Non-Hermitian evolution of two-level quantum systems

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In this paper we discuss in detail the dynamics of a two-state system ruled by a non-Hermitian Hamiltonian. The relevance of the results to multiphoton ionization is also discussed and a comparison to earlier works is presented.

I. INTRODUCTION

A feeling of reluctance is widespread among physicists against the use of a non-Hermitian Hamiltonian (NHH), and consequently of a non-Hermitian Schrödinger equation (NHSE), to treat physical problems. For this reason the literature on the subject suffered for a long time from strong limitations and fragmentation.[1] Only recently, in connection with various problems ranging from the multiphoton ionization[2] to transverse mode propagation in optical resonators[3] and to free-electron lasers[4] rigorous and practical methods have been developed to study the time dependence of non-Hermitian Schrödinger (NHS) equations.

The deep reasons underlying the non-Hermiticity of a Hamiltonian describing a physical process can be traced back to the Fock-Krylov theorem.[5] The theorem states that the necessary and sufficient condition for a quantum-mechanical state to be a truly decaying state [i.e., such that the probability $p(t)$ of a system in this state tends to zero for large times t] is that the energy distribution of the state be a continuum,

The possibility of treating any system, with a part or the whole of its motion in the continuum, as a decaying state is therefore ensured by virtue of this theorem. The finite lifetime of a decaying state and the uncertainty principle ensure that its energy cannot be sharply peaked, since the resulting width of the energy level is represented by an imaginary component of the energy parameter of the state. The complex energies can be shown to arise as eigenvalues of a NHH associated with the decaying process. Sometimes the NHH's are constructed heuristically, as in the case of two-level systems in which phenomenological diagonal decaying constants are introduced.[6] On the other hand, the use of the projection-operator technique[7,8] and that of optical potentials[9] allows one to derive a rigorous theory of a NHH, and a practical algorithm of nonperturbative nature has been developed to study the time dependence of a quantum-mechanical system ruled by a NHS equation.[10]

In this paper we will discuss the solution of the time-dependent NHSE of a two-level system using an algorithm developed by two of the present authors[11] (G.D. and A.T.) based on the spinor image solution technique of parabolic equations.[12] We will analyze the mathematical aspects of the two-state models with effective *decay* terms in a non-Hermitian Hamiltonian[13] because, due to their relative simplicity, they may represent a systematic tool for investigating the quantum theory of multiphoton ionization.[14]

II. NON-HERMITIAN HAMILTONIAN AND TWO-STATE EVOLUTION: THE SPINOR REPRESENTATION

In this section we will not dwell on a rigorous derivation of a NHS equation, which can be found in Refs. 1 and 10 where use has been made of the Feshbach[7] and Cohen-Tannoudji[8] projection-operator technique. For completeness, we will however recall some important properties of a NHH and NHSE; the interested reader is addressed to Ref. 15 for a more complete and rigorous treatment. In the following we will denote by $\hat{H}$ the Hamiltonian and by $\hat{H}^{+}(\neq\hat{H})$ its adjoint and suppose that they satisfy the following eigenvalue equations:

$$\hat{H}|\Psi_i\rangle=\lambda_i|\Psi_i\rangle\ , \tag{2.1a}$$

$$\hat{H}^{+}|\chi_i\rangle=\epsilon_i|\chi_i\rangle\ . \tag{2.1b}$$

We are now interested in the relationship between the eigenvalues (λ_i,ϵ_i) and the eigenvectors $(|\Psi_i\rangle,|\chi_i\rangle)$.

From (2.1b) we get

$$\langle\chi_i|\hat{H}=\langle\chi_i|\epsilon_i^*\ . \tag{2.2}$$

Projecting (2.2) onto $|\Psi_i\rangle$ we find

$$\langle\chi_i|\hat{H}|\Psi_i\rangle=\langle\chi_i|\epsilon_i^*|\Psi_i\rangle=\langle\chi_i|\lambda_i|\Psi_i\rangle \tag{2.3}$$

and thus

$$\lambda_i=\epsilon_i^*\ . \tag{2.4}$$

Furthermore, we also get

$$\langle\chi_i|\hat{H}|\Psi_j\rangle=\langle\chi_i|\lambda_j|\Psi_j\rangle=\langle\chi_i|\epsilon_i^*|\Psi_j\rangle\ , \tag{2.5}$$

whence it immediately follows that

$$(\lambda_j-\epsilon_i^*)\langle\chi_i|\Psi_j\rangle=0\ , \tag{2.6}$$

which together with (2.4) allows one to choose

$$\langle\chi_i|\Psi_j\rangle=\delta_{ij}\ . \tag{2.7}$$

Both results (2.4) and (2.7) are crucial in the theory of a NHH. The eigenvalues of $\hat{H}$ are a complex conjugate to those of its adjoint, while the concomitant eigenvectors are biorthogonal to each other. Accordingly, a generic state $|\Psi\rangle$ can be expanded as

$$\begin{aligned}|\Psi\rangle&=\sum_j a_j|\Psi_j\rangle\ ,\\ |\Psi\rangle&=\sum_j \bar{a}_j|\chi_j\rangle\ ,\end{aligned} \tag{2.8}$$

with the coefficients a_j and $\bar{a}_j$ given by

$$\begin{aligned}a_j&=\langle\chi_j|\Psi\rangle\ ,\\ \bar{a}_j&=\langle\Psi_j|\Psi\rangle\ .\end{aligned} \tag{2.9}$$

Finally, from (2.8) and (2.9) it follows that

$$|\Psi\rangle=\sum_j|\Psi_j\rangle\langle\chi_j|\Psi\rangle\ , \tag{2.10}$$

thus implying the closure or completeness property

$$\sum_j|\Psi_j\rangle\langle\chi_j|=1\ . \tag{2.11}$$

So far we have discussed the stationary case; the time evolution of a state ruled by a NHH can be studied using almost conventional tools. An evolution operator can be introduced as for the Hermitian case, and therefore well-documented and reliable methods can be exploited. The evolution operator $\hat{U}(t)$ associated with $\hat{H}$ is necessarily nonunitary; it is therefore convenient to introduce $\hat{\bar{U}}$, namely, the evolution operator of the adjoint, and it is straightforward to prove that

$$\hat{U}\hat{\bar{U}}^{+}=\hat{\bar{U}}^{+}\hat{U}=\hat{1}\ . \tag{2.12}$$

In fact,

$$\begin{aligned}i\hbar\frac{d}{dt}(\hat{\bar{U}}^{+}\hat{U})&=-\hat{\bar{U}}^{+}\hat{H}\hat{U}+\hat{\bar{U}}^{+}\hat{H}\hat{U}\\ &=-\hat{\bar{U}}^{+}(\hat{H}-\hat{H})\hat{U}=0\end{aligned}$$

and thus

$$\hat{\bar{U}}\hat{U}^{+}=\hat{1}\ .$$

The time evolution of a generic state $|\Psi(0)\rangle$ undergoing a non-Hermitian interaction is therefore given by

$$|\Psi(t)\rangle=\hat{U}(t)|\Psi(0)\rangle\ . \tag{2.13}$$

Owing to the *nonunitary* nature of $\hat{U}$, $|\Psi(0)\rangle$ develops a truly decaying probability configuration. In general, a time-dependent state can be expanded on an orthogonal or a biorthogonal basis as well; therefore, analogous to (2.9), we get

$$\begin{aligned}a_n(t)&=\langle\chi_n|\hat{U}(t)|\Psi(0)\rangle\ ,\\ \bar{a}_n(t)&=\langle\Psi_n|\hat{U}(t)|\Psi(0)\rangle\ .\end{aligned} \tag{2.14}$$

In order to stress the deep implications of the lack of unitarity of the evolution operator, let us notice that it can be shown [see Ref. 15(c)] that the operators $\hat{U}(t)$ and $\hat{\bar{U}}$ obey different semigroup (rather than group) properties, in the sense that

$$\begin{aligned}&\hat{U}(t)\hat{U}(s)=\hat{U}(t+s);t,s\geq 0\ ,\\ &\hat{\bar{U}}(t)\hat{\bar{U}}(s)=\hat{\bar{U}}(t+s);t,s\leq 0\ .\end{aligned}$$

In other words, the operators $\hat{H}$ and $\hat{H}^{\dagger}$ are forward and backward time generators, respectively (see Ref. 10 for a physical interpretation). This is just an example of the unconventional features arising when NHH's come into play. Another basic one is connected to the Heisenberg equations of motion for operators [see Eq. (2.5) below and the Appendix].

It is finally worth noticing that

$$\begin{aligned}|\Psi_n(t)\rangle&=\hat{U}(t)|\Psi_n\rangle\ ,\\ |\chi_n(t)\rangle&=\hat{\bar{U}}(t)|\chi_n\rangle\end{aligned} \tag{2.15}$$

form a biorthogonal basis at any time. In this paper we will be concerned with the explicit evaluation of the evolution operator for a NHH of two-level systems using the spinor image method of Ref. 12 and then ordering methods of Wei and Norman.[16] We must, however, underscore the fact that the possible non-Hermitian nature of a Hamiltonian is by no means a drawback to using conventional time-ordering methods of the type described, e.g., in Refs. 17 and 18.

In this section we will consider in detail the dynamical behavior of a non-Hermitian (NH) two-level system. According to Ref. 10 the Hamiltonian we shall investigate is

$$\begin{aligned}\hat{H}=&\frac{1}{2}\left[\delta(t)-\frac{i}{2}[\gamma_1(t)+\gamma_2(t)]\right]\hat{1}\\ &+\left[\delta(t)-\frac{i}{2}[\gamma_2(t)-\gamma_1(t)]\right]\hat{\sigma}_3\\ &-\Omega_1(t)\hat{\sigma}_+-\Omega_2(t)\hat{\sigma}_-\ ,\end{aligned} \tag{2.16}$$

where $\hat{1}$ is the unit matrix and $\hat{\sigma}_{3,\pm}$ are Pauli operators. The 2×2 matrix image of (2.16) is

$$\hat{H}(t)=\begin{bmatrix}\delta(t)-\frac{i}{2}\gamma_2(t) & -\Omega_1(t)\\ -\Omega_2(t) & -\frac{i}{2}\gamma_1(t)\end{bmatrix}\ . \tag{2.17}$$

All the matrix elements are assumed to be time dependent, thus accounting either for eventual *switching* and time profile of the interaction or because the interaction is included semiclassically. The time-dependent detuning $\delta(t)$ includes possible time-dependent Stark shifts. The evolution operator associated to the Hamiltonian (2.17) is specified by the following 2×2 matrix:[11]

$$\hat{U}=\begin{bmatrix}L & G\\ F & H\end{bmatrix}\ , \tag{2.18}$$

whose entries are related to those of (2.17) by the follow-

ing system of differential equations:

$$i\dot{L}=\left[\delta-\frac{i}{2}\gamma_2\right]L-\Omega_1 F,\quad L(0)=1$$
$$i\dot{F}=-\Omega_2 L-i\frac{\gamma_1}{2}F,\quad F(0)=0 \tag{2.19a}$$

and

$$i\dot{G}=\left[\delta-\frac{i}{2}\gamma_2\right]G-\Omega_1 H,\quad G(0)=0$$
$$i\dot{H}=-\Omega_2 G-\frac{i}{2}\gamma_1 H,\quad H(0)=1\ . \tag{2.19b}$$

The elements of the concomitant evolution operator $\hat{\bar{U}}$ are instead specified by

$$i\dot{\bar{L}}=\left[\delta+\frac{i}{2}\gamma_2\right]\bar{L}-\Omega_2^*\bar{F},\quad \bar{L}(0)=1$$
$$i\dot{\bar{F}}=-\Omega_1^*\bar{L}+\frac{i}{2}\gamma_1\bar{F},\quad \bar{F}(0)=0 \tag{2.20a}$$

and

$$i\dot{\bar{G}}=\left[\delta-\frac{i}{2}\gamma_2\right]\bar{G}-\Omega_2^*\bar{H},\quad \bar{G}(0)=0$$
$$i\dot{\bar{H}}=-\Omega_1^*\bar{G}+\frac{i}{2}\gamma_1\bar{H},\quad \bar{H}(0)=1\ . \tag{2.20b}$$

The unitarity condition according to (2.12) can be written in matrix form

$$\hat{\bar{U}}^{+}\hat{U}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{2.21}$$

and is an immediate consequence of the relations

$$L\bar{L}^*+F\bar{F}^*=1,\quad G\bar{G}^*+H\bar{H}^*=1\ ,$$
$$G\bar{L}^*+H\bar{F}^*=0,\quad L\bar{G}^*+F\bar{H}^*=0\ , \tag{2.22}$$

which can be deduced directly from Eqs. (2.19) and (2.20) [or otherwise simply stated once (2.12) is assumed].

It must be stressed that conditions (2.22) arise naturally in our formalism and are not imposed (see also Ref. 10). In the hypothesis that the Hamiltonian (2.16) and (2.17) is time independent all the entries of the $\hat{U}$ matrix can be explicitly derived and read

$$L=\left[-\frac{i}{\sqrt{\Delta}}\left[\delta-\frac{i}{2}(\gamma_2-\gamma_1)\right]\sin\left[\frac{\sqrt{\Delta}}{2}t\right]+\cos\left[\frac{\sqrt{\Delta}}{2}t\right]\right]e^{-(i/2)[\delta-(i/2)(\gamma_2+\gamma_1)]t}\ ,$$
$$G=+\frac{2i\Omega_1}{\sqrt{\Delta}}\sin\left[\frac{\sqrt{\Delta}}{2}t\right]e^{-(i/2)[\delta-(i/2)(\gamma_2+\gamma_1)]t}\ ,$$
$$F=+\frac{2i\Omega_2}{\sqrt{\Delta}}\sin\left[\frac{\sqrt{\Delta}}{2}t\right]e^{-(i/2)[\delta-(i/2)(\gamma_2+\gamma_1)]t}\ , \tag{2.23}$$
$$H=\left[\frac{i}{\sqrt{\Delta}}\left[\delta-\frac{i}{2}(\gamma_2-\gamma_1)\right]\sin\left[\frac{\sqrt{\Delta}}{2}t\right]+\cos\left[\frac{\sqrt{\Delta}}{2}t\right]\right]e^{-(i/2)[\delta-(i/2)(\gamma_2+\gamma_1)]t}\ ,$$
$$\Delta=\left[\delta-\frac{i}{2}(\gamma_2-\gamma_1)\right]^2+4\Omega_1\Omega_2\ .$$

Setting, for example, $\gamma_1=\gamma_2=\gamma$ and $\Omega_1=\Omega_2=\Omega$ and assuming that the system is initially specified by the spinor $\binom{1}{0}$ it is easy to realize that the evolution of the population inversion behaves like

$$|L|^2-|F|^2=e^{-\gamma t}\frac{\delta^2+4\Omega^2\cos(\Delta t)}{\Delta}\ . \tag{2.24}$$

We can, however, treat the problem in all its generality. Let us therefore represent the state describing the two-level system by the spinor

$$|\Psi\rangle=\begin{bmatrix}a_1\\ a_2\end{bmatrix}\ , \tag{2.25}$$

where the time-dependent components $a_{1,2}$ give the probability that the level 1,2 is occupied at the generic time t.

Then we introduce the Bloch vector **s** with components

$$\langle\hat{\sigma}_j\rangle=\langle\Psi|\hat{\sigma}_j|\Psi\rangle,\quad j=1,2,3 \tag{2.26}$$

where $\hat{\sigma}_j$ are the Pauli matrices. As usual,

$$\hat{\sigma}_1=\frac{\hat{\sigma}_+ +\hat{\sigma}_-}{2},\quad \hat{\sigma}_2=\frac{\hat{\sigma}_+ -\hat{\sigma}_-}{2i}\ .$$

the dynamics of the system is well described by the evolution of **s**. Noticing that

$$\langle\Psi|\hat{\sigma}_j|\Psi\rangle=\langle\Psi_0|\hat{U}^{+}\hat{\sigma}_j\hat{U}|\Psi_0\rangle\ , \tag{2.27}$$

with $|\Psi_0\rangle$ denoting the spinor (2.25) at the initial time, after some cumbersome algebra one gets

$$\langle\hat{\sigma}_j(t)\rangle=A_j(t)+\sum_{i=1}^{3}A_{j,i}(t)\langle\hat{\sigma}_i(0)\rangle\ , \tag{2.28}$$

where the time-dependent functions $A_j(t)$ and $A_{j,i}(t)$ are specified by

$$A_1(t)=\tfrac{1}{2}\mathrm{Re}(L^*F+HG^*)\ ,$$
$$A_2(t)=\tfrac{1}{2}\mathrm{Im}(L^*F+GH^*)\ , \tag{2.29}$$
$$A_3(t)=\tfrac{1}{4}(|L|^2-|F|^2+|G|^2-|H|^2)\ ;$$

and

$$A_{1,1}=\mathrm{Re}(L^*H+F^*G)\ ,$$
$$A_{1,2}=\mathrm{Im}(L^*H-F^*G)\ , \tag{2.30a}$$
$$A_{1,3}=\mathrm{Re}(LF^*-H^*G)\ ;$$
$$A_{2,1}=\mathrm{Im}(L^*H+G^*F)\ ,$$
$$A_{2,2}=\mathrm{Re}(L^*H-G^*F)\ , \tag{2.30b}$$
$$A_{2,3}=\mathrm{Im}(L^*F-G^*H)\ ;$$
$$A_{3,1}=\mathrm{Re}(L^*G-F^*H)\ ,$$
$$A_{3,2}=-\mathrm{Im}(L^*G-F^*H)\ , \tag{2.30c}$$
$$A_{3,3}=\tfrac{1}{2}[|L|^2-|F|^2-(|G|^2-|H|^2)]\ .$$

In deriving the above expressions it has been assumed that the spinor $|\Psi_0\rangle$ is normalized to unity. The solution (2.29) represents a kind of Rabi-rotation matrix for the non-Hermitian dynamical behavior of the Bloch vector.[6] It is worth stressing that, within the present context, the norm of $|\Psi\rangle$ and thus the modulus of **s**, is not a constant of motion, but evolves in time according to

$$|\Psi|^2=2\,\mathrm{Re}(L^*G+F^*H)\langle\sigma_1(0)\rangle-2\mathrm{Im}(L^*G+F^*H)\langle\sigma_2(0)\rangle+\tfrac{1}{2}(|L|^2+|F|^2-|G|^2-|H|^2)\langle\sigma_3(0)\rangle$$
$$+\tfrac{1}{2}(|L|^2+|F|^2+|G|^2+|H|^2)\ . \tag{2.31}$$

If $\hat{H}$ is Hermitian then we have

$$L=H^*,\quad F=-G^*\ ; \tag{2.32}$$

the A_j functions vanish and $A_{i,j}$ are just the elements of the generalized Rabi-rotation matrix derived in Ref. 19. Correspondingly the norm of $|\Psi\rangle$ does not change with time as it can be easily inferred from (2.32) and from the unitarity of $\hat{U}$ that implies[11]

$$|H|^2+|F|^2=1\ . \tag{2.33}$$

A law of conservation can be, however, associated to the NH Bloch dynamics. Defining indeed a Bloch vector **s** with components

$$\langle\hat{\sigma}_j\rangle=\langle\chi(t)|\hat{\sigma}_j|\Psi(t)\rangle \tag{2.34}$$

one can easily prove that the norm of **s** is a constant of motion.

It is useful to gain a geometrical insight in the evolution of the vector **s**. We derive the equations of motion for the expectation values of the operators $\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3$ using the following modified Heisenberg equations (see the Appendix for further comments):

$$\frac{d}{dt}\langle|\hat{A}|\rangle=\left\langle\left|\frac{\partial}{\partial t}\hat{A}\right|\right\rangle+i\langle|\hat{H}^{\dagger}\hat{A}-\hat{A}\hat{H}|\rangle\quad(\hbar=1)\ , \tag{2.35}$$

thus getting immediately

$$\frac{d}{dt}\begin{bmatrix}\langle\hat{\sigma}_1\rangle\\ \langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}=\begin{bmatrix}2\,\mathrm{Im}\omega_1 & -\mathrm{Re}\omega_2 & \mathrm{Im}(\Omega_1-\Omega_2) & \frac{1}{2}\mathrm{Im}(\Omega_1+\Omega_2)\\ \mathrm{Re}\omega_2 & 2\,\mathrm{Im}\omega_1 & \mathrm{Re}(\Omega_1+\Omega_2) & -\frac{1}{2}\mathrm{Re}(\Omega_1-\Omega_2)\\ -\mathrm{Im}(\Omega_1-\Omega_2) & -\mathrm{Re}(\Omega_1-\Omega_2) & 2\,\mathrm{Im}\omega_1 & 2\,\mathrm{Im}\omega_2\\ -2\,\mathrm{Im}(\Omega_1+\Omega_2) & -\mathrm{Re}(\Omega_1-\Omega_2) & 2\,\mathrm{Im}\omega_2 & 2\,\mathrm{Im}\omega_1\end{bmatrix}\begin{bmatrix}\langle\hat{\sigma}_1\rangle\\ \langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}\ . \tag{2.36}$$

For convenience we have defined

$$\omega_1=\delta-\tfrac{1}{2}(\gamma_1+\gamma_2)\ ,$$
$$\omega_2=\delta+\tfrac{1}{2}(\gamma_1-\gamma_2)\ .$$

In this case the evolution of the system (2.36) is specified by the further quantity $\langle\hat{I}\rangle$ due to the fact that *the operator* $\hat{U}^{+}\hat{U}$ is not a constant of motion [see Eq. (2.35)], from the physical point of view $\langle\hat{I}\rangle$ can be directly linked to the total bound population.

The motion of the three-component vector

$$\bar{\mathbf{S}}_i=\langle\hat{\sigma}_i\rangle\exp\left[2\int_{t_0}^{t}\mathrm{Im}\omega_1(t')dt'\right],\ i=1,2,3 \tag{2.37}$$

can be cast in the following Bloch-type form:

$$\dot{\bar{\mathbf{S}}}=\mathbf{Q}\times\bar{\mathbf{S}}+\mathbf{T}\ , \tag{2.38}$$

the vector **Q** is the torque vector given by

$$\mathbf{Q}=(-\mathrm{Re}(\Omega_1+\Omega_2),\mathrm{Im}(\Omega_1-\Omega_2),\mathrm{Re}\omega_2)\ , \tag{2.39}$$

and **T**, the damping vector, is defined as

$$\mathbf{T}=\left[\tfrac{1}{2}\exp\left[-2\int_{t_0}^{t}\mathrm{Im}\omega_1(t')dt'\right]\right]\times(\mathrm{Im}(\Omega_1+\Omega_2),-\mathrm{Re}(\Omega_1-\Omega_2),4\,\mathrm{Im}\omega_2)\ . \quad (2.40)$$

It is easy to realize that the presence of $\mathbf{T}$ causes the nonconservation of the norm of $\mathbf{S}$ whose time behavior is fixed by

$$\frac{d}{dt}|\bar{\mathbf{S}}|^2=2\mathbf{T}\cdot\bar{\mathbf{S}}\ . \quad (2.41)$$

We must underscore the fact that Eq. (2.28) represents the most general form of the transition amplitudes for a NH two-level problem.

As it happens for the Hermitian case, the evolution problem is completely solved once the time behavior of the (L,G,F,H) is totally specified, or, equivalently, when Eqs. (2.19) are solved. As is well known, this is a formidable problem and analytical solutions can be obtained in few cases.[20] Equations (2.19) can be cast in the form

$$\ddot{\tilde{L}}-\frac{\dot{\tilde{\Omega}}_1}{\tilde{\Omega}_1}\dot{\tilde{L}}+\tilde{\Omega}_1\tilde{\Omega}_2\tilde{L}=0\ ,\qquad \ddot{\tilde{F}}-\frac{\dot{\tilde{\Omega}}}{\tilde{\Omega}_1}\dot{\tilde{F}}+\tilde{\Omega}_1\tilde{\Omega}_2\tilde{F}=0\ , \quad (2.42)$$

after setting

$$F=\exp\left[-\tfrac{1}{2}\int_0^t\gamma_1 dt'\right]\tilde{F}\ , \quad (2.43a)$$

$$L=\exp\left[-i\int_0^t\left[\delta-\frac{i}{2}\gamma_2\right]dt'\right]\tilde{L}\ ,$$

and

$$\tilde{\Omega}_1=\Omega_1\exp\left[-i\int_0^t\delta(t')dt'\right]\exp\left[-\tfrac{1}{2}\int_0^t(\gamma_1+\gamma_2)dt'\right]\ , \quad (2.43b)$$

$$\tilde{\Omega}_2=\Omega_2\exp\left[+i\int_0^t\delta(t')dt'\right]\exp\left[+\tfrac{1}{2}\int_0^t(\gamma_1+\gamma_2)dt'\right]\ .$$

The initial conditions of Eqs. (2.42) can be inferred from (2.19) and (2.43) and read

$$\tilde{L}(0)=1,\quad \dot{\tilde{L}}(0)=0\ ,\qquad \tilde{F}(0)=0,\quad \dot{\tilde{F}}(0)=i\Omega_2\ . \quad (2.44)$$

As already remarked, Eq. (2.31) can be analytically solved in a very limited number of cases. Second-order differential equations with time-dependent coefficients have been the topics of intensive research during these last years.[20–22] Simple criteria have been developed to specify functional dependences of the coefficients which allow solutions in terms of known special functions. These methods, originated by the pioneering work of Bambini and Berman, consist in finding the appropriate change of variable which allows the mapping of the differential equation under study on an equation with known solutions. The first of equations (2.42) can be, for example, rewritten in the form

$$q''(x)+\frac{1}{\dot{x}}\left[\frac{d}{dt}\ln\dot{x}-\frac{d}{dt}\ln\tilde{\Omega}_1\right]q'(x)+\frac{\tilde{\Omega}_1\tilde{\Omega}_2}{x^2}q(x)=0\ , \quad (2.45)$$

where $x=x(t)$ is the mapping variable, $q(x)=L(t)$, the prime and the dot denote derivatives with respect to x and t, respectively. Equation (2.45) can be reduced to a Liouville standard form setting

$$q(x)=v(x)u(x)\ , \quad (2.46)$$

$$v(x)=\exp\left[-\frac{1}{2}\int_{x_0}^{x}A(x')dx'\right]\ ,\qquad A(x)=\frac{1}{\dot{x}}\frac{d}{dt}\ln(\dot{x}/\tilde{\Omega}_1)\ , \quad (2.47)$$

and with $u(x)$ satisfying the following pendulumlike equation:

$$u''(x)+\mu(x)u=0\ ,\qquad \mu(x)=\frac{\chi_1(x)\chi_2(x)}{x^2}-\frac{1}{4}\left[A(x)\right]^2-\frac{1}{2}\frac{d}{dx}A(x)\ , \quad (2.48)$$

$$\chi_{1,2}(x)=\Omega_{1,2}(t)$$

which can be mapped on a Whittaker equation,[23] provided that

$$x=e^{-|\eta|t},\quad \chi_\alpha^{(x)}=\Omega_a^0 x\qquad A(x)=\frac{a}{x}-b,\quad b=\frac{1}{|\eta|}(|\eta|^2+4\Omega_1^0\Omega_2^0)^{1/2}\ . \quad (2.49)$$

In the case $\delta=0$, we get from Eq. (2.47) that the relaxation functions $\gamma_{1,2}$ have the following time dependence:

$$\tfrac{1}{2}(\gamma_1+\gamma_2)=-a|\eta|+(|\eta|^2+4\Omega_1^0\Omega_1^0)^{1/2}e^{-|\eta|t}\ . \quad (2.50)$$

Choosing $a<0$ to get meaningful results we can finally write

$$\tilde{L}(t)=\exp\left[\frac{1}{2}(b-1)x-\frac{b}{2}\right]\left[\alpha M\left[\frac{a}{2}(1-b),a,x\right]+\beta U\left[\frac{a}{2}(1-b)a,x\right]\right]\quad (x=e^{-|\mu|t})\ , \quad (2.51)$$

where M and U are the independent solutions of the Kummer equation,[23] and α and β are constants depending on the initial conditions.

Within this framework, the solution of the second of Eqs. (2.31) is identical to (2.40), with the only difference being that, apart from the obvious differences due to the different initial conditions, a and b should be replaced by $-a+2$ and $-b$, respectively. More specific examples of physical interest will be discussed in the concluding section.

III. NON-HERMITIAN HAMILTONIAN AND TWO-STATE EVOLUTION: THE WAVE-FUNCTION METHOD

In Sec. II we have discussed a NH two-state evolution using the spinor representation and a generalized Bloch method accounting for both the evolution of the two-level vector and that of the total bounded population. The central results of this analysis have been the proof that the dynamics of the system depend on the coupled linear equations (2.19) and the derivation of the amplitude probability (2.31) which is directly linked to the ionization probability (see Ref. 14 and Sec. IV).

The (L,G,F,H) functions, namely, the characteristic functions of the two-level evolution, can be understood as a generalized version of the Cayley-Klein parameters of the Hermitian two-state problems.[11] The lack of unitarity and unimodularity of the evolution operator does not allow useful relations of the type (2.32) and therefore simple reinterpretations in terms of Euler angles are not possible.[24] In fact, as already stressed, and as implied by Eq. (2.38), the evolution dynamics of the NH two-level system cannot be simply reduced to a rotation of the Bloch vector around a torque, and useful laws of conservation cannot be easily stated. The nonconservation of the norm of the Bloch vector makes quite problematic the analysis of the evolution of the system using angular momentum eigenstates. It might be, however, interesting to deal directly with the evolution of the wave function rather than with the average values and transition matrix elements as discussed in Sec. II. To study the time behavior of the wave function it is worth having the evolution operator in the Wei-Norman ordered form[16]

$$\hat{U}=e^{+2h(t)\hat{J}_3}e^{g(t)\hat{J}_+}e^{-f(t)\hat{J}_-}e^{s(t)\hat{I}} . \tag{3.1}$$

According to Ref. 11 the (h,g,f,s) functions are related to (L,G,F,H) by the following relations (see also Ref. 24):

$$\begin{aligned} &e^{h+s}(1-fg)=L , \\ &e^{h+s}g=G , \\ &fe^{-h+s}=-F , \\ &e^{-h+s}=H ; \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &e^{2s}=HL-FG, \quad f=-\frac{F}{H} , \\ &e^{-2h}=\frac{HL-FG}{H^2}, \quad g=\frac{GH}{HL-FG} . \end{aligned} \tag{3.2b}$$

Suppose that the Hamiltonian (2.1) is that of two coupled Harmonic oscillators and that the $\hat{J}$ operators are written in the Schwinger angular momentum representation, namely,

$$\hat{J}_+=\hat{a}_1^{\dagger}\hat{a}_2, \quad \hat{J}_-=\hat{a}_2^{\dagger}\hat{a}_1, \quad \hat{J}_3=\tfrac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_1-\hat{a}_2^{\dagger}\hat{a}_2)] .$$

Assuming furthermore that the initial state is simply $|\Psi(0)\rangle=|n_1,0\rangle$, we easily find the wave function of the system at a later time t, just given by

$$\begin{aligned} |\Psi(t)\rangle&=\hat{U}(t)|n_1,0\rangle \\ &=e^{s}\sum_{m=0}^{n_1}\frac{[g(t)]^m}{m!}e^{-h(t)(n_1-2m)}|n_1-m,m\rangle . \end{aligned} \tag{3.3}$$

The ordering functions $(\bar{h},\bar{g},\bar{f},\bar{s})$ relevant to the evolution operator associated with $\hat{H}^{+}$ can be obtained using the same method leading to (3.2) and are also straightforwardly derived. If, otherwise, the $\hat{J}$ operators are written in the coordinate representation as

$$\hat{J}_+=x\frac{\partial}{\partial y}, \quad \hat{J}_-=y\frac{\partial}{\partial x}, \quad \hat{J}_3=\frac{1}{2}\left[x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right] , \tag{3.4}$$

and assuming that the initial condition of the Schrödinger problem is $g(x,y)$ we can write[12]

$$\Psi(t;x,y)=e^{s(t)}g[x(t),y(t)] , \tag{3.5}$$

where

$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}L & G\\F & H\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} . \tag{3.6}$$

For practical purposes and for a deeper physical insight the geometrical picture of Sec. II is recommended.

IV. CONCLUSIONS

The results we have presented in Sec. II offer a general and comprehensive treatment of the non-Hermitian dynamics of two-level systems. NHH operators have been the subject of considerable interest during the last years within the framework of the theory of the multiphoton ionization.[10] They in fact permit exact solution by the resolvent-operator technique[14] or by iteration in orders of adiabaticity.[13] Furthermore, a considerable body of literature has been devoted to the time-dependent effects implied by different models and by different shapes of the external couplings Ω.[25] It is therefore worth concluding this paper by adding some further remarks on the generality of our results and their connection to earlier treatments of multiphoton ionization.

The operator (2.16) is a model Hamiltonian for a two-photon ionization process involving two states only, the ground state $|1\rangle$, and one of the excited states $|2\rangle$, resonant with the applied external field. The effect of the remaining nonresonant states is accounted for introducing an effective Hamiltonian into the atom-field interaction part or equivalently the quantities γ_1 and γ_2, physically understood as the rate of two-photon ionization of the ground state, not involving the intermediate state, and the rate of ionization of the excited state, respectively. One of the advantages of the present formulation is that the dynamics of the two-photon ionization can be fully determined, once Eqs. (2.19) specifying the time behavior of the evolution operator are solved. According to the usual procedure the ionization probability $P(t)$ should be given by

$$P(t)=1-|\Psi|^2 , \tag{4.1}$$

where $|\Psi|^2$ has been derived in Sec. II [Eq. (2.31)]. In the hypothesis that the atom-field interaction, abruptly turned on at $t=t_i$ and off at $t=t_f$, can be regarded as constant during the time t_f-t_i, Eqs. (2.19) are solved exactly [see Eqs. (2.23)] so that

$$P(t)=1-\frac{e^{-\Gamma t/2}}{2a^2}\left[\cosh(at\sin\theta)\left[a^2+\delta^2+\frac{\rho^2}{4}+4|\Omega|^2\right]+2a\sinh(at\sin\theta)(\delta\cos\theta+\tfrac{1}{2}\rho\sin\theta)\right.$$
$$\left.+\left[a^2-\delta^2-\frac{\rho^2}{4}-4|\Omega|^2\right]\cos^2(at\cos\theta)+2a\sin(at\cos\theta)(\tfrac{1}{2}\rho\cos\theta-\delta\sin\theta)\right]. \tag{4.2}$$

We have assumed that initially $\langle\hat{\sigma}_1\rangle=\langle\hat{\sigma}_2\rangle=0$ and $\langle\hat{\sigma}_3\rangle=1$; furthermore we have introduced the following notation:

$$\gamma_1+\gamma_2=\Gamma,\quad \gamma_1-\gamma_2=\rho\ ,$$
$$Q^2=\delta^2-\tfrac{1}{4}\rho^2+4|\Omega|^2+i\delta\rho=a^2e^{2i\theta}\ . \tag{4.3}$$

Equation (4.2) is the ionization probability given by Beers and Armstrong and derived using a more cumbersome method based on the so-called resolvent technique.[14] However, within this concern the question about the validity of the procedures, usually adopted in the Hermitian quantum mechanics, in calculating for instance operator expectation values, naturally arises. Accordingly, the theory developed in Ref. 10 suggests that one evaluate the ionization probability as

$$P(t)=1-\sum_r|\langle\chi_r(t)|\Psi(t)\rangle|^2\ , \tag{4.4}$$

the summation involving all the eigenstates $\chi_r(t)$ of the adjoint Hamiltonian $\hat{H}^{\dagger}(t)$.

In particular, for the Hamiltonian (2.16) we are considering, the eigenstates of the adjoint can be represented by the spinors

$$|\chi_-(t)\rangle=\begin{bmatrix}\chi_{2-}\\ \chi_{1-}\end{bmatrix},\quad |\chi_+(t)\rangle=\begin{bmatrix}\chi_{2+}\\ \chi_{1+}\end{bmatrix} \tag{4.5}$$

belonging to the eigenvalues

$$\lambda_{\mp}=\frac{1}{2}\left[\delta+\frac{i}{2}\Gamma\mp\sigma Q^*\right], \tag{4.6}$$

respectively, where $\sigma=\mathrm{sgn}(\delta)$ and Q is given by (4.3).

Correspondingly the components of (4.5) can be inferred from the eigenvalue equation associated with $\hat{H}^{+}$, thus getting

$$\chi_{2\pm}=\frac{2\Omega}{\delta-i\frac{\rho}{2}-\sigma Q^*}\chi_{1\pm}\ . \tag{4.7}$$

For an initial state,

$$|\Psi(t)\rangle=\begin{bmatrix}1\\ 0\end{bmatrix}, \tag{4.8}$$

using the explicit expression of the U-matrix elements L and F. After some algebra we end up with the following $P(t)$:

$$P(t)=1-e^{-\Gamma t/2}\left\{\left[p\cos^2\left[\frac{at}{2}\cos\theta\right]+q\sin^2\left[\frac{at}{2}\cos\theta\right]\right]\cosh^2\left[\frac{at}{2}\sin\theta\right]\right.$$
$$+\left[p\sin^2\left[\frac{at}{2}\cos\theta\right]+q\cos^2\left[\frac{at}{2}\cos\theta\right]\right]\sinh^2\left[\frac{at}{2}\sin\theta\right]$$
$$+\frac{1}{a}\sin(at\cos\theta)\left[(p-2)\frac{\rho}{2}\cos\theta-(p+2)\delta\sin\theta\right]$$
$$\left.+\frac{1}{a}\sinh(at\sin\theta)\left[(p-2)\frac{\rho}{2}\sin\theta+(p+2)\delta\sin\theta\right]\right\}, \tag{4.9}$$

with

$$p=\frac{1}{2|\Omega|^2}\left[\delta^2+a^2+\frac{\rho^2}{4}\right],$$
$$q=\frac{1}{a^2}\left[p\left[\delta^2+\frac{\rho^2}{4}\right]+8|\Omega|^2+4\delta^2-\rho^2\right]. \tag{4.10}$$

It is evident that the expressions (4.2) and (4.9) are quite different. Just to further display the differences, let us consider the simple case $\gamma_1=\gamma_2=\gamma$, which yields the simplified form

$$P(t)=1-\frac{e^{-\gamma t}}{2}\left[1+\frac{4|\Omega|^2}{Q^2}\right] \tag{4.11}$$

for (4.2) and

$$P(t)=1-2e^{-\gamma t}\left[1-\frac{\delta^2}{2|\Omega|^2}\right] \tag{4.12}$$

for (4.9).

A further example stressing both the generality and the flexibility of our method is the comparison with the Ackerhalt model of ionizing resonant pulse excitation.[25] In

this model the atom consists of two bound states and essentially a third level which represents the continuum. The laser pulse excites the atom; the ionization is due to some mechanism other than the exciting pulse, which is time independent during the excitation. Ionization occurs only from the atomic excited state, giving the state a width proportional to the inverse of the ionization rate. The model assumes that the laser is resonant with the intermediate state and that no Stark shifts occur. In this case since $\gamma_2=\gamma,\gamma_1=0,\delta=0$, Eq. (2.36) specializes to

$$\frac{d}{dt}\begin{bmatrix}\langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}=\begin{bmatrix}-\gamma & -2\Omega & 0\\ 2\Omega & \gamma & -\gamma\\ 0 & -\gamma & -\gamma\end{bmatrix}\begin{bmatrix}\langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}, \tag{4.13}$$

which are just the equations of the Ackerhalt model. It is, however, worth further dwelling on the generalized Bloch equations (2.36) in the realistic hypothesis of $\Omega_1=\Omega_2$ real function of time. In this case we have

$$\frac{d}{dt}\begin{bmatrix}\langle\hat{\sigma}_1\rangle\\ \langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}=\begin{bmatrix}2\,\mathrm{Im}\omega_1 & -\mathrm{Re}\omega_2 & 0 & 0\\ \mathrm{Re}\omega_2 & 2\,\mathrm{Im}\omega_1 & 2\Omega & 0\\ 0 & -2\Omega & 2\,\mathrm{Im}\omega_1 & 2\,\mathrm{Im}\omega_2\\ 0 & 0 & 2\,\mathrm{Im}\omega_2 & 2\,\mathrm{Im}\omega_1\end{bmatrix}\begin{bmatrix}\langle\hat{\sigma}_1\rangle\\ \langle\hat{\sigma}_2\rangle\\ \langle\hat{\sigma}_3\rangle\\ \langle\hat{I}\rangle\end{bmatrix}. \tag{4.14}$$

Introducing the four-vector

$$\sigma_\mu=(\langle\hat{\sigma}\rangle,\langle\hat{I}\rangle) \tag{4.15}$$

with metric

$$\sigma_\mu\sigma^\mu=|\langle\sigma\rangle|^2-\langle I\rangle^2, \tag{4.16}$$

we get from (4.14)

$$\frac{d}{dt}(\sigma_\mu\sigma^\mu)=(4\,\mathrm{Im}\omega_1)(\sigma_\mu\sigma^\mu). \tag{4.17}$$

We must, however, emphasize that, since the solution of the problem can be always traced back to Eqs. (2.19), it is convenient to deal directly with them rather than with the system (2.36), which is significantly more complicated. In fact, in the Ackerhalt model, Eqs. (2.19) reduce to the simpler form [see also Eqs. (2.19)]

$$i\dot{\tilde{L}}=-\tilde{\Omega}_1\bar{F},\quad i\dot{\bar{F}}=-\tilde{\Omega}_2\tilde{L}. \tag{4.18}$$

Then, if γ is a constant, the solution, in analytical form, can be found for different Ω shapes, as we will discuss elsewhere in a more quantitative paper.

As a concluding remark we notice that the results of the paper are not confined to quantum optics. They can be indeed extended to the analysis of other phenomena, like K^0-$\bar{K}^0$ mixing, where non-Hermitian Hamiltonians are currently used,[26] and, for example, CP violations could receive in the Bloch model we have developed a simple geometrical picture. It is, however, worth adding a few words of caution against the use of average values of operators in non-Hermitian quantum mechanics as exploited throughout the paper. According to the already quoted criticism by Baker, the usual definitions should be indeed carefully reconsidered.

APPENDIX

In Sec. II we have touched on the fact that for quantum systems ruled by NHH operators the Heisenberg equations of motion should be modified. This is an immediate consequence of the nonunitary nature of the evolution operator, which, on the other hand, causes the nonequivalence between the Heisenberg and Schrödinger pictures.

Consider, for example, an operator $\hat{A}_s(t)$ in the Schrödinger picture; the average value of $\hat{A}_s$ is

$$\langle\hat{A}_s\rangle=\langle\Psi(t)|\hat{A}_s|\Psi(t)\rangle=\langle\Psi(0)|\hat{U}^+\hat{A}_s\hat{U}(|\Psi(0)\rangle. \tag{A1}$$

We assume that $\hat{U}$ is nonunitary since the relevant Hamiltonian operator is non-Hermitian. Taking the Schrödinger derivative ($i\hbar d/dt$) on both sides of (A1) we find

$$i\hbar\frac{d}{dt}\langle\hat{A}_s\rangle=\langle\Psi(0)|-\hat{U}^+\hat{H}^+\hat{A}_s\hat{U}+\hat{U}^+\hat{A}_s\hat{H}\hat{U}+i\hbar\hat{U}^+\frac{\partial}{\partial t}A_s\hat{U}|\Psi(0)\rangle, \tag{A2}$$

from which Eq. (2.35) immediately follows. It is worth stressing that the modified Heisenberg equation (2.35) for NHH suffers from some drawbacks, the most serious of which is the lack of an algebraic structure. In fact, it is well known that in the Hermitian case, the expression at the right-hand side of Heisenberg's equation is nothing but the commutator $[\hat{A},\hat{H}]$, i.e., the associative product of a Lie algebra. This property is obviously lost in Eq. (2.35), since the corresponding term entering it is trilinear in the operators $\hat{A},\hat{H},\hat{H}^+$.

It is clear that owing to the nonunitary of $\hat{U}$ it is not convenient to introduce the Heisenberg counterpart of $\hat{A}_s$, namely,

$$\hat{A}_H=\hat{U}^+\hat{A}_s\hat{U}. \tag{A3}$$

Indeed, as it is easy to check, transformation (A3) does not even preserve the form invariance of the evolution law (2.35). One can, however, take advantage from the fact that

$$\hat{\bar{U}}^+\hat{U}=\hat{U}^+\hat{\bar{U}}=\hat{1} \tag{A4}$$

and use the new representation

$$\hat{A}_{H'}=\hat{\bar{U}}^{+}\hat{A}_s\hat{U}\ . \tag{A5}$$

Taking the Schrödinger derivative on both sides of (A3) we find

$$i\hbar\frac{d}{dt}\hat{A}_{H'}=-\hat{\bar{U}}^{+}\hat{H}\hat{A}_s\hat{U}+\hat{\bar{U}}^{+}\hat{A}_s\hat{H}\hat{U}+i\hbar\hat{\bar{U}}^{+}\frac{\partial}{\partial t}\hat{A}_s\hat{U}\ . \tag{A6}$$

Using (A4) we immediately get

$$i\hbar\frac{d}{dt}\hat{A}_{H'}=[\hat{A}_{H'},\hat{H}_{H'}]+i\hbar\hat{\bar{U}}+\frac{\partial}{\partial t}\hat{A}_s\hat{U}\ , \tag{A7}$$

thus recovering the familiar form of the Heisenberg equations. It is also worth stressing that the commutation relations takes the same from in both pictures, as it can be easily inferred from the following simple considerations. Let

$$[\hat{A}_s,\hat{B}_s]=i\hat{C}_s\ , \tag{A8}$$

then

$$[\hat{A}_{H'},\hat{B}_{H'}]=\hat{\bar{U}}^{+}[\hat{A}_s,\hat{B}_s]\hat{U}=i\hat{C}_{H'}\ . \tag{A9}$$

On account of Eqs. (2.15) and (A9) the transformation (A5) plays, for non-Hermitian Hamiltonians and biorthogonal states, a role analogous to the standard, quantum canonical transformations (i.e., it preserves mean values, commutation relations, and biorthonormality of states). Therefore, we may name it biunitary (or bicanonical) transformation.

We must, however, conclude that for practical calculations the most convenient equation for the operator evolution is that provided by (A2). The mixed representation (A3) is just a mathematical curiosity, which may become useful for problems involving the concomitant evolution of biorthogonal states.

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