum_j N(t,t_j) + g^2 \int_{-\infty}^{t} dt' \sum_j N(t,t_j)N(t',t_j)(\rho_{aa} - \rho_{bb})a(t) - ig \sum_j N(t,t_j)[\sigma^j(t_j) - \langle \sigma^j(t_j) \rangle] + F_\gamma . \tag{11}$$

We now perform the sum over the atoms by integrating over the injection times, i.e.,

$$\sum_j \rightarrow R \int_{-\infty}^{\infty} dt_j ,$$

in which R is the atomic injection rate. This is possible because we assumed a regular injection of the atoms. Such an injection can be achieved, for example, by preparing the lasing atoms through a train of narrow and equally spaced laser pulses. The evaluation of the sums in Eq. (11) is performed in Appendix A and we obtain the result

$$\dot{a} = -iS_M\rho_{ab} - \frac{\gamma}{2}a + \frac{\alpha_M}{2}(\rho_{aa} - \rho_{bb}) + F_a , \tag{12}$$

with

$$F_a = F_\gamma - ig \sum_j N(t,t_j)[\sigma^j(t_j) - \langle \sigma^j(t_j) \rangle] . \tag{13}$$

The parameters in Eq. (12) are defined by

$$S_M = Rg\tau, \quad \alpha_M = Rg^2\tau^2 . \tag{14}$$

We note that the first term in Eq. (12) is a klystron-type contribution which drives the electromagnetic field. Its strength is specified by the initial atomic coherence and the parameter S_M. The third term in Eq. (12) is the linear gain of the system due to stimulated emission with α_M being the familiar gain coefficient for a maser.[10] We next turn to the evaluation of the noise correlation functions for F_a. From the definition (13) we obtain

$$\begin{aligned}\langle F_a^\dagger(t)F_a(t') \rangle &= \langle F_\gamma^\dagger(t)F_\gamma(t') \rangle + g^2 \sum_{j,k} N(t,t_j)N(t',t_k)[\langle \sigma^{\dagger j}(t_j)\sigma^k(t_k) \rangle - \langle \sigma^{\dagger j}(t_j) \rangle \langle \sigma^k(t_k) \rangle] \\ &= g^2 \sum_j N(t,t_j)N(t',t_j)(\rho_{aa} - |\rho_{ab}|^2) \\ &= \alpha_M(\rho_{aa} - |\rho_{ab}|^2)T(t,t') .\end{aligned} \tag{15}$$

Here we have assumed that the damping reservoir for the field is at zero temperatures so that the normally ordered product of F_γ is equal to zero. Furthermore, we have used the fact that the atoms are independent of each other so that $\langle (\sigma^j)^\dagger(t_j)\sigma^k(t_k) \rangle = \langle (\sigma^j)^\dagger(t_j) \rangle \langle \sigma^k(t_k) \rangle$ for $j \neq k$. The function $T(t,t')$ is the triangularly shaped correlation function which we evaluated in Appendix A. In a similar way we find

$$\langle F_a(t)F_a(t') \rangle = \alpha_M \rho_{ab}^2 T(t,t') , \tag{16}$$

$$\langle F_a^\dagger(t)F_a^\dagger(t') \rangle = \alpha_M \rho_{ba}^2 T(t,t') . \tag{17}$$

If we restrict ourselves to averages of normally ordered products of a and $a^\dagger$, we can identify Eq. (12) with a corresponding c-number stochastic differential equation. Equations (15)–(17) then specify the correlation functions of the classical stochastic noise sources. Therefore we make the identification $a \rightarrow \mathcal{E} = re^{i\phi}$, which is a convenient choice, since we are interested in the phase and amplitude of the electromagnetic field. Furthermore, we define the amplitude and phase of the atomic coherence by

$$\rho_{ab} = |\rho_{ab}| e^{i\theta} . \tag{18}$$

We then find from Eq. (12) the following stochastic differential equations for r and φ:

$$\dot{r} = S_M |\rho_{ab}| \cos\left[\varphi - \left[\theta - \frac{\pi}{2}\right]\right] + \tfrac{1}{2}[\alpha_M(\rho_{aa} - \rho_{bb}) - \gamma] r + F_r , \tag{19}$$

$$\dot{\varphi} = -\frac{S_M}{r} |\rho_{ab}| \sin\left[\varphi - \left[\theta - \frac{\pi}{2}\right]\right] + F_\varphi , \tag{20}$$

with

$$F_r = \tfrac{1}{2}(F_{\mathcal{E}} e^{-i\varphi} + F_{\mathcal{E}^*} e^{i\varphi}) , \tag{21}$$

$$F_\varphi = \frac{1}{2ir}(F_{\mathcal{E}} e^{-i\varphi} - F_{\mathcal{E}^*} e^{i\varphi}) . \tag{22}$$

The function $F_{\mathcal{E}}$ is the c-number noise force which corresponds to the noise operator F_a. The moments of $F_{\mathcal{E}}$ are given by the corresponding relations for the noise operator F_a.

If we neglect the small noise-induced drift terms,[11] we find from Eqs. (19) and (20) the steady-state values for phase and amplitude to be

$$\varphi_0 = \theta - \frac{\pi}{2} , \tag{23}$$

and

$$r_0 = \frac{2S_M |\rho_{ab}|}{\gamma - \alpha_M(\rho_{aa} - \rho_{bb})} . \tag{24}$$

We note that in contrast to the ordinary maser case, we here find a steady-state value for the amplitude in a linear theory, provided $\alpha_M(\rho_{aa} - \rho_{bb}) < \gamma$. If the last inequality is not fulfilled the steady-state value for the amplitude has to be determined from a nonlinear analysis. We next evaluate the correlation functions for r and φ. From Eq. (21) we obtain

$$\langle F_r(t) F_r(t') \rangle = \tfrac{1}{4}[2\langle F_{\mathcal{E}^*}(t) F_{\varepsilon}(t') \rangle + \langle F_{\mathcal{E}}(t) F_{\mathcal{E}}(t') \rangle e^{-2i\varphi} + \langle F_{\mathcal{E}^*}(t) F_{\mathcal{E}^*}(t') \rangle e^{2i\varphi}] , \tag{25}$$

and making use of the relations (15)–(17), we find

$$\langle F_r(t) F_r(t') \rangle = \frac{\alpha_M}{2}[\rho_{aa} - 2|\rho_{ab}|^2 \sin^2(\varphi - \theta)] T(t,t') . \tag{26}$$

In an analogous way, the noise correlation function for the phase is calculated as

$$\langle F_\varphi(t) F_\varphi(t') \rangle = \frac{\alpha_M}{2\bar{n}}[\rho_{aa} - 2|\rho_{ab}|^2 \cos^2(\varphi - \theta)] T(t,t') . \tag{27}$$

Here we have substituted r^2 by the mean number of photons $\bar{n}$ inside the cavity. Since we assumed that the field is slowly varying on the atomic time scale, we can approximate the triangularly shaped correlation function by the δ function

$$T(t,t') \sim \delta(t - t') . \tag{28}$$

Such an approximation neglects atomic memory effects[12] which are not relevant in this context. If we define the diffusion coefficient for the phase by $\langle F_\varphi(t) F_\varphi(t') \rangle = 2D_{\varphi\varphi}\delta(t - t')$ and use the steady-state values φ_0 and r_0 we obtain, from Eq. (27),

$$D_{\varphi\varphi}(\varphi_0) = \frac{\alpha_M}{4\bar{n}}[\rho_{aa} - 2|\rho_{ab}|^2 \cos^2(\varphi_0 - \theta)] = \frac{\alpha_M}{4\bar{n}} \rho_{aa} . \tag{29}$$

This is the same result as for an ordinary, incoherently pumped maser. We therefore see that the phase diffusion of the polarization-correlated-emission maser is unaffected by the atomic coherence.

The diffusion coefficient of the amplitude at the steady-state operation is found to be

$$D_{rr}(\varphi_0) = \frac{\alpha_M}{4}[\rho_{aa} - 2|\rho_{ab}|^2 \sin^2(\varphi_0 - \theta)] = \frac{\alpha_M}{4}(\rho_{aa} - 2|\rho_{ab}|^2) . \tag{30}$$

Equation (30) clearly demonstrates the noise-reducing feature of the atomic coherence. Remarking that the atomic coherence can be written as $|\rho_{ab}| = \sqrt{\rho_{aa}\rho_{bb}}$ the expression for the phase diffusion can be cast into the form

$$D_{rr}(\varphi_0) = \frac{\alpha_M}{4} \rho_{aa}(1 - 2\rho_{bb}) . \tag{31}$$

We can now see that for an equal, coherent population of the two lasing levels, i.e., $\rho_{aa} = \rho_{bb} = \frac{1}{2}$, the diffusion coefficient for the amplitude vanishes. This corresponds to a complete quieting of spontaneous emission noise. In the case of $\rho_{bb} > \frac{1}{2}$, the diffusion coefficient even becomes negative, indicating squeezing of the amplitude fluctuations below the shot-noise limit. For this we recall that the diffusion coefficients are related to the phase and amplitude fluctuations of the field by[13]

$$(\Delta r)^2 = \tfrac{1}{4} + \frac{D_{rr}}{\left| \frac{\partial}{\partial r} d_r \right|_{r_0, \varphi_0}} , \tag{32}$$

$$(\Delta\varphi)^2=\frac{1}{4\bar{n}}+\frac{D_{\varphi\varphi}}{\left|\frac{\partial}{\partial\varphi}d_\varphi\right|_{r_0,\varphi_0}} . \quad (33)$$

Here d_r and d_φ are the drift coefficients which can be taken from Eqs. (19) and (20), respectively. The terms 1/4 and $1/4\bar{n}$ are the contributions from the vacuum fluctuations. They arise from the commutation relation between the operators a and $a^\dagger$ and account for the fact that our equations for r and φ correspond to normally ordered products of the operators.[13] We finally conclude that the phase fluctuations in the polarization-correlated-emission maser are the same as for an ordinary phase-locked maser. Thus there is no phase noise reduction due to the injected atomic coherence. On the other hand, the amplitude noise caused by spontaneous emission can be totally suppressed. For $\rho_{bb}>\frac{1}{2}$, the amplitude fluctuations for the polarization-correlated-emission maser can be made even smaller than the vacuum limit.

B. Laser

We again start with the Hamiltonian in Eq. (1), but now include the atomic decay by coupling the atoms to a heat reservoir. The interaction function $f(t,t_j)$ then specifies only the start of the interaction for the jth atom and is given by

$$f(t,t_j)=\Theta(t-t_j)=\begin{cases}1, & t\ge t_j\\ 0 & \text{otherwise} .\end{cases} \quad (34)$$

We can now write down the Heisenberg equations of motion for the field and the atomic operators. We again chose a rotating coordinate system in which all operators are slowly time varying

$$\dot a=-\frac{\gamma}{2}a-ig\sum_j\Theta(t-t_j)\sigma^j+F_\gamma , \quad (35)$$

$$\dot\sigma^j=-\Gamma\sigma^j+ig\Theta(t-t_j)\sigma_z^j a+F^j , \quad (36)$$

$$\dot\sigma_z^j=-\Gamma\sigma_z^i+O(g)+\mathcal{N} , \quad (37)$$

where $\mathcal{N}$ represents noise. Our restriction to an analysis up to second order in the coupling constant allows us to keep only the lowest-order term in Eq. (37). The parameter Γ is the atomic decay rate and F^j the corresponding Langevin noise force. The noise correlation function for F^j, which is relevant in this context, is given by[14]

$$\langle (F^j)^\dagger(t)F^j(t')\rangle=\Gamma\langle\sigma_{aa}\rangle\delta(t-t') . \quad (38)$$

As a first step to the solution of the above equations we formally integrate Eqs. (36) and (37) and obtain

$$\sigma^j(t)=\sigma^j(t_j)e^{-\Gamma(t-t')}+\int_{-\infty}^{t}dt'\Theta(t'-t_j)e^{-\Gamma(t-t')}\times[ig\sigma_z^i(t')a(t')+F^j(t')] , \quad (39)$$

$$\sigma_z^j(t)=e^{-\Gamma(t-t_j)}\sigma_z^j(t_j)+O(g)+\mathcal{N} , \quad (40)$$

We next substitute these results into Eq. (35) for the electromagnetic field operator and find

$$\dot a=-\frac{\gamma}{2}a-ig\rho_{ab}\sum_j\Theta(t-t_j)e^{-\Gamma(t-t_j)}+g^2\int_{-\infty}^{t}dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t-t_j)}\times\sigma_z^j(t_j)a(t')+F_a , \quad (41)$$

with

$$F_a=F_\gamma-ig\sum_j\Theta(t-t_j)e^{-\Gamma(t-t_j)}[\sigma^j(t_j)-\langle\sigma^j(t_j)\rangle]-ig\int_{-\infty}^{t}dt'\sum_j\Theta(t'-t_j)\Theta(t-t_j)e^{-\Gamma(t-t')}F^j(t') . \quad (42)$$

In Eqs. (41) and (42) we have also added and subtracted the average value of the driving term which involves the atomic dipole operators $\sigma^j(t_j)$. This enables us to separate the drift from the noise contributions. We now follow a similar procedure as in the maser case. We assume that the radiation field is slowly varying during the lifetime of an atom and approximate $a(t')$ and $a(t)$. Furthermore, we again substitute $\sigma_z^j(t_j)$ by $\rho_{aa}-\rho_{bb}$ which neglects the correction terms of higher order in the coupling constant. The remaining sums over all atoms are then carried out in Appendix B. The final result is

$$\dot a=-iS_L\rho_{ab}-\frac{\gamma}{2}a+\frac{\alpha_L}{2}(\rho_{aa}-\rho_{bb})a+F_a , \quad (43)$$

in which we made the definition

$$S_L=\frac{Rg}{\Gamma},\quad \alpha_L=\frac{2Rg^2}{\Gamma} . \quad (44)$$

The parameter S_L again specifies the strength of the driving term, while α_L is the linear gain coefficient for the laser. When comparing Eq. (43) with the corresponding result for the maser case [Eq. (12)] we see that the drift terms for the electromagnetic field are the same. Therefore the maser analysis of the steady-state values for phase and amplitude, as given by Eqs. (19),(20) and (23),(24), is also valid for the laser case. The only difference is the definition of the driving and linear gain parameters S and α which we denoted with the subscripts M and L for the maser and laser, respectively. We now turn to the evaluation of the noise correlation function for F_a,

$$\langle F_a^\dagger(t)F_a(t')\rangle=\langle F_\gamma^\dagger(t)F_\gamma(t')\rangle+g^2\sum_{j,k}\Theta(t-t_j)\Theta(t'-t_k)e^{-\Gamma(t-t_j)}e^{-\Gamma(t'-t_k)}(\langle(\sigma^j)^\dagger\sigma^k\rangle-\langle(\sigma^j)^\dagger\rangle\langle\sigma^k\rangle)$$
$$+g^2\int_{-\infty}^{t}ds\int_{-\infty}^{t'}ds'\sum_{j,k}\Theta(t-t_j)\Theta(s-t_j)\Theta(t'-t_k)\Theta(s'-t_k)e^{-\Gamma(t+t'-s-s')}\langle F^{j\dagger}(s)F^k(s')\rangle$$
$$=g^2\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t+t'-2t_j)}(\rho_{aa}-|\rho_{ab}|^2)$$
$$+g^2\int_{-\infty}^{t}ds\int_{-\infty}^{t'}ds'\sum_j\Theta(s-t_j)\Theta(s'-t_j)e^{-\Gamma(t+t'-s-s')}\Gamma\langle\sigma_{aa}(s)\rangle\delta(s-s')\ . \tag{45}$$

In the last step we have again made use of the fact that different atoms are independent of each other. We remark that the operator σ_{aa} has an equation of motion similar to that of the operator σ_z in Eq. (40). Therefore we can make the substitution $\langle\sigma_{aa}(s)\rangle=\rho_{aa}e^{-\Gamma(s-t_j)}+O(g)$. The remaining sums over the atoms and the time integrations in Eq. (45) are then carried out in Appendix B. The result is

$$\langle F_a^\dagger(t)F_a(t')\rangle=\alpha_L(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2)E(t,t')\ . \tag{46}$$

The function $E(t,t')$ is an exponential time correlation function which we defined in Appendix B.

We see that the main difference between this result and the corresponding maser result [Eq. (15)] is a reduction of the atomic coherence term by a factor of $\frac{1}{2}$. This effect is caused by the additional noise contribution introduced by the atomic decay.

In a similar way, we find

$$\langle F_a(t)F_a(t')\rangle=\frac{\alpha_L}{2}\rho_{ab}^2E(t,t')\ , \tag{47}$$

$$\langle F_a^\dagger(t)F_a^\dagger(t')\rangle=\frac{\alpha_L}{2}\rho_{ba}^2E(t,t')\ . \tag{48}$$

We now want to discuss the phase and amplitude diffusion in the polarization-correlated-emission laser. This is done in an analogous way as in the maser case. We identify the operator a with the classical variables $re^{i\varphi}$ and obtain the relations (21) and (22) for the noise forces F_φ and F_r. Then we approximate the time-correlation function by a δ function

$$E(t,t')\sim\delta(t-t')\ , \tag{49}$$

and find the diffusion coefficients to be

$$D_{\varphi\varphi}=\frac{\alpha_L}{4\bar{n}}\left[\rho_{aa}-|\rho_{ab}|^2\frac{1+\cos2(\varphi-\theta)}{2}\right]$$
$$=\frac{\alpha_L}{4\bar{n}}[\rho_{aa}-|\rho_{ab}|^2\cos^2(\varphi-\theta)]\ , \tag{50}$$

$$D_{rr}=\frac{\alpha_L}{4}\left[\rho_{aa}-|\rho_{ab}|^2\frac{1-\cos2(\varphi-\theta)}{2}\right]$$
$$=\frac{\alpha_L}{4}[\rho_{aa}-|\rho_{ab}|^2\sin^2(\varphi-\theta)]\ . \tag{51}$$

We can now substitute the steady-state value for the phase and obtain our final expressions for the diffusion coefficients:

$$D_{\varphi\varphi}=\frac{\alpha_L}{4\bar{n}}\rho_{aa}\ , \tag{52}$$

$$D_{rr}=\frac{\alpha_L}{4}(\rho_{aa}-|\rho_{ab}|^2)\ . \tag{53}$$

We observe that the phase diffusion constant is the same as the one for an ordinary laser. Therefore we again find no noise reduction in the phase due to the atomic coherence. In contrast, the amplitude diffusion is affected by the atomic coherence, however the reduction is less than in the maser case. Comparing the above results to the corresponding maser equations (29) and (30) we see that the term proportional to $|\rho_{ab}|^2$ is reduced by a factor of 2. This constitutes an important difference between the polarization-correlated-emission maser and laser. The maximum noise reduction one can achieve in the polarization-correlated-emission laser is a total elimination of the spontaneous emission noise in the amplitude. A squeezing of the fluctuations beyond the shot-noise limit, which was found in the maser case, is not present in the case of a laser.

III. DENSITY-MATRIX ANALYSIS

In this section we present an alternative approach to the polarization-correlated-emission maser and laser which offers a different point of view for these radiation devices. Again, we discuss the maser and laser case separately.

A. Maser

We start with the Hamiltonian given in Eq. (1). The equation of motion for the density operator in the interaction picture then obeys the equation

$$\dot{\rho}=-ig\sum_j N(t,t_j)[V_j,\rho]\ . \tag{54}$$

As we are mainly interested in the electromagnetic field of the system, we trace over the atoms and find the equation for the reduced density operator for the field:

$$\dot{\rho}^f=-ig\sum_j N(t,t_j)\mathrm{Tr}_{A^j}[V_j,\rho_j^f]\ . \tag{55}$$

Here ρ_j^f is the density operator traced over all atoms but the jth one. For ρ_j^f we find from Eq. (54)

$$\dot{\rho}_j^f=-igN(t,t_j)[V_j,\rho_j^f]-ig\sum_{k\neq j}N(t,t_k)\mathrm{Tr}_{A^k}[V_k,\rho_{j,k}^f]\ , \tag{56}$$

in which $\rho^f_{j,k}$ is the density matrix which has been traced over all atoms except for the jth and kth one. Note that in the above equation we explicitly accounted for possible correlations between different atoms. This enables us to demonstrate that the noise-reducing coherence terms, proportional to $|\rho_{ab}|^2$, are single-atom effects and do not originate from a correlation among atoms. Integrating Eq. (56) yields

$$\rho^f_j(t)=\rho^f_j(t_j)-ig\int_{-\infty}^{t}dt' N(t',t_j)[V_j,\rho^f_j(t')] -ig\int_{t_j}^{t}dt'\sum_{\substack{k\\k\neq j}}N(t',t_k)\mathrm{Tr}_{A^k}[V_k,\rho^f_{j,k}(t')] \ . \tag{57}$$

If we substitute this result into Eq. (55), we get

$$\dot\rho^f=-ig\sum_j N(t,t_j)\mathrm{Tr}_{a^j}[V_j,\rho^f_j(t_j)]-g^2\int_{-\infty}^{t}dt'\sum_j N(t,t_j)N(t',t_j)\mathrm{Tr}_{A^j}[V_j,[V_j,\rho^f_j(t')]] -g^2\sum_j N(t,t_j)\int_{t_j}^{t}dt'\sum_{\substack{k\\k\neq j}}N(t',t_k)\mathrm{Tr}_{A^j}[V_j,\mathrm{Tr}_{A^k}[V_k,\rho^f_{j,k}(t')]] \ . \tag{58}$$

We next use the fact that at the initial injection time t_j the atomic and field density operator for the jth atom factorizes

$$\rho^f_j(t_j)=\rho_j(t_j)\otimes\rho^f(t_j) \ . \tag{59}$$

As a further step we express $\rho^f(t_j)$ in terms of $\rho^f(t)$. For this we integrate Eq. (55), solve for $\rho^f(t_j)$, and obtain

$$\rho^f(t_j)=\rho^f(t)+ig\int_{t_j}^{t}dt'\sum_k N(t',t_k)\mathrm{Tr}_{A^k}[V_k,\rho^f_k(t')] \ . \tag{60}$$

Substituting Eq. (59) together with Eq. (60) into the expression (58) leads to

$$\dot\rho^f(t)=-ig\sum_j N(t,t_j)\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\rho^f(t)]-g^2\int_{-\infty}^{t}dt'\sum_j N(t,t_j)N(t',t_j)\mathrm{Tr}_{A^j}[V_j,[V_j,\rho^f_j(t')]] -g^2\sum_j N(t,t_j)\int_{t_j}^{t}dt'\sum_{\substack{k\\k\neq j}}N(t',t_k)\mathrm{Tr}_{A^j}[V_j,\mathrm{Tr}_{A^k}[V_k,\rho^f_{j,k}(t')]] +g^2\sum_j N(t,t_j)\int_{t_j}^{t}dt'\sum_k N(t',t_k)\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\mathrm{Tr}_{A^k}[V_k,\rho^f_k(t')]] \ . \tag{61}$$

If we again restrict ourselves to terms in second order in the coupling constant g, this expression can be greatly simplified. We note that for $j\neq k$

$$\rho_j(t_j)\otimes\mathrm{Tr}_{A^k}[V_k,\rho^f_k(t')]=\mathrm{Tr}_{A^k}[V_k,\rho_j(t_j)\otimes\rho^f_k(t')]=\mathrm{Tr}_{A^k}[V_k,\rho^f_{j,k}(t')]+O(g) \ . \tag{62}$$

Substituting this relation into Eq. (61) we find that the double sums, which involve pairs of atoms j,k with $j\neq k$, cancel in second order in the coupling constant. Thus the correlations between atoms do not contribute in a linear analysis for the electromagnetic field.

If we further use the relation

$$\rho^f_j(t')=\rho_j(t_j)\otimes\rho^f(t_j)+O(g)=\rho_j(t_j)\otimes\rho^f(t)+O(g) \ , \tag{63}$$

we find for Eq. (61) the simpler expression

$$\dot\rho^f(t)=-ig\sum_j N(t,t_j)\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\rho^f(t)]-g^2\int_{-\infty}^{t}dt'\sum_j N(t,t_j)N(t',t_j)\mathrm{Tr}_{A^j}[V_j,[V_j,\rho_j(t_j)\otimes\rho^f(t)]] +g^2\int_{-\infty}^{t}dt'\sum_j N(t,t_j)N(t',t_j)\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\rho^f(t)]]+O(g^3) \ . \tag{64}$$

The traces over the atomic variables appearing in Eq. (64) have been evaluated in Appendix C and the remaining sums are found in Appendix A. We then obtain the following master equation for the field:

$$\dot\rho^f=-iS_M[\rho_{ab}(a^\dagger\rho^f-\rho^f a^\dagger)+\rho_{ba}(a\rho^f-\rho^f a)] -\frac{\alpha_M}{2}[(\rho_{aa}-|\rho_{ab}|^2)(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a)+(\rho_{bb}-|\rho_{ab}|^2)(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) -\rho_{ab}^2(a^\dagger a^\dagger\rho^f+\rho^f a^\dagger a^\dagger-2a^\dagger\rho^f a^\dagger)-\rho_{ba}^2(aa\rho^f+\rho^f aa-2a\rho^f a)] \ . \tag{65}$$

So far we have only considered the change in the radiation field due to the atomic gain. In order to take the cavity losses into account we have to add the loss contribution[15]

$$\dot{\rho}^f_{\text{loss}}=-\frac{\gamma}{2}(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) . \tag{66}$$

We now want to convert the master equation for the reduced density operator into a corresponding Fokker-Planck equation. This enables us to make easy comparison with the results of Sec. II. We again choose the normal ordering of the operators a and $a^\dagger$ and use the Glauber P representation[16] defined by

$$\rho^f(t)=\int d^2\mathcal{E}\, P(\mathcal{E},\mathcal{E}^*,t)|\mathcal{E}\rangle\langle\mathcal{E}| . \tag{67}$$

Substituting Eq. (67) into the master equation for the field density operator we arrive after a straightforward calculation at the following Fokker-Planck equation:

$$\frac{\partial P(\mathcal{E},\mathcal{E}^*,t)}{\partial t}=\left[-\frac{\partial}{\partial\mathcal{E}}\left[-iS_M\rho_{ab}-\frac{\gamma}{2}\mathcal{E}+\frac{\alpha_M}{2}(\rho_{aa}-\rho_{bb})\mathcal{E}\right]+\text{c.c.}+\frac{\partial^2}{\partial\mathcal{E},\partial\mathcal{E}^*}\alpha_M(\rho_{aa}-|\rho_{ab}|^2)\right.$$
$$\left.+\frac{\partial^2}{\partial\mathcal{E}^2}\frac{\alpha_M}{2}\rho_{ab}^2+\frac{\partial^2}{\partial\mathcal{E}^{*2}}\frac{\alpha_M}{2}\rho_{ba}^2\right]P(\mathcal{E},\mathcal{E}^*t) . \tag{68}$$

Changing to polar coordinates by defining $\mathcal{E}=re^{i\varphi}$ we obtain the equivalent Fokker-Planck equation

$$\frac{\partial P(r,\varphi,t)}{\partial t}=\left[-\frac{1}{r}\frac{\partial}{\partial r}(rd_r)-\frac{\partial}{\partial\varphi}d_\varphi+\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\varphi}(2rD_{r\varphi})+\frac{1}{r}\frac{\partial^2}{\partial r^2}(rD_{rr})+\frac{\partial^2}{\partial\varphi^2}D_{\varphi\varphi}\right]P(r,\varphi,t) . \tag{69}$$

The drift and diffusion coefficients which appear in Eq. (69) are given by

$$d_r=S_M|\rho_{ab}|\cos\left[\varphi-\left[\theta-\frac{\pi}{2}\right]\right]-\frac{\gamma}{2}r$$
$$+\frac{\alpha_M}{2}(\rho_{aa}-\rho_{bb})r+\frac{1}{r}D_{\varphi\varphi} , \tag{70}$$

$$d_\varphi=-\frac{S_M}{r}|\rho_{ab}|\sin\left[\varphi-\left[\theta-\frac{\pi}{2}\right]\right]-\frac{2}{r}D_{r\varphi} , \tag{71}$$

$$D_{rr}=\frac{\alpha_M}{4}[\rho_{aa}-2|\rho_{ab}|^2\sin^2(\varphi-\theta)] , \tag{72}$$

$$D_{\varphi\varphi}=\frac{\alpha_M}{4r^2}[\rho_{aa}-2|\rho_{ab}|^2\cos^2(\varphi-\theta)] , \tag{73}$$

$$D_{r\varphi}=-\frac{\alpha_M}{4r}|\rho_{ab}|^2\sin2(\varphi-\theta) . \tag{74}$$

If we again neglect in Eqs. (70) and (71) the small, noise-induced drift terms we find perfect agreement with the results of our previous Langevin analysis.

B. Laser

We start from the Hamiltonian (1), in which $f(t,t_j)$ is the step function $\Theta(t-t_j)$, and derive an equation of motion for the total density operator. Here we also have to take into account the atomic decay from the levels a and b to some ground-state level c. This is done in the standard way by coupling the atoms to a heat reservoir. If we make the simplifying assumption that the decay rates for the two upper levels are equal, the equation of motion for the density operator in the interaction picture is given by[17]

$$\dot{\rho}=-ig\sum_j\left[\Theta(t-t_j)[V_j,\rho]\right.$$
$$-\frac{\Gamma}{2}(|a\rangle\langle a|\rho+\rho|a\rangle\langle a|+|b\rangle$$
$$\times\langle b|\rho+\rho|b\rangle\langle b|-2|c\rangle\langle a|\rho|a\rangle$$
$$\left.\times\langle c|-2|c\rangle\langle b|\rho|b\rangle\langle c|)_j\right] , \tag{75}$$

in which c denotes the inert ground state of the atoms. Tracing Eq. (75) over all atoms results in an equation for the reduced density operator for the field

$$\dot{\rho}^f=-ig\sum_j\Theta(t-t_j)\text{Tr}_{A^j}[V_j,\rho_j^f] . \tag{76}$$

For ρ_j^f we find

$$\dot{\rho}_j^f=-ig\Theta(t-t_j)[V_j,\rho_j^f]-ig\sum_{\substack{k\\k\neq j}}\Theta(t-t_k)\text{Tr}_{A^k}[V_k,\rho_{j,k}^f]$$
$$-\frac{\Gamma}{2}(|a\rangle\langle a|\rho+\rho|a\rangle\langle a|+|b\rangle\langle b|\rho+\rho|b\rangle\langle b|-2|c\rangle\langle a|\rho|a\rangle\langle c|-2|c\rangle\langle b|\rho|b\rangle\langle c|)_j . \tag{77}$$

The matrix elements of ρ_j^f that we need for Eq. (76) only involve the atomic levels a and b. Therefore we can restrict ourselves to those two levels and simplify the contributions of the radiative decay in Eq. (77) by writing the effective equation

$$\dot{\rho}_j^f = -ig\Theta(t-t_j)[V_j,\rho_j^f] - ig\sum_{\substack{k\\k\neq j}}\Theta(t-t_k)\mathrm{Tr}_{A^k}[V_k,\rho_{j,k}^f] - \Gamma\rho_j^f \ . \tag{78}$$

It is useful in the following analysis to define the new operators

$$\rho_j^f = e^{-\Gamma(t-t_j)}\tilde{\rho}_j^f, \quad \rho_{j,k}^f = e^{-\Gamma(t-t_j)}\tilde{\rho}_{j,k}^f \ . \tag{79}$$

The equations of motion (76) and (78) then become

$$\dot{\rho}^f = -ig\sum_j \Theta(t-t_j)e^{-\Gamma(t-t_j)}\mathrm{Tr}_{A^j}[V_j,\tilde{\rho}_j^f] \ , \tag{80}$$

$$\frac{d}{dt}\tilde{\rho}_j^f = -ig\Theta(t-t_j)[V_j,\tilde{\rho}_j^f] - ig\sum_{\substack{k\\k\neq j}}\Theta(t-t_k)\mathrm{Tr}_{A^k}[V_k,\tilde{\rho}_{j,k}^f] \ . \tag{81}$$

These equations are very similar to the Eqs. (55) and (56) in the maser case. Therefore we can follow analogous steps and arguments as for the maser and find the expression

$$\begin{aligned}\dot{\rho}^f = &-ig\sum_j \Theta(t-t_j)e^{-\Gamma(t-t_j)}\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\rho^f(t)]\\ &-g^2\int_{-\infty}^t dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t-t_j)}\mathrm{Tr}_{A^j}[V_j,[V_j,\rho_j(t_j)\otimes\rho^f(t)]]\\ &+g^2\int_{-\infty}^t dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t+t'-2t_j)}\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\mathrm{Tr}_{A^j}[V_j,\rho_j(t_j)\otimes\rho^f(t)]]+O(g^3) \ .\end{aligned} \tag{82}$$

If we compare Eq. (82) for the laser with the corresponding maser result (64), we notice that the commutator expressions are identical. However, the integral kernels are different. Using the results from Appendixes B and C, we find from eq. (82) the master equation for the reduced density matrix to be

$$\begin{aligned}\dot{\rho}^f = &-iS_L[\rho_{ab}(a^\dagger\rho^f-\rho^f a^\dagger)+\rho_{ba}(a\rho^f-\rho^f a)]\\ &-\frac{\alpha_L}{2}[(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2)(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a)+(\rho_{bb}-\tfrac{1}{2}|\rho_{ab}|^2)(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger)\\ &\quad-\tfrac{1}{2}\rho_{ab}^2(a^\dagger a^\dagger\rho^f+\rho^f a^\dagger a^\dagger-2a^\dagger\rho^f a^\dagger)-\tfrac{1}{2}\rho_{ba}^2(aa\rho^f+\rho^f aa-2a\rho^f a)] \ .\end{aligned} \tag{83}$$

The coefficients S_L and α_L are the ones defined by Eq. (44). If we again add the cavity losses to the master Eq. (83) and convert it into a Fokker-Planck equation we readily get

$$\begin{aligned}\frac{\partial P}{\partial t}(\mathcal{E},\mathcal{E}^*,t) = &\left[-\frac{\partial}{\partial\mathcal{E}}\left[-iS_L\rho_{ab}-\frac{\gamma}{2}\mathcal{E}+\frac{\alpha_L}{2}(\rho_{aa}-\rho_{bb})\mathcal{E}\right]+\text{c.c.}+\frac{\partial^2}{\partial\mathcal{E}\,\mathcal{E}^*}[\alpha_L(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2)]\right.\\ &\left.+\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\left[\frac{\alpha_L}{2}\rho_{ab}^2\right]+\text{c.c.}\right]P(\mathcal{E},\mathcal{E}^*,t) \ .\end{aligned} \tag{84}$$

Comparing these drift and diffusion coefficients with the corresponding Eqs. (43) and (46)–(48) of the Langevin theory in Sec. III, we find a perfect agreement. Furthermore, we note again that the terms proportional to the square of the atomic coherence differ by a factor of 2 in the maser and laser case.

IV. INHOMOGENEOUS BROADENING

So far, we only considered the case in which the radiation frequency is resonant with the center of the homogeneously broadened atomic line. We now want to relax these constraints and investigate the case of inhomogeneous broadening. Because of the similarities between the laser and the maser results we will only discuss the laser in detail and quote the corresponding maser results.

We start with the general Heisenberg equations of motion for the laser

$$\dot{a} = -i\Omega a - \frac{\gamma}{2}a - ig\sum_j\Theta(t-t_j)\sigma^j + F_\gamma \ , \tag{85}$$

$$\dot{\sigma}^j = -i\omega_j\sigma^j - \Gamma\sigma^j + ig\Theta(t-t_j)\sigma_z^j a + F_j \ . \tag{86}$$

Next, we eliminate the quickly time varying contributions by moving into a rotating frame. We define

$$a(t) = e^{-i\omega_0 t}\tilde{a}(t), \quad \sigma^j(t) = e^{-i\omega_0 t}\tilde{\sigma}^j(t) \ , \tag{87}$$

in which ω_0 is the frequency of the laser field. Substituting these expressions into the Eqs. (85) and (86) we obtain the following equations:

$$\frac{d}{dt}\tilde{a}=-i(\Omega-\omega_0)\tilde{a}-\frac{\gamma}{2}\tilde{a}-ig\sum_j\Theta(t-t_j)\tilde{\sigma}^j+\tilde{F}_\gamma\ , \tag{88}$$

$$\frac{d}{dt}\tilde{\sigma}^j=-[\Gamma+i(\omega_j-\omega_0)]\tilde{\sigma}^j+ig\Theta(t-t_j)\sigma_z^j\tilde{a}+\tilde{F}_j\ . \tag{89}$$

For convenience we will drop the tilde on the operators in the following discussion. We next integrate Eq. (89) and substitute the result into Eq. (88). We then obtain

$$\dot{a}=-i(\Omega-\omega_0)a-\frac{\gamma}{2}a-ig\sum_j\Theta(t-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t_j)}\sigma^j(t_j)$$
$$+g^2\int_{-\infty}^t dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t')}\sigma_z^j(t')a(t')$$
$$+F_\gamma-ig\int_{-\infty}^t dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma+i(\omega_j-\omega_o)](t-t')}F_j(t')\ . \tag{90}$$

We can now make analogous simplifications as in the homogeneously broadened case. Neglecting terms of higher order than g^2 we can substitute $a(t')$ by $a(t)$ and $\sigma_z^j(t')$ by $(\rho_{aa}-\rho_{bb})e^{-\Gamma(t-t_j)}$ [see the discussion following Eq. (42)]. We then find

$$\dot{a}=-(\Omega-\omega_0)a-\frac{\gamma}{2}a-ig\sum_j\Theta(t-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t_j)}\rho_{ab}$$
$$+g^2\int_{-\infty}^t dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t'-t_j)}e^{-[\Gamma+i(\omega_j-\omega_0)](t-t')}(\rho_{aa}-\rho_{bb})a(t)+F_a(t)\ , \tag{91}$$

with

$$F_a(t)=F_\gamma-ig\int_{-\infty}^t dt'\sum_j\Theta(t'-t_j)\Theta(t-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t')}F_j(t')$$
$$-ig\sum_j\Theta(t-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t_j)}[\sigma^j(t_j)-\langle\sigma^j(t_j)\rangle]\ . \tag{92}$$

In Eqs. (91) and (92) we have again added and subtracted the average value of the driving term.

We proceed to evaluate he sum over all the atoms. We note that every atom has two characteristic features: the injection time t_j and the frequency ω_j. Therefore we first group the atoms according to different frequencies ω_k which corresponds, for example, to grouping the atoms into various velocity groups in the case of Doppler broadening. Then we integrate over the different injection times in each subgroup. Thus the sum over the atoms is substituted by

$$\sum_{\substack{j\\ \text{atoms}}}\rightarrow\sum_{\substack{k\\ \text{frequencies}}} r_k\int_{-\infty}^{\infty}dt_j\ . \tag{93}$$

Here r_k is the injection rate of the kth frequency group. If p_k is the probability of an atom to have the frequency ω_k, it is easy to see that $r_k=Rp_k$ with R being the total atomic injection rate.

We have placed the detailed evaluation of the sums in Eq. (91) over all atoms in Appendix D. The final result is

$$\dot{a}=-i(\Omega-\omega_0)a-\frac{\gamma}{2}a-iS_L\rho_{ab}(W_1-iW_2)$$
$$+\frac{\alpha_L}{2}(W_1-iW_2)(\rho_{aa}-\rho_{bb})a+F_a\ , \tag{94}$$

in which W_1 and W_2 denote different weight factors. If we define $x_k=(\omega_k-\omega_0)/\Gamma$, these weight factors are given by

$$W_1=\sum_k p_k\frac{1}{1+x_k^2}=\left\langle\frac{1}{1+x^2}\right\rangle\ , \tag{95}$$

$$W_2=\sum_k p_k\frac{x_k}{1+x_k^2}=\left\langle\frac{x}{1+x^2}\right\rangle\ . \tag{96}$$

We next calculate the noise correlation functions. Following the same procedure as for the homogeneously broadened laser, we find

$$\langle F_a^\dagger(t)F_a(t')\rangle=g^2(\rho_{aa}-|\rho_{ab}|^2)\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma-i(\omega_j-\omega_0)](t-t_j)}e^{[\Gamma+i(\omega_j-\omega_0)](t'-t_j)}$$
$$+g^2\rho_{aa}\int_{-\infty}^t ds\int_{-\infty}^{t'}ds'\sum_j\Theta(s-t_j)\Theta(s'-t_j)e^{-[\Gamma-i(\omega_j-\omega_0)](t-s)}$$
$$\times e^{-[\Gamma+i(\omega_j-\omega_0)](t'-s')}\Gamma e^{-\Gamma(s-t_j)}\delta(s-s')\ . \tag{97}$$

The remaining sums and integrals are again evaluated in Appendix D. The result is

$$\langle F_a^\dagger(t)F_a(t')\rangle=\alpha_L(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2)W_1\delta(t-t') \ . \quad (98)$$

In a similar way, we find the other correlation functions to be

$$\langle F_a(t)F_a(t')\rangle=\frac{\alpha_L}{2}\rho_{ab}^2(W_3-iW_4)\delta(t-t') \ , \quad (99)$$

and

$$\langle F_a^\dagger(t)F_a^\dagger(t')\rangle=\frac{\alpha_L}{2}\rho_{ba}^2(W_3+iW_4)\delta(t-t') \ . \quad (100)$$

The weight functions W_3 and W_4 are defined by

$$W_3=\left\langle\frac{1-x^2}{(1+x^2)^2}\right\rangle \ , \quad (101)$$

$$W_4=\left\langle\frac{2x}{(1+x^2)^2}\right\rangle \ . \quad (102)$$

The detailed calculation of the last two correlation functions is also presented in Appendix D. Comparing the results given by Eqs. (98)–(100) with the corresponding equations for the homogeneously broadened laser (46)–(48), we notice that the only difference are the weight factors W_1, W_3, and W_4 which are due to the frequency distribution over the atoms.

We can now use the above results to find expressions for the phase and amplitude diffusion in an inhomogeneously broadened polarization CEL. Proceeding in the same way as for the homogeneous case we find

$$D_{\varphi\varphi}=\frac{\alpha_L}{4\bar{n}}\{\rho_{aa}W_1-\tfrac{1}{2}|\rho_{ab}|^2[W_1+W_3\cos2(\theta-\varphi)+W_4\sin2(\theta-\varphi)]\} \ , \quad (103)$$

$$D_{rr}=\frac{\alpha_L}{4}\{\rho_{aa}W_1-\tfrac{1}{2}|\rho_{ab}|^2[W_1-W_3\cos2(\theta-\varphi)-W_4\sin2(\theta-\varphi)]\} \ . \quad (104)$$

In order to obtain specific results for the diffusion coefficients let us assume that the frequency distribution of the atoms is a Gaussian, centered around an atomic frequency ω_a. For simplicity we assume this frequency to be equal to the cavity frequency Ω. It is easy to see that in such a case the laser frequency ω_0 also coincides with Ω. If we again use the notation $x=(\omega-\omega_0)/\Gamma$, the frequency distribution $P(x)$ acquires the form

$$P(x)=\frac{1}{(2\pi\sigma^2)^{1/2}}e^{-x^2/2\sigma^2} \ , \quad (105)$$

in which σ^2 is the characteristic variance of the distribution. Note that σ^2 is essentially the ratio of the inhomogeneous to the homogeneous linewidth. We immediately see that because of the symmetry of $P(x)$ around the frequency ω_0 the weight factors W_2 and W_4 are equal to zero. The drift term for the electromagnetic field in Eq. (94) then simplifies to

$$d_a=-iS_L\rho_{ab}W_1-\frac{\gamma}{2}a+\frac{\alpha_L}{2}W_1(\rho_{aa}-\rho_{bb})a \ . \quad (106)$$

This is the same as in the homogeneously broadened case, except for the weight factor W_1. Therefore the locking phase of the laser is still given by $\varphi_0=\theta-\pi/2$. Substituting this angle into the diffusion coefficients for phase and amplitude, we get

$$D_{\varphi\varphi}=\frac{\alpha_L}{4\bar{n}}[\rho_{aa}W_1-\tfrac{1}{2}|\rho_{ab}|^2(W_1-W_3)] \ , \quad (107)$$

$$D_{rr}=\frac{\alpha_L}{4}[\rho_{aa}W_1-\tfrac{1}{2}|\rho_{ab}|^2(W_1+W_3)] \ . \quad (108)$$

We see that the inhomogeneous broadening of the atoms does not destroy the noise-reducing effects of the atomic coherence. In fact, it leads to a redistribution of the $|\rho_{ab}|^2$ terms between the phase and the amplitude. To illustrate this effect let us discuss the following two cases.

(a) *Narrow distribution* ($\sigma^2\ll1$). In this limit, we can approximate $P(x)$ by $\delta(x)$, which corresponds to the homogeneously broadened case. As expected, we find $W_1=W_3=1$ and we recover the results of the homogeneously broadened laser given by Eqs. (52) and (53).

(b) *Broad distribution* ($\sigma^2\gg1$). In this case, the distribution function $P(x)$ does not vary much over the range over which the functions $1/(1+x^2)$ and $(1-x^2)/(1+x^2)$ are appreciably different from zero. We can therefore make the approximation

$$W_1=\left\langle\frac{1}{1+x^2}\right\rangle\approx P(0)\int_{-\infty}^{\infty}dx\frac{1}{1+x^2}=\pi P(0) \ , \quad (109)$$

$$W_3=\left\langle\frac{1-x^2}{(1+x^2)^2}\right\rangle=P(0)\int_{-\infty}^{\infty}dx\frac{1-x^2}{(1+x^2)^2}=0 \ . \quad (110)$$

Substituting these results into Eqs. (107) and (108) yields

$$D_{\varphi\varphi}=\frac{\alpha_L}{4\bar{n}}W_1(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2) \ , \quad (111)$$

$$D_{rr}=\frac{\alpha_L}{4}W_1(\rho_{aa}-\tfrac{1}{2}|\rho_{ab}|^2) \ . \quad (112)$$

We find the unexpected result that we have equal noise reduction in phase and amplitude due to the atomic coherence. This can be most easily understood by recalling the results for the homogeneously broadened laser in Eqs. (50) and (51). The broadband distribution over the atomic frequencies has essentially the effect of eliminating the phase-dependent contribution of the $|\rho_{ab}|^2$ terms, which enter the expressions for phase and amplitude diffusion coefficients with opposite signs. The remaining, phase insensitive part of the atomic coherence then leads to equal noise reduction in both quadratures.

We finally quote the results for the inhomogeneously broadened polarization-correlated-emission maser. Following analogous steps as in the laser case, we find the equation of motion for the field operator a to be

$$\dot{a}=-i(\Omega-\omega_0)a-\frac{\gamma}{2}a-iS_M\rho_{ab}(W_1^M-iW_2^M)+\frac{\alpha_M}{2}(\tilde{W}_1^M-i\tilde{W}_2^M)(\rho_{aa}-\rho_{bb})+F_a \ , \quad (113)$$

with

$$W_1^M=\left\langle\frac{\sin x}{x}\right\rangle,\quad W_2^M=\left\langle\frac{1-\cos x}{x}\right\rangle, \tag{114}$$

and

$$\tilde{W}_1^M=2\left\langle\frac{1-\cos x}{x^2}\right\rangle,\quad \tilde{W}_2^M=2\left\langle\frac{x-\sin x}{x^2}\right\rangle. \tag{115}$$

The noise correlation functions are found to be

$$\langle F_a^\dagger(t)F_a(t')\rangle=\alpha_M(\rho_{aa}-|\rho_{ab}|^2)\tilde{W}_1^M\delta(t-t'), \tag{116}$$

$$\langle F_a(t)F_a(t')\rangle=\alpha_M\rho_{ab}^2(W_3^M-iW_4^M)\delta(t-t'), \tag{117}$$

and

$$\langle F_a^\dagger(t)F_a^\dagger(t')\rangle=\alpha_M\rho_{ba}^2(W_3^M+iW_4^M)\delta(t-t'), \tag{118}$$

in which the weight functions W_3^M and W_4^M are defined by

$$W_3^M=2\left\langle\frac{\cos x-\cos^2 x}{x^2}\right\rangle, \tag{119}$$

$$W_4^M=\left\langle\frac{2\sin x-\sin 2x}{x^2}\right\rangle. \tag{120}$$

From the correlation functions (116)–(118), we can calculate the phase and amplitude diffusion coefficients for the inhomogeneously broadened maser case. The result is

$$D_{\varphi\varphi}=\frac{\alpha_M}{4\bar{n}}\{\rho_{aa}\tilde{W}_1^M-|\rho_{ab}|^2[\tilde{W}_1^M+W_3^M\cos2(\theta-\varphi)+W_4^M\sin2(\theta-\varphi)]\}, \tag{121}$$

$$D_{rr}=\frac{\alpha_M}{4}\{\rho_{aa}\tilde{W}_1^M-|\rho_{ab}|^2[\tilde{W}_1^M-W_3^M\cos2(\theta-\varphi)-W_4^M\sin2(\theta-\varphi)]\}. \tag{122}$$

Comparing these expressions with the corresponding ones in the laser case [Eqs. (103) and (104)], we find that both results are completely analogous, apart from a factor of 2 in front of the $|\rho_{ab}|^2$ terms. This factor is a significant difference between the coherently pumped laser and maser and was also observed in the homogeneously broadened case.

The discussion of the diffusion coefficients (121) and (122) is completely analogous to the laser case. In particular, for a broadband distribution we find

$$D_{\varphi\varphi}=\frac{\alpha_M}{4\bar{n}}\tilde{W}_1^M(\rho_{aa}-|\rho_{ab}|^2), \tag{123}$$

$$D_{rr}=\frac{\alpha_m}{4}\tilde{W}_1^M(\rho_{aa}-|\rho_{ab}|^2), \tag{124}$$

which corresponds to equal noise reduction in both the amplitude and the phase of the electromagnetic field.

V. SUMMARY AND DISCUSSION

We have analyzed in detail the effect of atomic coherence for the polarization-correlated-emission maser and laser in the low-intensity regime. We find through a Langevin and master equation analysis that the phase diffusion coefficient of the homogeneously broadened laser or maser is unaffected by the injected atomic coherence. In fact, its value corresponds to that of an ordinary, incoherently pumped laser or maser. On the other hand, the injected atomic coherence can lead to a reduction of the amplitude diffusion coefficient. We can achieve a complete elimination of the spontaneous emission noise in the field amplitude for the laser. In the maser case this noise reduction is twice as large as in the laser case. In fact, we can get negative values for the diffusion coefficient D_{rr} which corresponds to a squeezing of the amplitude fluctuations.

Furthermore, we have considered the effects of inhomogeneous broadening. We find that for a broad atomic linewidth the noise reduction in the diffusion coefficients, which in the homogeneously broadened medium was only effective for the amplitude coefficient, is now distributed between the phase and the amplitude. All of these results are summarized in Table I.

TABLE I. Diffusion coefficients for phase and amplitude of the electromagnetic field. The specified cases are as follows: the ordinary, incoherently pumped maser and laser, the resonant polarization-correlated-emission maser and laser and the inhomogeneously broadened polarization-correlated-emission maser and laser. The results for the second and third case are valid for the low-intensity regime and for a regular injection of the atoms.

Diffusion coefficients		Maser	Laser
Ordinary	$D_{\varphi\varphi}$	$\frac{\alpha}{4\bar{n}}\rho_{aa}$	$\frac{\alpha}{4\bar{n}}\rho_{aa}$
	D_{rr}	$\frac{\alpha}{4}\rho_{aa}$	$\frac{\alpha}{4}\rho_{aa}$
Resonant	$D_{\varphi\varphi}$	$\frac{\alpha}{4\bar{n}}\rho_{aa}$	$\frac{\alpha}{4\bar{n}}\rho_{aa}$
polarization	D_{rr}	$\frac{\alpha}{4}(\rho_{aa}-2\|\rho_{ab}\|^2)$	$\frac{\alpha}{4}(\rho_{aa}-\|\rho_{ab}\|^2)$
Broadened	$D_{\varphi\varphi}$	$\frac{\alpha}{4\bar{n}}(\rho_{aa}-\|\rho_{ab}\|^2)$	$\frac{\alpha}{4\bar{n}}(\rho_{aa}-\frac{1}{2}\|\rho_{ab}\|^2)$
polarization	D_{rr}	$\frac{\alpha}{4}(\rho_{aa}-\|\rho_{ab}\|^2)$	$\frac{\alpha}{4}(\rho_{aa}-\frac{1}{2}\|\rho_{ab}\|^2)$

APPENDIX A

In this appendix we evaluate the sums over the notch function $N(t,t_j)$ which appear in Eq. (11) of the maser case:

$$\sum_j N(t,t_j)=R\int_{-\infty}^{\infty}dt_jN(t,t_j)=R\int_{t-\tau}^{t}dt_j=R\tau. \tag{A1}$$

Furthermore,

$$\sum_j N(t,t_j)N(t',t_j)=R\int_{-\infty}^{\infty}dt_jN(t,t_j)N(t',t_j)\equiv R\tau^2T(t,t'). \tag{A2}$$

Here $T(t,t')$ is a triangularly shaped time correlation function which we defined as

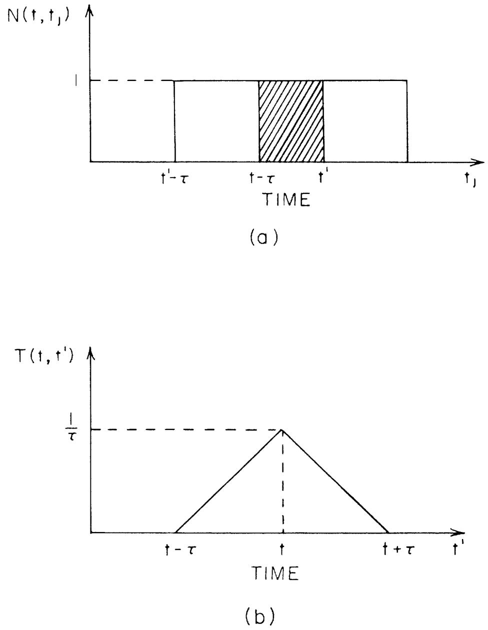


FIG. 2. Integration over the overlap between $N(t,t_j)$ and $N(t',t_j)$, as shown in (a), yields the triangularly shaped correlation function (b) $T(t,t')$.

$$T(t,t')=\frac{1}{\tau^2}\begin{cases}0, & |t-t'|\geq\tau\\ t'-(t-\tau), & t-\tau\leq t'<t\\ t+\tau-t', & t\leq t'\leq t+\tau\ .\end{cases} \tag{A3}$$

Note that $T(t,t')$ is normalized such that its integral is equal to 1. A sketch of this function is given in Fig. 2.

APPENDIX B

In this appendix we evaluate the sums which appear in Eqs. (41), (42), and (45):

$$\sum_j \Theta(t-t_j)e^{-\Gamma(t-t_j)}=R\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)e^{-\Gamma(t-t_j)}=\frac{R}{\Gamma}\ . \tag{B1}$$

Furthermore,

$$\sum_j \Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t-t_j)}=R\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t-t_j)}=\frac{R}{\Gamma}e^{-\Gamma|t-t'|}\ . \tag{B2}$$

Finally

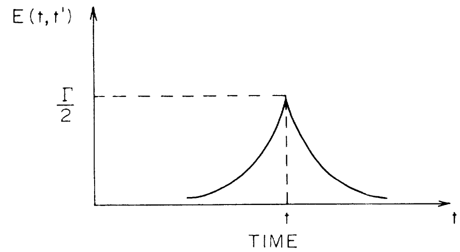


FIG. 3. Exponential noise correlation function $E(t,t')$.

$$\sum_j \Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t+t'-2t_j)}=\frac{R}{2\Gamma}e^{-\Gamma|t-t'|}\equiv\frac{R}{\Gamma^2}E(t,t')\ . \tag{B3}$$

The function $E(t,t')$ is an exponential time correlation function, defined by

$$E(t,t')=\frac{\Gamma}{2}e^{-\Gamma|t-t'|}\ . \tag{B4}$$

We have again normalized the time correlation function such that its integral is equal to 1. A sketch of $E(t,t')$ is shown in Fig. 3.

APPENDIX C

Here we calculate the commutators of Eqs. (64) and (82). For this we use a matrix representation for the atomic operators. If we choose the two atomic levels a and b as a basis, the atomic dipole operator σ^j can be written as

$$\sigma^j=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}_j\ . \tag{C1}$$

The interaction part V_j of the Hamiltonian then acquires the form

$$V_j=\begin{bmatrix}0 & a\\ a^\dagger & 0\end{bmatrix}_j\ . \tag{C2}$$

The initial condition for the jth atom is given by

$$\rho_j(t_j)=\begin{bmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{bmatrix}_j\ . \tag{C3}$$

Using Eqs. (C2) and (C3) the commutators and traces can be easily evaluated

$$\mathrm{Tr}_{A_j}[V_j,\rho_j(t_j)\otimes\rho^f] = \mathrm{Tr}\begin{bmatrix}\rho_{ba}a\rho^f-\rho_{ab}\rho^f a^\dagger & \rho_{bb}a\rho^f-\rho_{aa}\rho^f a\\ \rho_{aa}a^\dagger\rho^f-\rho_{bb}\rho^f a^\dagger & \rho_{ab}a^\dagger\rho^f-\rho_{ba}\rho^f a\end{bmatrix} = \rho_{ab}(a^\dagger\rho^f-\rho^f a^\dagger)+\rho_{ba}(a\rho^f-\rho^f a)\ , \tag{C4}$$

in which we have suppressed the explicit time dependence of the field operator ρ^f.

In a similar way, we find

$$\mathrm{Tr}_{A_j}[V_j,[V_j,\rho_j(t_j)\otimes\rho^f]] = \rho_{aa}(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a) + \rho_{bb}(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) \tag{C5}$$

and

$$\mathrm{Tr}_{A_j}[V_j,\rho_j(t_j)\otimes \mathrm{Tr}_{A_j}[V_j,\rho_j(t_j)\otimes\rho^f]] = |\rho_{ab}|^2(aa^\dagger\rho^f+\rho^f aa^\dagger-2a^\dagger\rho^f a+a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) + \rho_{ab}^2(a^\dagger a^\dagger\rho^f+\rho^f a^\dagger a^\dagger-2a^\dagger\rho^f a^\dagger)+\rho_{ba}^2(aa\rho^f+\rho^f aa-2a\rho^f a)\ . \tag{C6}$$

APPENDIX D

In this appendix we evaluate the various sums and integral appearing in the calculations for the inhomogeneously broadened polarization CEL. For this we again note that we can group the atoms according to their frequency and their injection time. Assuming a regular injection of the atoms, we can then make the substitution

$$\sum_{\substack{j\\ \text{atoms}}} \rightarrow \sum_{\substack{k\\ \text{frequencies}}} r_k\int_{-\infty}^{\infty} dt_j\ . \tag{D1}$$

Here the integration over the injection times t_j is performed in each frequency subgroup. We can now evaluate the sums and integrals in Eqs. (91) and (97):

$$\begin{aligned}\sum_j \Theta(t-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t_j)} &= R\sum_k p_k\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)e^{-[\Gamma+i(\omega_k-\omega_0)](t-t_j)}\\ &= R\sum_k p_k\frac{1}{\Gamma+i(\omega_k-\omega_0)}\\ &= \frac{R}{\Gamma}\sum_k p_k\frac{\Gamma^2-i\Gamma(\omega_k-\omega_0)}{\Gamma^2+(\omega_k-\omega_0)^2}=\frac{R}{\Gamma}(W_1-iW_2)\ .\end{aligned} \tag{D2}$$

The weight functions W_1 and W_2 are defined by

$$W_1=\sum_k p_k\frac{\Gamma^2}{\Gamma^2+(\omega_k-\omega_0)^2}\ , \tag{D3}$$

$$W_2=\sum_k p_k\frac{\Gamma(\omega_k-\omega_0)}{\Gamma^2+(\omega_k-\omega_0)^2}\ . \tag{D4}$$

If we adopt the shorthand notation $x_k=\omega_k-\omega_0/\Gamma$, then Eqs. (D3) and (D4) can be written as

$$W_1=\sum_k p_k\frac{1}{1+x_k^2}=\left\langle\frac{1}{1+x^2}\right\rangle\ , \tag{D5}$$

$$W_2=\sum_k p_k\frac{x_k}{1+x_k^2}=\left\langle\frac{x}{1+x^2}\right\rangle\ , \tag{D6}$$

in which $\langle\ \rangle$ denotes an average over the variable x. For the expression of the linear gain term in Eq. (91) we obtain

$$\begin{aligned}&\int_{-\infty}^{t}dt'\sum_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t'-t_j)}\times e^{-[\Gamma+i(\omega_j-\omega_0)](t-t')}\\ &=R\sum_k p_k\int_{-\infty}^{t}dt'e^{-[\Gamma+i(\omega_k-\omega_0)](t-t')}\times\int_{-\infty}^{\infty}dt_j\sum_j\Theta(t-t_j)\Theta(t'-t_j)\times e^{-\Gamma(t'-t_j)}\\ &=\frac{R}{\Gamma}\sum_k p_k\int_{-\infty}^{t}dt'e^{-[\Gamma+i(\omega_k-\omega_0)](t-t')}\\ &=\frac{R}{\Gamma^2}(W_1-iW_2)\ .\end{aligned} \tag{D7}$$

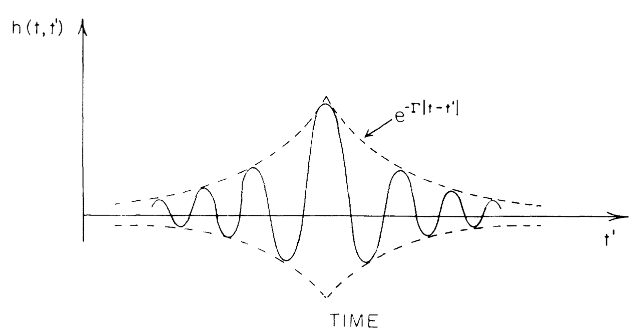


FIG. 4. Correlation function $h(t-t')$ as defined by Eq. (D9).

For the noise correlation functions we have to evaluate the following sums:

$$\sum_j \Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma-i(\omega_j-\omega_0)](t-t_j)}e^{-[\Gamma+i(\omega_j-\omega_0)](t'-t_j)}$$

$$=R\sum_k p_k e^{i(\omega_k-\omega_0)(t-t')}\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)\Theta(t'-t_j)e^{-\Gamma(t+t'-2t_j)}$$

$$=\frac{R}{2\Gamma}\sum_k p_k e^{-\Gamma|t-t'|}e^{i(\omega_k-\omega_0)(t-t')}$$

$$=\frac{R}{2\Gamma}\sum_k p_k\{\cos[(\omega_k-\omega_0)(t-t')]e^{-\Gamma|t-t'|}+i\sin[(\omega_k-\omega_0)(t-t')]e^{-\Gamma|t-t'|}\} . \quad \text{(D8)}$$

Let us discuss the time dependence of the first terms in the last expression of Eq. (D8). The first term is given by the function

$$h(t-t')\equiv e^{-\Gamma|t-t'|}\cos[(\omega_k-\omega_0)(t-t')] , \quad \text{(D9)}$$

which is sketched in Fig. 4.

If the atomic lifetime Γ^{-1} is very short as compared with the time scale of interest, we can make the Markov approximation:

$$h(t-t')\simeq\left[\int_{-\infty}^{\infty}d\tau\, h(\tau)\right]\delta(t-t')=\frac{2\Gamma}{\Gamma^2+(\omega_k-\omega_0)^2}\delta(t-t') . \quad \text{(D10)}$$

Under the same approximation we find

$$e^{-\Gamma|t-t'|}\sin[(\omega_k-\omega_0)(t-t')]\simeq 0 . \quad \text{(D11)}$$

Substituting Eqs. (D10) and (D11) into (D8) yields

$$\sum_j \Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma-i(\omega_j-\omega_0)](t-t_j)}e^{-[\Gamma+i(\omega_j-\omega_0)](t'-t_j)}=\frac{R}{\Gamma^2}\left[\sum_k p_k\frac{\Gamma^2}{\Gamma^2+(\omega_k-\omega_0)^2}\right]\delta(t-t')$$

$$=\frac{R}{\Gamma^2}W_1\delta(t-t') . \quad \text{(D12)}$$

Finally we have to evaluate the integrals and sums which appear in the coherence terms of the noise correlation functions $\langle F_a^\dagger F_a\rangle$, $\langle F_a F_a\rangle$, and $\langle F_a^\dagger F_a^\dagger\rangle$ [c.f. Eq. (97)]:

$$\int_{-\infty}^{t}ds\int_{-\infty}^{t'}ds'\sum_j\Theta(s-t_j)\Theta(s'-t_j)e^{-[\Gamma-i(\omega_j-\omega_0)](t-s)}e^{-[\Gamma+i(\omega_j-\omega_0)](t'-s')}\Gamma e^{-\Gamma(s-t_j)}\delta(s-s')$$

$$=R\sum_k p_k e^{i(\omega_k-\omega_0)(t-t')}\int_{-\infty}^{\min(t,t')}ds' e^{-\Gamma(t+t'-2s)}\int_{-\infty}^{\infty}dt_j\Theta(s-t_j)\Gamma e^{-\Gamma(s-t_j)}$$

$$=\frac{R}{2\Gamma}\sum_k p_k e^{i(\omega_k-\omega_0)(t-t')}e^{-\Gamma|t-t'|}$$

$$=\frac{R}{\Gamma^2}W_1\delta(t-t') . \quad \text{(D13)}$$

Furthermore,

$$\sum_j \Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma+i(\omega_j-\omega_0)](t-t_j)}e^{-[\Gamma+i(\omega_j-\omega_0)](t'-t_j)}$$

$$=R\sum_k p_k\int_{-\infty}^{\infty}dt_j\Theta(t-t_j)\Theta(t'-t_j)e^{-[\Gamma+i(\omega_k-\omega_0)](t+t'-2t_j)}$$

$$=\frac{R}{2\Gamma}\sum_k p_k\frac{\Gamma}{\Gamma+i(\omega_k-\omega_0)}e^{-[\Gamma+i(\omega_k-\omega_0)]|t-t'|}$$

$$=\frac{R}{2\Gamma}\sum_k p_k\frac{\Gamma}{\Gamma+i(\omega_k-\omega_0)}\left[\cos[(\omega_k-\omega_0)(t-t')]e^{-\Gamma|t-t'|}-i\sin[(\omega_k-\omega_0)(t-t')]e^{-\Gamma|t-t'|}\right]$$

$$\simeq\frac{R}{2\Gamma}\sum_k p_k\frac{\Gamma}{\Gamma+i(\omega_k-\omega_0)}\left[\frac{2\Gamma}{\Gamma^2+(\omega_k-\omega_0)^2}-i\frac{2(\omega_k-\omega_0)}{\Gamma^2+(\omega_k-\omega_0)^2}\right]\delta(t-t')$$

$$=\frac{R}{\Gamma^2}(W_3-iW_4)\delta(t-t') \,. \tag{D14}$$

In the last step we have defined the weight function W_3 and W_4 as

$$W_3=\left\langle \frac{1-x^2}{(1+x^2)^2} \right\rangle \,, \tag{D15}$$

$$W_4=\left\langle \frac{2x}{(1+x^2)^2} \right\rangle \,. \tag{D16}$$

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