# Perturbed ladder operator method: An algebraic recursive solution of perturbed wave equations

N. Bessis and G. Bessis
*Laboratoire de Physique des Lasers, Université Paris-Nord, avenue J.-B. Clement, 93430 Villetaneuse France*
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The Schrödinger-Infeld-Hull factorization method is extended within the perturbation scheme in order to treat nonfactorizable Sturm-Liouville eigenequations in the same way as factorizable ones. It is shown that, provided suitable choices of the expansion basis set for the perturbing potential and for the associated perturbed ladder function are made, the solution of the factorizability condition associated with the perturbed eigenequation can be achieved by using an elementary finite difference calculus. An algebraic manufacturing process allowing the determination of the perturbed ladder and factorization functions, capable of handling any order of the perturbation and any type of factorization (Infeld-Hull types A to E), is given. This procedure, well adapted for computer algebra, allows an analytical determination of the perturbed eigenvalues and eigenfunctions without calculation of either the excited unperturbed eigenfunctions or any matrix element. This extension of the exact factorization method within the perturbation scheme can be applied to many model equations of current interest in quantum physics. Special attention is paid to perturbed factorizations that correspond to unperturbed ladder operators that are linear functions of the quantum number (types A to D). Illustrative applications are given. Particularly, the perturbed harmonic-oscillator ladder operators and eigenenergies are obtained in closed form.

## I. INTRODUCTION

In many problems of current interest in quantum mechanics, particularly in atomic and molecular physics, one has to determine wave functions and calculate matrix elements of Hermitian operators between these wave functions. In this respect, one is usually led to the solution of wave equations and, at many stages of the physical modelization, after exact or approximate separation of variables, one requires the solution of one-variable linear second-order differential equations with associated boundary conditions. In many cases, these equations are, or are amenable to, factorizable eigenequations. Without being exhaustive, let us recall that the generalized spherical harmonics $Y_{lm,\gamma}(\theta,\varphi)$, the symmetric top functions $D^{(j)}_{mm'}(\varphi,\theta,\phi)$, the harmonic oscillator. Morse oscillator, Poschl-Teller, Manning-Rosen, and Rosen-Morse diatomic vibration-rotation functions, the hydrogenic and generalized Kepler functions, and more generally the Gauss hypergeometric and the confluent hypergeometric functions, can be directly related to solutions of factorizable equations.[1] Let us add that Weyl's spherical harmonics, Dirac's radial functions in the usual Euclidean space[1] and also in a space of constant curvature[2] can be expressed in terms of solutions of factorizable equations. Hence, when dealing with problems involving these functions, one can apply the Schrödinger-Infeld-Hull factorization method.[3,1]

When a given equation is factorizable, one has at one's disposal ladder operators that generate the eigenfunctions, step by step, downward or upward, and allow the determination of any eigenfunction $\Psi_{jm}(x)$ from the knowledge of the top or bottom eigenfunction $\Psi_{jj}(x)$, i.e., the "key" function, which is the solution of a first-order differential equation. Moreover, analytical expressions of the eigenvalues in terms of the quantum numbers are readily obtained as well as closed-form expressions of the eigenfunctions[4] involving orthogonal polynomials, and closed-form expressions of matrix elements are easily obtainable.[4-6] Let us mention that the factorization method is closely related to the theory of supersymmetry quantum mechanics (SSQM): as pointed out recently,[7-9] both methods are special cases of an old procedure developed by Darboux[10] for generating isospectral second-order linear differential equations. Particularly, it can be shown[11] that the shape-invariance condition for supersymmetric potentials is equivalent to the factorizability condition.

If real problems encountered in physics, or even many sufficiently elaborate model problems, do not lead, at once, to the solution of factorizable eigenequations, nevertheless, they can be conveniently described by a kernel potential, corresponding to a factorizable equation, together with an additional perturbation. In such cases, it is possible to extend the original range of applicability of the exact factorization process within the perturbation scheme. Summarizing *grosso modo* the principle of the perturbed factorization technique, one assumes that the perturbed potential function, as well as the associated ladder and factorization functions to be found, can be expanded in a perturbation series where the unperturbed potential leads to a factorizable equation. Then, one tries to build up the perturbed ladder operators and the associated factorization functions allowing the factorization of the perturbed equation at any rank of the perturbation. Once the "factorization instruments" have been found, the exact factorization scheme applies: one obtains closed-form expressions of the perturbed eigenvalues and of the perturbed eigenfunctions to the required accuracy without having to calculate either the excited unperturbed functions or any matrix element.

The straightforward extension of the unperturbed scheme, which tries to determine *ab initio* the perturbed ladder operators as a series of powers of the quantum number $m$, was recognized early.[3,1,12] However, this procedure leads to rather intricate calculations involving antidifferences in the quantum number. This is probably why Infeld and Hull have limited their pioneering use of this procedure to second-order Stark effect calculations. Many years after, we have proposed several procedures of determination of the perturbed ladder and factorization functions which, according to the factorization type under consideration, involve either successive integrations[13-15] or successive finite summations.[16] Unfortunately, the complexity of these integrations (or summations) rapidly increases with the order of the perturbation.

In the present paper the perturbed ladder operator method is revisited. An efficient procedure is proposed for the determination of the perturbed eigenvalues and associated ladder operators. Indeed, when assuming particular "ladderlike" properties for the $x$ dependence of the perturbed ladder functions to be found, general algebraic formulas can be derived. These formulas, well adapted for computer algebra, valid for any factorization type and at any order of the perturbation, allow an algebraic recursive determination of the perturbed ladder and factorization functions associated with the given perturbation. Hence, closed-form expressions of the "perturbed" eigenvalues and eigenfunctions can be obtained in the same way as within the "unperturbed" exact factorization scheme.

After a brief recall of the exact and perturbed factorization methods (Sec. II), the novel procedure is described and general formulas allowing the determination of the perturbed ladder and factorization functions are derived (Sec. III). The perturbed factorizations (Infeld-Hull types A to D), which correspond to unperturbed ladder operators which are linear functions of the quantum number, are considered in detail and several illustrative applications are worked out (Sec. IV).

## II. EXACT AND PERTURBED FACTORIZATION SCHEMES

In order to set up the definitions and notations, it is first necessary to briefly recall the exact and perturbed factorization schemes.

### A. Exact factorization

After exact or approximate separation of variables, many problems of current interest in quantum mechanics lead to the solution of eigenequations of the Sturm-Liouville type. By an appropriate transformation of variable and function, these equations can be reduced to the standard form

$$\left[\frac{d^2}{dx^2}+U(x,m)+\Lambda_j\right]\Psi_{jm}(x)=0 \tag{2.1}$$

associated with the boundary conditions $(x_1 \le x \le x_2)$

$$|\Psi(x_1)|^2=|\Psi(x_2)|^2=0,\quad \int_{x_1}^{x_2}|\Psi(x)|^2dx=1\ , \tag{2.2}$$

where $m=m_0+1,m_0+2,\ldots$ is a quantum number which takes successive discrete values labeling the eigenfunctions.

Such an equation as (2.1) is factorizable when it can be replaced by each of the following two difference-differential equations:

$$\begin{aligned} H_{m+1}^{-}H_{m+1}^{+}\Psi_{jm}&=[\Lambda_j-L(m+1)]\Psi_{jm}\ ,\\ H_m^{+}H_m^{-}\Psi_{jm}&=[\Lambda_j-L(m)]\Psi_{jm}\ , \end{aligned} \tag{2.3}$$

where $j$ is the quantum number associated with the eigenvalues $\Lambda_j$, $H_m^{\mp}=K(x,m)\pm(d/dx)$ are the ladder operators, and $L(m)$ is the factorization function that does not depend on $x$.

Owing to the mutual adjointness of the ladder operators $H_m^{+}$ and $H_m^{-}$, the necessary condition for the existence of quadratically integrable solutions of Eq. (2.1), i.e., the quantization condition, is $\epsilon(j-m)=v$ equal to a non-negative integer and $\epsilon=+1$ (or $\epsilon=-1$) according to whether $L(m)$ is an increasing (or decreasing) function of $m$.

The associated eigenvalues are

$$\Lambda_j=L\left[j+\frac{\epsilon}{2}+\frac{1}{2}\right]\ . \tag{2.4}$$

The normalized eigenfunctions $\Psi_{jm}(x)$ are solutions of the following pair of difference-differential equations:

$$\begin{aligned} \left[K(x,m)+\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m)\Psi_{jm-1}\ ,\\ \left[K(x,m+1)-\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m+1)\Psi_{jm+1}\ , \end{aligned} \tag{2.5}$$

with $\mathcal{N}_j(m)=[\Lambda_j-L(m)]^{1/2}$.

These "ladder" equations allow the determination of any $\Psi_{jm}(x)$ function from the knowledge of the "key" function $\Psi_{jj}(x)$ which is the solution of the first-order differential equation

$$\left[K\left[x,j+\frac{\epsilon}{2}+\frac{1}{2}\right]-\epsilon\frac{d}{dx}\right]\Psi_{jj}=0\ . \tag{2.6}$$

From the comparison of Eqs. (2.3) and (2.1), it is easily shown that the necessary and sufficient condition to be satisfied by $K(x,m)$ and $L(m)$ allowing the factorization of equation (2.1) is

$$\begin{aligned} &[K(x,m+1)]^2+\frac{d}{dx}K(x,m+1)+L(m+1)=-U(x,m)\ ,\\ &[K(x,m)]^2-\frac{d}{dx}K(x,m)+L(m)=-U(x,m)\ . \end{aligned} \tag{2.7}$$

There are six fundamental types of factorization

TABLE I. Infeld-Hull exact factorization types.

| Types | $U^{(0)}(x,m)$ | $K^{(0)}(x,m)$ | $L^{(0)}(m)$ |
|---|---|---|---|
| A | $-\frac{a^2}{\sin^2 ax}[m(m+1)+d^2+(2m+1)d\cos(ax)]$ | $am\cot ax+\frac{ad}{\sin ax}$ | $a^2m^2$ |
| B | $-a^2d^2e^{2ax}+a^2(2m+1)de^{ax}$ | $-am+ade^{ax}$ | $-a^2m^2$ |
| C | $-\frac{m(m+1)}{x^2}-b^2x^2$ | $\frac{m}{x}+bx$ | $-4bm$ |
| D | $-b^2x^2+b(2m+1)$ | bx | $-2bm$ |
| E | $-\frac{a^2m(m+1)}{\sin^2 ax}-2aq\cot ax$ | $am\cot ax+\frac{q}{m}$ | $a^2m^2-\frac{q^2}{m^2}$ |
| F | $-\frac{m(m+1)}{x^2}-\frac{2q}{x}$ | $\frac{m}{x}+\frac{q}{m}$ | $-\frac{q^2}{m^2}$ |

(denoted types A to F, within the Infeld-Hull nomenclature) with potential functions $U^{(0)}(x,m)$, and associated ladder and factorization functions $K^{(0)}(x,m)$ and $L^{(0)}(m)$ which are summarized in Table I. Closed-form expressions of the eigenfunctions are known[4] and involve classical orthogonal polynomials (see Table II).

As pointed out by Infeld and Hull,[1] when direct factorization is not possible solely because of the inadequate $m$ dependence of the potential function $U(x,m)$ under consideration, one can resort to "artificial" factorization, i.e., one can consider $U(x,m)$ as "embedded" in a new potential function $u(x,m;\mu)$ which depends on a supplementary artificial parameter $\mu$ such that $u(x,m;\mu)$ can be identified in $m$ with a factorizing potential $U^{(0)}(x,m)$ and that $u(x,m;\mu=m)=U(x,m)$. Then, Eq. (2.1) is factorized using $u(x,m;\mu)$, and the eigenvalues $\Lambda_j(\mu)=L(j+\epsilon/2+\frac{1}{2};\mu)$ are determined as well as the eigenfunctions $\Psi_{jm}(x;\mu)$, both depending on the parameter $\mu$. At the end of the ladder procedure (2.5), one merely sets $\mu=m$ and obtains the required solutions $\Lambda_j(m)=\Lambda_j(\mu=m)$ and $\Psi_{jm}(x)=\Psi_{jm}(x;\mu=m)$. This artificial or embedded factorization device is widely used all along the "perturbed ladder" scheme.

### B. Perturbed factorization

Now, let us consider the Sturm-Liouville differential equation (2.1) involving a potential function $U(x,m)$

TABLE II. Eigenfunctions of Infeld-Hull factorizable equations. $P_v^{(\alpha,\beta)}(\ )$, $L_v^{\alpha}(\ )$, and $H_v(\ )$ are, respectively, a Jacobi, Laguerre, and Hermite polynomial of degree $v=\epsilon(j-m)$. $\bar{j}=j+\epsilon/2+1/2$; $\epsilon=+1$ or $-1$ for class-I or class-II problems; $N_{jm}$ is a normalization constant.

| Types | $\Psi_{jm}^{(0)}(x)$ | Parameters |
|---|---|---|
| A | $N_{jm}\left[\sin\frac{ax}{2}\right]^{\alpha+1/2}\left[\cos\frac{ax}{2}\right]^{\beta+1/2}P_v^{(\alpha,\beta)}(\cos ax)$ | $\alpha=\epsilon(m+d+1/2)$ |
| | | $\beta=\epsilon(m-d+1/2)$ |
| B | $N_{jm}\exp[\frac{1}{2}(\alpha ax-\beta e^{ax})]L_v^{\alpha}(\beta e^{ax})$ | $\alpha=-2\epsilon j$ |
| | | $\beta=-2\epsilon d$ |
| C | $N_{jm}x^{\epsilon(m+1/2)+1/2}\exp\left[\frac{\epsilon bx^2}{2}\right]L_v^{\epsilon(m+1/2)}(-\epsilon bx^2)$ | |
| D | $N_{jm}\exp\left[\frac{\epsilon bx^2}{2}\right]H_v[(-\epsilon b)^{1/2}x]$ | |
| E | $N_{jm}(\sin ax)^{\epsilon\bar{j}}\exp\left[\frac{\epsilon aqx}{\bar{j}}\right]P_v^{(\alpha,\beta)}(-i\cot ax)$ | $\alpha=-\epsilon\bar{j}-iq/\bar{j}$ |
| | | $\beta=-\epsilon\bar{j}+iq/\bar{j}$ |
| F | $N_{jm}x^{m+1}\exp\left[\frac{qx}{j+1}\right]L_v^{2m+1}\left[-\frac{2qx}{j+1}\right]$ | |

which does not belong to any of the six Infeld-Hull factorization types and let us assume that this potential function, as well as the associated ladder and factorization functions $K(x,m)$ and $L(m)$ to be found can be expanded in a perturbation series with a parameter $\eta$

$$\begin{aligned} U(x,m) &= U^{(0)}(x,m)+\eta U^{(1)}(x,m)+\eta^2 U^{(2)}(x,m) + \cdots , \\ K(x,m) &= K^{(0)}(x,m)+\eta K^{(1)}(x,m)+\eta^2 K^{(2)}(x,m) + \cdots , \\ L(m) &= L^{(0)}(m)+\eta L^{(1)}(m)+\eta^2 L^{(2)}(m)+ \cdots , \end{aligned} \tag{2.8}$$

where $K^{(0)}(x,m)$ and $L^{(0)}(m)$ are the ladder and factorization functions allowing an exact factorization of Eq. (2.1) with $U^{(0)}(x,m)$.

Then, one has to satisfy the factorizability condition (2.7) up to a given power of the parameter $\eta$. The required $K^{(N)}(x,m)$, $L^{(N)}(m)$, and $U^{(N)}(x,m)$ are found to be solutions of the following equations:

$$\begin{aligned} &\sum_{v=0}^{N} K^{(v)}(x,m+1)K^{(N-v)}(x,m+1)+\frac{d}{dx}K^{(N)}(x,m+1)+L^{(N)}(m+1)=-U^{(N)}(x,m) , \\ &\sum_{v=0}^{N} K^{(v)}(x,m)K^{(N-v)}(x,m)-\frac{d}{dx}K^{(N)}(x,m)+L^{(N)}(m)=-U^{(N)}(x,m) . \end{aligned} \tag{2.9}$$

These equations will be solved recursively, i.e., when considering the determination of $K^{(N)}(x,m)$ and $U^{(N)}(x,m)$, it is assumed that all the $K^{(v)}(x,m)$, for $v=1,2,\ldots,N-1$, have already been found. Their finite difference aspect determines the $m$ dependence of the functions while their differential aspect determines their $x$ dependence.

Since, in the present paper, the solution of the factorizability condition (2.9) is worked out by means of finite difference calculus, it is convenient to introduce the usual first difference $\Delta$ operator in $m$

$$\Delta F(m)=F(m+1)-F(m) . \tag{2.10}$$

Then, the difference-differential equations (2.9) can be written again

$$2\Delta[K^{(0)}(x,m)K^{(N)}(x,m)]+\frac{d}{dx}[K^{(N)}(x,m+1)+K^{(N)}(x,m)]=-\Delta L^{(N)}(m)-\Delta\sum_{v=1}^{N-1} K^{(v)}(x,m)K^{(N-v)}(x,m) , \tag{2.11}$$

$$U^{(N)}(x,m)=\left[\frac{d}{dx}-2K^{(0)}(x,m)\right]K^{(N)}(x,m)-L^{(N)}(m)-\sum_{v=1}^{N-1} K^{(v)}(x,m)K^{(N-v)}(x,m) . \tag{2.12}$$

Equation (2.11) is used to determine the ladder and factorization functions $K^{(N)}(x,m)$ and $L^{(N)}(m)$. Once they are known the required potential functions $U^{(N)}(x,m)$ are given by Eq. (2.12) and one obtains the required "factorizing" potential function $U(x,m)$ of the eigenequation (2.1).

Thus one can solve physicomathematical problems with a potential function $V(x,m)$ such as

$$V(x,m)=U^{(0)}(x,m)+\eta V^{(1)}(x)+\eta^2 V^{(2)}(x)+ \cdots , \tag{2.13}$$

where the $V^{(v)}(x)$ have the same dependence in $x$ as the $U^{(v)}(x,m)$ and, in most cases, do not depend on $m$.[17]

In order to match $V(x,m)$ with the factorizing potential $U(x,m)$, one has to resort to the artificial factorization process. The following condition must hold, for any value of $x$:

$$V^{(v)}(x)=U^{(v)}(x;m=\mu) . \tag{2.14}$$

Finally, one can factorize an eigenequation (2.1) with a given potential function $V(x,m)$ by determining the associated perturbed ladder and factorization functions which are solutions of the difference-differential equation (2.11) and, as a consequence of Eqs. (2.12) and (2.14), which satisfy the following condition:

$$\left[\frac{d}{dx}-2K^{(0)}(x,\mu)\right]K^{(N)}(x,\mu)-L^{(N)}(\mu) = V^{(N)}(x)+\sum_{v=1}^{N-1} K^{(v)}(x,\mu)K^{(N-v)}(x,\mu) . \tag{2.15}$$

Once the perturbed ladder and factorization functions $K^{(v)}(x,m;\mu)$ and $L^{(v)}(m;\mu)$, both depending on the artificial parameter $\mu$, have been found, the perturbed problems (up to the $N$th order) may be handled in the same way as the exact factorizable (unperturbed) problem.

The total perturbed eigenvalue and associated ladder function are [see Eqs. (2.4) and (2.8)]

$$\begin{aligned} \Lambda_j(m) &= L^{(0)}\left[j+\frac{\epsilon}{2}+\frac{1}{2}\right] + \sum_{v=1}^{N} \eta^v L^{(v)}\left[m=j+\frac{\epsilon}{2}+\frac{1}{2};\ \mu=m\right] , \\ K(x,m;\mu) &= K^{(0)}(x,m)+\sum_{v=1}^{N} \eta^v K^{(v)}(x,m;\mu) , \end{aligned} \tag{2.16}$$

where $\epsilon=+1$ (or $\epsilon=-1$) according as the unperturbed factorization function $L^{(0)}(m)$ is an increasing (or decreasing) function of $m$.

The normalized key $(m=j)$ perturbed eigenfunction $\Psi_{jj}(x;\mu)$ is the solution of the first-order differential equation

$$\left[K\left(x,j+\frac{\epsilon}{2}+\frac{1}{2};\mu\right)-\epsilon\frac{d}{dx}\right]\Psi_{jj}(x;\mu)=0 . \tag{2.17}$$

In terms of the unperturbed normalized key eigenfunction $\Psi_{jj}^{(0)}(x)$, one gets

$$\Psi_{jj}(x;\mu)=\Psi_{jj}^{(0)}(x)\exp\left[\epsilon\sum_{v=1}^{N}\eta^{v}\int K^{(v)}\left[x,j+\frac{\epsilon}{2}+\frac{1}{2};\mu\right]dx\right] . \tag{2.18}$$

The closed-form expression of any normalized perturbed $\Psi_{jm}(x;\mu)$ function can be obtained stepwise from the key eigenfunction $\Psi_{jj}(x;\mu)$ by using the $\mu$-dependent ladder equations

$$\left[K(x,m+1;\mu)-\frac{d}{dx}\right]\Psi_{jm}(x;\mu) =[\Lambda_j(\mu)-L(m+1;\mu)]^{1/2}\Psi_{j,m+1}(x;\mu) , \tag{2.19}$$

$$\left[K(x,m;\mu)+\frac{d}{dx}\right]\Psi_{jm}(x;\mu) =[\Lambda_j(\mu)-L(m;\mu)]^{1/2}\Psi_{j,m-1}(x;\mu) .$$

At the final stage of the ladder process, one sets $\mu=m$ and obtains the required perturbed eigenfunction $\Psi_{jm}(x)=\Psi_{jm}(x;\mu=m)$.

When perturbed eigenfunctions $\Psi_{jm}(x)$ far from the key eigenfunction $\Psi_{jj}(x)$ are required, it may be convenient to use an alternative procedure which provides the perturbed eigenfunctions as linear combinations of the unperturbed eigenfunctions [see, for instance, Ref. 15 or 16].

Let us now return to the determination of the perturbed ladder and factorization functions, $K^{(N)}(x,m;\mu)$ and $L^{(N)}(m;\mu)$, and show the interest of choosing, for the perturbed potential $V^{(N)}(x)$ and ladder functions $K^{(N)}(x,m)$, associated expansion basis sets satisfying particular "ladderlike" properties.

## III. ALGEBRAIC DETERMINATION OF THE PERTURBED LADDER FUNCTIONS AND EIGENVALUES

### A. Determination of the perturbed ladder functions

Let us first consider the $x$ dependence of the difference-differential equation (2.11) and assume that, at each order $N$ of the perturbation, the perturbed ladder function can be written

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N}\gamma_s^{(N)}(m)Y_s(x) , \tag{3.1}$$

The $Y_s(x)$ basis set, specific to each factorization type, is to be chosen so that all terms appearing in Eq. (2.11) can be expanded on a common basis set $y_s(x)$. Namely, let us assume that one can find suitable associated basis sets $Y_s(x)$ and $y_s(x)$ such that

$$2K^{(0)}(x,m)Y_s(x)=A_s(m)y_s(x)+B_s(m)y_{s+1}(x) ,$$
$$\frac{dY_s}{dx}=\alpha_s y_s(x)+\beta_s y_{s+1}(x) , \tag{3.2}$$

$$Y_s(x)Y_t(x)=\sum_r h(s,t,r)y_r(x) , \tag{3.3}$$

and that, as a consequence of Eq. (3.3), one can set

$$\sum_{v=1}^{N-1}K^{(v)}(x,m)K^{(N-v)}(x,m)=\sum_{s=0}^{S_N}w_s^{(N)}(m)y_s(x) , \tag{3.4}$$

where the $w_s^{(N)}(m)$ functions originate from the preceding orders of the perturbation.

Let us note that, if at the first order $N=1$ of the perturbation, $w_s^{(1)}(m)=0$ and the upper bound $S_1$ which is involved in $K^{(1)}(x,m)$ can be chosen arbitrarily, this is not true for the higher orders $S_N$. Indeed, owing to the presence of the term (3.4) in the factorizability condition (2.11), it can be easily inferred that the value of $S_N$ will depend both on the values of $N$ and of $S_1$.

When conditions (3.2)–(3.4) have been fulfilled, after substituting for $K^{(N)}$ and $\sum_{v=1}^{N-1}K^{(v)}K^{(N-v)}$ from Eqs. (3.1) and (3.4) into Eq. (2.11) and by equating the coefficients of $y_s(x)$ in both sides, one obtains the following finite difference equations allowing the determination of the $\gamma_s^{(N)}(m)$ and $L^{(N)}(m)$ functions:

$$[B_{S_N}(m+1)+\beta_{S_N}]\gamma_{S_N}^{(N)}(m+1)-[B_{S_N}(m)-\beta_{S_N}]\gamma_{S_N}^{(N)}(m)=0 , \tag{3.5}$$

$$[B_{s-1}(m+1)+\beta_{s-1}]\gamma_{s-1}^{(N)}(m+1)-[B_{s-1}(m)-\beta_{s-1}]\gamma_{s-1}^{(N)}(m) =-[A_s(m+1)+\alpha_s]\gamma_s^{(N)}(m+1)+[A_s(m)-\alpha_s]\gamma_s^{(N)}(m)-\Delta w_s^{(N)}(m) , \tag{3.6}$$

$$\Delta L^{(N)}(m)=-[A_0(m+1)+\alpha_0]\gamma_0^{(N)}(m+1)+[A_0(m)-\alpha_0]\gamma_0^{(N)}(m)-\Delta w_0^{(N)}(m) . \tag{3.7}$$

The solution of the first-order linear homogeneous finite difference equation (3.5) is obtainable in closed form. Using elementary results of finite difference calculus,[18] one gets

$$\gamma_{S_N}^{(N)}(m)=Q_{S_N}(m)k_{S_N}^{(N)} , \tag{3.8}$$

where $k_{S_N}^{(N)}$ is an arbitrary constant of summation and

$$Q_s(m)=\prod_{j=1}^{m-1}\frac{B_s(j)-\beta_s}{B_s(j+1)+\beta_s} . \tag{3.9}$$

Then, the inhomogeneous finite difference equations (3.6) can be solved recursively, the integer $s$ descending stepwise from $s=S_N$ down to zero. For each value of $s$, one obtains the general solution[18]

$$\gamma_s^{(N)}(m)=Q_s(m)[k_s^{(N)}+F_s^{(N)}(m)] , \tag{3.10}$$

where $k_s^{(N)}$ is an arbitrary summation constant and

$$F_s^{(N)}(m)=\Delta^{-1}[R_{s+1}^{(N)}(m)/Q_s(m+1)] ,$$
$$R_s^{(N)}(m)=\frac{-[A_s(m+1)+\alpha_s]\gamma_s^{(N)}(m+1)+[A_s(m)-\alpha_s]\gamma_s^{(N)}(m)-\Delta w_s^{(N)}(m)}{B_{s-1}(m+1)+\beta_{s-1}} . \tag{3.11}$$

At this level, since the $F_s^{(N)}(m)$ functions are defined within an additive arbitrary summation constant, it is well advised to impose the following vanishing condition:

$$F_s^{(N)}(m=\mu)=0 . \tag{3.12}$$

As will be shown hereafter, this imposed condition greatly simplifies the determination of the $K^{(N)}(x,m;\mu)$ perturbed ladder functions satisfying Eq. (2.15).

In order to determine the $F_s^{(N)}(m)$ functions so that condition (3.12) be fulfilled, it is rewarding to use Newton's formula (see Appendix A) and set

$$F_s^{(N)}(m)=\sum_{n=1}^{n_s^{(N)}}\begin{bmatrix} m-\mu \\ n\end{bmatrix}\Delta^n F_s^{(N)}(m=\mu) , \tag{3.13}$$

where

$$\begin{bmatrix} m-\mu \\ n\end{bmatrix}=\frac{\Gamma(m-\mu+1)}{\Gamma(m-\mu-n+1)\Gamma(n+1)}$$

is a generalized binomial coefficient and $n_s^{(N)}$ is the degree in $m$ of $F_s^{(N)}(m)$. Hence, at each order $N$ of the perturbation, the determination of the $F_s^{(N)}(m)$ functions amounts to the determination of the values, for $m=\mu$, of the $n$th finite differences $\Delta^n F_s^{(N)}(m)$ $(n=1,n_s^{(N)})$.

Using elementary results of finite difference calculus together with some algebraic manipulations (see Appendix B), one obtains the following relation allowing an algebraic recursive determination of any $\Delta^n F_s^{(N)}=\Delta^n F_s^{(N)}(m=\mu)$ coefficient in terms of the arbitrary summation constants $k_u^{(N)}=\Delta^0 F_u^{(N)}$ and of the contributions $\Delta^i w_u^{(N)}=\Delta^i w_u^{(N)}(m=\mu)$ generated from the preceding orders of the perturbation:

$$\Delta^n F_s^{(N)}=\sum_{u=s+1}^{S_N}\sum_{i=0}^{n-1}\begin{bmatrix} n-1 \\ i\end{bmatrix}[I(u,s;i,n)\Delta^i F_u^{(N)}+J(u,s;i,n)\Delta^{i+1}w_u^{(N)}] , \tag{3.14}$$

where

$$I(u,s;i,n)=\Delta^{n-i-1}f_{us}(m=\mu+i) ,$$
$$J(u,s;i,n)=\Delta^{n-i-1}g_{us}(m=\mu+i) , \tag{3.15}$$

$$f_{us}(m)=\{[A_u(m+1)+\alpha_u]Q_u(m+1)-[A_u(m)-\alpha_u]Q_u(m)\}g_{us}(m) ,$$
$$g_{us}(m)=(-1)^{u-s}\prod_{t=s+1}^{u-1}[A_t(m+1)+\alpha_t]\Bigg/\left[Q_s(m+1)\prod_{t=s}^{u-1}[B_t(m+1)+\beta_t]\right] . \tag{3.16}$$

Particularly, when $A_s(m)$ is a linear function of $m$ (factorization types A to D) and when also $Q_0(m)=Q_0$ does not depend on $m$, the following "reduced" relation holds for the determination of the $\Delta^n F_s^{(N)}$ coefficients in terms of the constant $k_u^{(N)}=\Delta^0 F_u^{(N)}$ (see Appendix B):

$$\Delta^n F_s^{(N)}=\sum_{u=s+1}^{S_N-n+1}[\mathcal{I}(u,s,n)\Delta^{n-1}F_u^{(N)}+\mathcal{J}(u,s,n)\Delta^n w_u^{(N)}] , \tag{3.17}$$

where

$$\mathcal{A}(u,s,n) = -\frac{(-1)^{u-s}}{B_{u-1}} \prod_{t=s+1}^{u-1} \frac{(n+\mu)\Delta A_t + A_t(m=0)+\alpha_t}{B_{t-1}} ,$$

$$\mathcal{I}(u,s,n)=(n\Delta A_u+2\alpha_u)\mathcal{A}(u,s,n) .$$

The coefficients $I(u,s;i,n)$ and $J(u,s;i,n)$, or the reduced ones $\mathcal{I}(u,s,n)$ and $\mathcal{A}(u,s,n)$, do not depend on the order $N$ of the perturbation; as a consequence, for a given problem, one can calculate once for all the expressions of the $\Delta^n F_s^{(N)}$ in terms of $S_N$ (see Appendix B) and, then, use the same expressions at the successive orders of the perturbation. At the first order ($N=1$) of the perturbation $w_u^{(1)}(m)=0$, and the final expressions of the $\Delta^n F_s^{(1)}$ will involve only the $k_u^{(1)}=\Delta^0 F_u^{(1)}$ constants. At the higher orders ($N>1$) of the perturbation, the $\Delta^n F_S^{(N)}$ will contain, in addition, the contributions $\Delta^{i+1}w_u^{(N)}$ which are already known functions of the $\Delta^i F_u^{(v)}$ of the preceding orders ($v=1$ to $N-1$) of the perturbation (see Appendix C).

Once the $\Delta^n F_s^{(N)}$ coefficients have been computed, the perturbed ladder function $K^{(N)}(x,m)$ is completely known and involves the arbitrary summation constants $k_u^{(v)}$ ($v=1,N;u=0,S_v$)

$$K^{(N)}(x,m;\mu) = \sum_{s=0}^{S_N} Y_s(x)Q_s(m)\left[k_s^{(N)}+\sum_{n=1}^{n_s^{(N)}}\begin{bmatrix} m-\mu \\ n\end{bmatrix}\Delta^n F_s^{(N)}\right] . \quad (3.18)$$

The value of the upper bound $n_s^{(N)}$ in (3.18), i.e., the degree in $m$ of $F_s^{(N)}(m)$, is easily obtainable from expression (B3) of $\Delta F_s^{(N)}$.

The factorizing perturbed potential $U^{(N)}(x,m;\mu)$ associated with $K^{(N)}(x,m;\mu)$ is given by Eq. (2.12) and, as well as $K^{(N)}(x,m;\mu)$, depends on the arbitrary constants $k_u^{(N)}$.

Let us now consider the perturbed factorization of eigenequation (2.1) with a given $V(x,m)$ physical model potential (2.13). In the same way as within the exact factorization scheme, in order to apply the method to a given problem, one has to determine the expressions of the $k_u^{(N)}$ constants in terms of the data specific to that problem by matching $V^{(N)}(x)$ with $U^{(N)}(x,m;\mu)$. From expression (2.12), it is easily seen that the theoretical factorizing perturbed potential $U^{(N)}(x,m;\mu)$ associated with $K^{(N)}(x,m;\mu)$ can be written as a finite expansion on the $y_s(x)$ basis set. Consequently, in order to match $V^{(N)}(x)$ with $U^{(N)}(x,m;\mu)$, one has first to expand the $V^{(N)}(x)$ on the $y_s(x)$ basis and to set

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1} b_s^{(N)}y_s(x) , \quad (3.19)$$

where the $b_s^{(N)}$ constants are specific to the physical model potential under consideration.

Hence, using the artificial factorization device, the determination of the $K^{(N)}(x,m;\mu;b_i^{(v)})$ ladder function associated with $V^{(N)}(x)$ amounts to the determination of the $K^{(N)}(x,m;\mu;k_u^{(v)})$ function satisfying condition (2.15). Introducing the expressions (3.1), (3.2), (3.4), and (3.19) of the perturbed ladder and potential functions into the condition (2.15) and equating the coefficients of $y_s(x)$ in both sides, one obtains the following relations to be satisfied by the $\gamma_s^{(N)}(m)$ and $L^{(N)}(m)$ functions:

$$[B_{S_N}(\mu)-\beta_{S_N}]\gamma_{S_N}^{(N)}(\mu)=-b_{S_N+1}^{(N)} , \quad (3.20)$$

$$[A_s(\mu)-\alpha_s]\gamma_s^{(N)}(\mu)+[B_{s-1}(\mu)-\beta_{s-1}]\gamma_{s-1}^{(N)}(\mu) = -b_s^{(N)}-w_s^{(N)}(\mu) , \quad (3.21)$$

$$L^{(N)}(\mu)=-[A_0(\mu)-\alpha_0]\gamma_0^{(N)}(\mu)-w_0^{(N)}(\mu) . \quad (3.22)$$

As a consequence of the vanishing condition $F_s^{(N)}(m=\mu)=0$, it follows that $\gamma_s^{(N)}(\mu)=Q_s(\mu)k_s^{(N)}$ [see Eq. (3.10]; then, Eqs. (3.20) and (3.21) allow the determination of closed-form expressions of the arbitrary constants $k_s^{(N)}$ in terms of $\mu$ and of the $b_u^{(N)}$ expansion coefficients of the $V^{(N)}(x)$ potential.

Indeed, for $s=S_N$, using (3.20), one gets

$$k_{S_N}^{(N)}=-\frac{b_{S_N+1}^{(N)}}{Q_{S_N}(\mu)[B_{S_N}(\mu)-b_{S_N}]} . \quad (3.23)$$

For $s\neq S_N$, one applies ($S_N-s$) times the two-terms recursive relation (3.21) and obtains

$$k_s^{(N)}=\sum_{u=s+1}^{S_N+1} C_{us}(\mu)[b_u^{(N)}+w_u^{(N)}(\mu)] , \quad (3.24)$$

where

$$C_{us}(\mu)=-\frac{\prod_{t=s+1}^{u-1}[A_t(\mu)-\alpha_t]}{Q_s(\mu)\prod_{t=s}^{u-1}[B_t(\mu)-\beta_t]} .$$

Finally, the required expression of the perturbed ladder function $K^{(N)}(x,m;\mu)$ associated with each given physical model perturbation term $V^{(N)}(x)$, readily follows from the general expression (3.18). It involves, by means of expression (3.24) of the $k_s^{(N)}$, the expansion coefficients $b_s^{(N)}$ of the perturbations $V^{(N)}(x)$ on the suitable basis set $y_s(x)$ and, also, via the already known expressions of the $w_u^{(N)}(\mu)$ and $\Delta^{i+1}w_u^{(N)}$ in terms of the $k_j^{(v)}$ (see Appendix C), the $b_u^{(v)}$ coefficients originating from the preceding orders of the perturbation ($v=1,N-1$).

### B. Determination of the perturbed eigenvalues

At each order $N$ of the perturbation, the perturbed factorization function $L^{(N)}(m;\mu)$, associated with $V^{(N)}(x)$, is the solution of the first-order difference equation (3.7). Let us use again the Newton's formula (see Appendix A) and set

$$L^{(N)}(m;\mu)=L^{(N)}(\mu)+\sum_{n=1}^{n_0^{(N)}}\begin{bmatrix} m-\mu \\ n\end{bmatrix}\Delta^n L^{(N)}(m=\mu) ,$$

where $L^{(N)}(\mu)$ is given by Eq. (3.22).

Introducing the $(n-1)$th discrete derivative of expression (3.7) and using few algebraic manipulations, one obtains

$$L^{(N)}(m;\mu)=-w_0^{(N)}(m)-\lambda_0(m;\mu)k_0^{(N)} -\sum_{n=1}^{n_0^{(N)}}\lambda_n(m;\mu)\Delta^n F_0^{(N)} , \qquad (3.25)$$

where

$$\lambda_n(m;\mu)=\sum_{j=n}^{n_0^{(N)}}\begin{bmatrix} m-\mu \\ j \end{bmatrix}\sum_{i=n}^{j}\begin{bmatrix} i \\ n \end{bmatrix} Z_{ij}(\mu) \times\Delta^{i-n}Q_0(m=\mu+n) , \qquad (3.26)$$

$$Z_{00}(\mu)=A_0(\mu)-\alpha_0 ,$$

and, for $j>0$,

$$Z_{jj}(\mu)=\alpha_0+A_0(\mu+j) ,$$

$$Z_{j-1,j}(\mu)=2\alpha_0+j\Delta A_0(m=\mu+j-1) ,$$

$$Z_{ij}(\mu)=\begin{bmatrix} j \\ i \end{bmatrix}\Delta^{j-i}A_0(m=\mu+j-1), \quad i<j-1 .$$

Particularly, when $A_0(m)$ is a linear function of $m$ (factorization types A to D) and when, also, $Q_0(m)=Q_0$ does not depend on $m$, the expression (3.26) reduces to

$$\lambda_0(m;\mu)=Q_0(A_0(\mu)-\alpha_0+(m-\mu)(2\alpha_0+\Delta A_0)) , \qquad (3.27)$$

$$\lambda_n(m;\mu)=Q_0\left[\begin{bmatrix} m-\mu \\ n \end{bmatrix}[\alpha_0+A_0(\mu+n)] + \begin{bmatrix} m-\mu \\ n+1 \end{bmatrix}[2\alpha_0+(n+1)\Delta A_0]\right] , \quad n>0 .$$

Closed-form expressions of the $\Delta^n F_0^{(N)}$ have been obtained in terms of the expansion coefficients $b_i^{(v)}$ of $V^{(N)}(x)$, when computing the perturbed ladder function $K^{(N)}(x,m;\mu)$; therefore the perturbed eigenvalue is readily found: $\Lambda_j^{(N)}(m)=L^{(N)}(m=j+\epsilon/2+1/2;m=\mu)$.

It is interesting to note that, when rearranging the terms in order to put in evidence the coefficients of $(b_u^{(N)}+w_u^{(N)})$ in the expression of $\Lambda_j^{(N)}(m)$, one obtains

$$\Lambda_j^{(N)}(m)=\sum_{s=1}^{S_N+1}\langle y_s\rangle(b_s^{(N)}+w_s^{(N)})+(\ ) , \qquad (3.28)$$

where ( ) stands for the contributions involving the $\Delta^i w_u^{(N)}$ and the $\langle y_s\rangle$ do not depend on the order $N$ of the perturbation.

When comparing the first-order ($N=1$) expression $\Lambda_j^{(1)}(m)$ with its alternative first-order expression within the classical Rayleigh-Schrödinger framework, it follows that $\langle y_s\rangle$ is merely the expectation value of $y_s(x)$ between the unperturbed eigenfunctions $\Psi_{jm}^{(0)}(x)$ of Table II; in other words, one obtains, as a by-product of the computation of the eigenvalues, closed-form expressions of the diagonal matrix elements

$$\langle y_s\rangle=\int_{x_1}^{x_2}|\Psi_{jm}^{(0)}|^2 y_s(x)dx . \qquad (3.29)$$

Such integrals, involving products of Jacobi, Laguerre, or Hermite polynomials, are not always trivial and easy to compute by brute termwise integration. Let us mention that they can also be obtained in closed form by means of an algebraic procedure.[5,6]

### C. Practical use of the perturbed ladder operator method

General expressions (3.18) and (3.25) of the perturbed factorization instruments $K^{(N)}(x,m;\mu)$ and $L^{(N)}(m;\mu)$ associated with a given perturbation $V^{(N)}(x)$ have been derived. They are valid for any order $N$ of the perturbation and allow perturbed factorization of any eigenequation which can be viewed as a factorizable equation with an additional perturbation $V(x)$

$$\left[\frac{d^2}{dx^2}+U^{(0)}(x,m)+V(x)+\Lambda_j(m)\right]\Psi_{jm}(x)=0 , \qquad (3.30)$$

where

$$V(x)=\eta V^{(1)}(x)+\eta^2V^{(2)}(x)+\cdots ,$$

$$\Lambda_j(m)=\eta\Lambda_j^{(1)}(m)+\eta^2\Lambda_j^{(2)}(m)+\cdots .$$

Let us briefly summarize the main steps of the computation.

(a) Once the unperturbed potential $U^{(0)}(x,m)$ has been chosen in Table I and a suitable expansion basis set $y_s(x)$, satisfying the selective conditions (3.2) and (3.3), has been found for the perturbating potential $V(x)$, closed-form expressions of the functions $Q_s(m)$, $\lambda_n(m;\mu)$, $C_{us}(\mu)$ and of the bounds $S_N$ and $n_s^{(N)}$ appearing in the expressions (3.18) and (3.25) of the factorizable instruments $K^{(N)}(x,m;\mu)$ and $L^{(N)}(m;\mu)$ are easily obtainable [see Eqs. (3.9), (3.26), and (3.24)]. They are valid and will serve at the successive orders of the perturbation.

(b) At each order $N$ of the perturbation, the determination of the theoretical perturbed ladder function $K^{(N)}(x,m;\mu)$ and of the perturbed factorization function $L^{(N)}(m;\mu)$ amounts to the algebraic recursive determination of the required $\Delta^n F_s^{(N)}$ coefficients ($s=0,S_N;n=1,n_s^{(N)}$) in terms of the arbitrary constants $k_u^{(N)}=\Delta^0F_u^{(N)}$; one uses either the algebraic recursive relation (3.14) or the reduced one (3.17). At the first order $N=1$ of the perturbation, $\Delta^n w_u^{(1)}=0$ for any $n$ and $u$. At the higher orders $N$, one has to take into account the contributions of the $\Delta^n w_u^{(N)}$ increments originating from the preceding orders ($v=1,N-1$) of the perturbation and involving the $k_j^{(v)}$ constants ($j=0,S_v$) (Appendix C).

(c) One uses the closed-form expressions (3.24) of the $k_u^{(N)}$ constants in terms of the expansion coefficients $b_s^{(N)}$ of the given perturbations $V^{(N)}(x)$ on the $y_s(x)$ basis set and obtains the required perturbed eigenvalue $A_j^{(N)}(m)=L^{(N)}(m=j+\epsilon/2+1/2;\mu=m)$ and ladder function associated with $V^{(N)}(x)$. Following the class of factorization of the unperturbed problem: $\epsilon=+1$ (class I) or $\epsilon=-1$ (class II).

TABLE III. Perturbed type-A and -B factorizations. Suitable basis sets and associated data.

| Type | A | B |
|---|---|---|
| $Y_s(x)$ | $a^2\left[\tan\frac{ax}{2}\right]^{2s}$ | $a^2\exp(asx)$ |
| $Y_s(x)$ | $a\left[\tan\frac{ax}{2}\right]^{2s+1}$ | $a\exp(asx)$ |
| $A_s(m)$ | $d+m$ | $-2m$ |
| $B_s(m)$ | $d-m$ | $2d$ |
| $\alpha_s$ | $s+1/2$ | $s$ |
| $\beta_s$ | $s+1/2$ | $0$ |
| $Q_s(m)$ | $\frac{(s-d+m-1/2)_{2s}}{(s-d+1/2)_{2s}}$ | $1$ |
| $Y_sY_t$ | $Y_{s+t+1}$ | $Y_{s+t}$ |
| $S_N$ | $NS_1+N-1$ | $NS_1$ |
| $n_s^{(N)}$ | $2(S_N-s)$ | $S_N-s$ |
| $\lambda_0(m;\mu)$ | $2m-\mu+d-1/2$ | $2m$ |
| $\lambda_n(m;\mu)$ | $(\mu+d+n+1/2)\left[\begin{matrix}m-\mu\\ n\end{matrix}\right]+(n+2)\left[\begin{matrix}m-\mu\\ n+1\end{matrix}\right]$ | $2(\mu+n)\left[\begin{matrix}m-\mu\\ n\end{matrix}\right]+2(n+1)\left[\begin{matrix}m-\mu\\ n+1\end{matrix}\right]$ |
| $C_{us}(\mu)$ | $\frac{(s-d+1/2)_{2s}(u-\mu-d-1/2)_{u-s-1}}{(u+\mu-d-1/2)_{u+s}}$ | $-\left[-\frac{1}{2d}\right]^{u-s}(u+2\mu)_{u-s-1}$ |

Common functions such as powers of $x$, powers of trigonometric functions, or exponentials will satisfy the selective conditions (3.2) and (3.3), and will provide adequate expansion basis functions $y_s(x)$ for the perturbations $V^{(N)}(x)$. Several illustrative examples, concerning factorization types A to D, have been reported in Tables III and IV. Let us remark that, in several cases, since $Q_s(m)=1$, the computation of the $\Delta^n F_s^{(N)}$ and of the $\lambda_n(m;\mu)$ can be performed by means of reduced formulas, such as (3.17) instead of (3.14) and (3.27) instead of (3.26).

Let us emphasize that the main formulas to be used are the expressions (3.18) and (3.25) of the factorization instruments $K^{(N)}(x,m;\mu)$ and $L^{(N)}(m;\mu)$ together with

TABLE IV. Perturbed type-C and -D factorizations. Suitable basis sets and associated data.

| Type | C | | D |
|---|---|---|---|
| $y_s(x)$ | $x^{2s}$ | $x^{2s}$ | $\chi_s H_{2s}(b^{1/2}x)$ |
| $Y_s(x)$ | $x^{2s+1}$ | $x^{2s+1}$ | $b^{-1/2}\chi_{s+1}H_{2s+1}(b^{1/2}x)$ |
| $A_s(m)$ | $2m$ | $0$ | $1$ |
| $B_s(m)$ | $2b$ | $2b$ | $1$ |
| $\alpha_s$ | $2s+1$ | $2s+1$ | $1$ |
| $\beta_s$ | $0$ | $0$ | $0$ |
| $Y_sY_t$ | $y_{s+t+1}$ | $y_{s+t+1}$ | $\sum_{u=\lvert s-t\rvert}^{s+t+1} h(s,t,u)y_u$ |
| $Q_s(m)$ | $1$ | $1$ | $1$ |
| $S_N$ | $NS_1+N-1$ | $NS_1+N-1$ | $NS_1+N-1$ |
| $n_s^{(N)}$ | $S_N-s$ | $S_N-s$ | $S_N-s$ |
| $\lambda_0(m;\mu)$ | $4m-2\mu-1$ | $2m-2\mu-1$ | $2(m-\mu)$ |
| $\lambda_n(m;\mu)$ | $(2\mu+2n+1)\left[\begin{matrix}m-\mu\\ n\end{matrix}\right]+(2n+4)\left[\begin{matrix}m-\mu\\ n+1\end{matrix}\right]$ | $\left[\begin{matrix}m-\mu\\ n\end{matrix}\right]+2\left[\begin{matrix}m-\mu\\ n+1\end{matrix}\right]$ | $2\left[\begin{matrix}m-\mu\\ n\end{matrix}\right]+2\left[\begin{matrix}m-\mu\\ n+1\end{matrix}\right]$ |
| $C_{us}(\mu)$ | $-\frac{(\mu-s-3/2)_{u-s-1}}{2b^{u-s}}$ | $-\frac{(u-1/2)_{u-s-1}}{2b^{u-s}}$ | $\delta_{u,s+1}$ |

formulas (3.14), (3.26), and (3.24) (or their reduced version). Other equations are mostly intermediate ones required for the demonstrations.

## IV. ILLUSTRATIVE APPLICATIONS

In the present paper our attention has been focused on the factorization types A to D with unperturbed ladder functions $K^{(0)}(x,m)$ which are linear functions of the quantum number $m$ (see Table I). The type-A factorizable equation, i.e., the transformed Jacobi eigenequation, is of particular interest in computational physics; it is then interesting to work out at least one example of perturbed type-A factorization. As already known,[1,19–21] the Morse potential model leads to the solution of an exact type-B factorizable eigenequation. Thus, perturbed type-B factorization is of particular interest for an elaborate computation of diatomic rotation-vibration perturbed Morse-Pekeris energies, wave functions, and matrix elements. Particularly, it is expected to be useful for comparative studies of the rotation-vibration intensities of diatomic molecules, and/or for calculating the centrifugal distortion contributions to their rotational spectra, when the radial dependence of the fine-structure interaction terms is taken into account.[22] In the present paper, for sake of brevity, we give only the starting data concerning one example of perturbed type-B factorization (see the second column of Table III). Perturbed factorizations of harmonic-oscillator eigenequations are studied in detail; let us recall that the radial equation of the $D$-dimensional simple harmonic oscillator is simply related to a type-C factorizable equation (see Appendix D), while the one-dimensional linear harmonic oscillator is relevant to type-D factorization (see Table I).

### A. Perturbed type-A factorization with the associated basis functions $y_s(x)=a^2[\tan(ax/2)]^{2s}$

Let us assume that the perturbations can be written

$$V^{(N)}(x)=a^2\sum_{s=1}^{S_N+1} b_s^{(N)}\left[\tan\frac{ax}{2}\right]^{2s} . \tag{4.1}$$

The perturbed factorization and ladder functions associated with $V^{(N)}(x)$ are, respectively [see Eqs. (3.18), (3.26), and the first column of Table III],

$$\begin{aligned} L^{(N)}(m;\mu)&=-(2m-\mu+d-\tfrac{1}{2})k_0^{(N)}-\sum_{n=1}^{2S_N}\lambda_n\Delta^n F_0^{(N)} ,\\ K^{(N)}(x,m;\mu)&=\sum_{s=0}^{S_N}\left[\tan\frac{ax}{2}\right]^{2s+1}Q_s(m)\left[k_s^{(N)}+\sum_{n=1}^{2S_N-2s}\begin{bmatrix} m-\mu\\ n\end{bmatrix}\Delta^n F_s^{(N)}\right] ,\end{aligned} \tag{4.2}$$

where

$$\lambda_n=(\mu+d+n+\tfrac{1}{2})\begin{bmatrix} m-\mu\\ n\end{bmatrix}+(n+2)\begin{bmatrix} m-\mu\\ n+1\end{bmatrix} .$$

Since $Q_s(m)$ does depend on $m$ (see Table III), one has to compute the required $\Delta^n F_s^{(N)}$ in terms of the $k_u^{(N)}=\Delta^0 F_u^{(N)}$ by means of the general relation (3.14). Using Eq. (3.16), one gets

$$\begin{aligned} f_{us}(m)&=[(2m+1)(2u+1)-2d](u+d+m+1/2)_{u-s-1}(u-d+m-1/2)_{u-s-1} ,\\ g_{us}(m)&=(s-d+1/2)_{2s}(u+d+m+1/2)_{u-s-1}/(s-d+m+1/2)_{u+s} ,\end{aligned} \tag{4.3}$$

where

$$(m)_u=m(m-1)\cdots(m-u+1) .$$

Closed-form expressions of the $I(u,s;i,n)$ and $J(u,s;i,n)$ coefficients appearing in (3.14) are obtainable without special difficulty by using both Eqs. (4.3) and the expressions (A4) and (A5) of the $n$th discrete derivative of $(m)_u$ and $1/(m)_u$.

As an illustrative example, let us apply these results to the determination of the perturbed type-A eigenvalues associated with the following perturbing potential function:

$$V(x)=a^2\eta b_1^{(1)}\left[\tan\frac{ax}{2}\right]^2+a^2\eta^2\left[b_1^{(2)}\left[\tan\frac{ax}{2}\right]^2+b_2^{(2)}\left[\tan\frac{ax}{2}\right]^4\right] . \tag{4.4}$$

In order to avoid reproducing cumbersome expressions, we have limited ourselves to the second order ($N=2$) of the perturbation and chosen the low values $S_1=0$ and, consequently (see Table III), $S_2=2S_1+1=1$.

#### 1. *First order ($N=1$) of the perturbation: $S_1=0$*

The theoretical perturbed factorization and ladder functions are [see Eq. (4.2)]

$$\begin{aligned} L^{(1)}(m;\mu)&=-(2m-\mu+d-1/2)k_0^{(1)} ,\\ K^{(1)}(x,m;\mu)&=k_0^{(1)}\tan\frac{ax}{2} .\end{aligned} \tag{4.5}$$

#### 2. *Second order ($N=2$) of the perturbation: $S_2=1$*

The perturbed factorization and ladder functions are

$$L^{(2)}(m;\mu)=-(2m-\mu+d-1/2)k_0^{(2)}-\left[(\mu+d+3/2)\begin{bmatrix}m-\mu\\1\end{bmatrix}+3\begin{bmatrix}m-\mu\\2\end{bmatrix}\right]\Delta F_0^{(2)}$$
$$-\left[(\mu+d+5/2)\begin{bmatrix}m-\mu\\2\end{bmatrix}+4\begin{bmatrix}m-\mu\\3\end{bmatrix}\right]\Delta^2F_0^2\ ,$$
$$K^{(2)}(x,m;\mu)=\left[k_0^{(2)}+\begin{bmatrix}m-\mu\\1\end{bmatrix}\Delta F_0^{(2)}+\begin{bmatrix}m-\mu\\2\end{bmatrix}\Delta^2F_0^{(2)}\right]\tan\frac{ax}{2}+k_1^{(2)}Q_1(m)\left[\tan\frac{ax}{2}\right]^3\ , \qquad (4.6)$$

where

$$Q_1(m)=(m-d+1/2)_2/(3/2-d)_2\ .$$

In order to obtain the required $\Delta F_0^{(2)}$ and $\Delta^2F_0^{(2)}$ coefficients, one has first to calculate the contributions $w_u^{(2)}(m;\mu)$ originated from the first order ($N=1$) of the perturbation. Let us use their definition (3.4). Since on the one hand, $(K^{(1)})^2=(k_0^{(1)})^2[\tan(ax/2)]^2$, and on the other hand, $(K^{(1)})^2=w_1^{(2)}(m)[\tan(ax/2)]^2$, one gets

$$w_1^{(2)}=(k_0^{(1)})^2,\quad \Delta^i w_1^{(2)}=0 \quad \text{for any } i\neq 0\ . \qquad (4.7)$$

Then, one uses the general expressions (B4) with $S_N=1$, $I(1,0;0,1)=f_{10}(m=\mu)$, $I(1,0;0,2)=\Delta f_{10}$. From Eq. (4.11) one gets $f_{10}(m)=(6m+3-2d)/(3/2-d)_2$; $\Delta f_{10}=6/(3/2-d)_2$ and, since $\Delta w_1^{(2)}=\Delta^2 w_1^{(2)}=0$, one obtains

$$\Delta F_0^{(2)}=\frac{6\mu+3-2d}{(3/2-d)_2}k_1^{(2)}\ ,$$
$$\Delta^2F_0^{(2)}=\frac{6}{(3/2-d)_2}k_1^{(2)}\ . \qquad (4.8)$$

### 3. Expression of the total eigenvalue and ladder function

One now has to express the arbitrary constants $k_u^{(N)}$ in terms of the expansion coefficients $b_i^{(N)}$ of the perturbed potential $V^{(N)}(x)$. Since the maximum value of $S_N$ is $S_1=1$, one first writes down, once for all, the required expressions to be used for $N=1$ and 2. Picking up the expression of $C_{us}(\mu)$ from the first column of Table III and using Eq. (3.24), one gets

$$k_{S_N}^{(N)}=\frac{(S_N-d+1/2)_{2S_N}}{(S_N+\mu-d+1/2)_{2S_N+1}}b_{S_N+1}^{(N)}\ ,$$
$$k_{S_N-1}^{(N)}=(S_N-d-1/2)_{2S_N-2}\left[\frac{(S_N-\mu-d+1/2)b_{S_N+1}^{(N)}}{(S_N+\mu-d+1/2)_{2S_N}}+\frac{b_{S_N}^{(N)}+w_{S_N}^{(N)}}{(S_N+\mu-d-1/2)_{2S_N-1}}\right]\ . \qquad (4.9)$$

Setting, successively, $N=1$, $S_N=S_1=0$ and $N=2$, $S_N=S_2=1$ in these expressions, one finds

$$k_0^{(1)}=\frac{b_1^{(1)}}{(\mu-d+1/2)}\ ,$$
$$k_1^{(2)}=-\frac{(3/2-d)_2}{(\mu-d+3/2)_3}b_2^{(2)}\ ,$$
$$k_0^{(2)}=-\frac{(\mu+d-3/2)}{(\mu-d+3/2)(\mu-d+1/2)}b_2^{(2)}+\frac{1}{\mu-d+1/2}\left[b_1^{(2)}+\frac{(b_1^{(1)})^2}{(\mu-d+1/2)_2}\right]\ . \qquad (4.10)$$

The expressions of the perturbed eigenvalues $\Lambda_j^{(1)}(m)$ and $\Lambda_j^{(2)}(m)$ directly follow from the expressions (4.5) of $L^{(1)}(m;\mu)$ and (4.6) of $L^{(2)}(m;\mu)$; one replaces $m$ by $\bar{j}=j+\epsilon/2+1/2$, $\mu$ by $m$ and, rearranging the terms in order to put in evidence the coefficients of the given potential expansion coefficients $b_i^{(v)}$, one obtains

$$\Lambda_j^{(1)}(m)=\left\langle\left[\tan\frac{ax}{2}\right]^2\right\rangle b_1^{(1)}\ ,$$
$$\Lambda_j^{(2)}(m)=\left\langle\left[\tan\frac{ax}{2}\right]^2\right\rangle\left[b_1^{(2)}+\frac{(b_1^{(1)})^2}{(m-d+1/2)_2}\right]+\left\langle\left[\tan\frac{ax}{2}\right]^4\right\rangle b_2^{(2)}\ , \qquad (4.11)$$

where

$$a^2\left\langle\left[\tan\frac{ax}{2}\right]^2\right\rangle=\frac{2\bar{j}-m+d-1/2}{m-d+1/2}\ ,$$

$$a^2\left\langle\left[\tan\frac{ax}{2}\right]^4\right\rangle=\frac{1}{(m-d+3/2)_2}\Bigg[(m+d-1/2)_2(m-d-1/2)$$
$$+[(2m+2d-3)(m-d-1/2)+(m+d+3/2)(6m+3-2d)]\begin{bmatrix}\bar{j}-m\\1\end{bmatrix}$$
$$+24(m+1)\begin{bmatrix}\bar{j}-m\\2\end{bmatrix}+24\begin{bmatrix}\bar{j}-m\\3\end{bmatrix}\Bigg]\ .$$

Finally, the total eigenvalue and total ladder function of a type-A eigenequation (3.30) with a perturbing potential (4.4) are

$$\Lambda_j(m)=a^2(j+\epsilon/2+1/2)^2+\eta\Lambda_j^{(1)}(m)+\eta^2\Lambda_j^{(2)}(m)\ ,$$
$$K(x,m;\mu)=am\ \cot ax+\frac{ad}{\sin ax}+\eta K^{(1)}(x,m;\mu)+\eta^2K^{(2)}(x,m;\mu)\ . \tag{4.12}$$

The computation could be pursued up to higher orders of the perturbation without special difficulty, the central point being, at each order $N$ of the perturbation, the determination of the $\Delta^i w_u^{(N)}$ in terms of the $k_u(v)$ of the preceding orders ($v=1$ to $N-1$; $u=0$ to $S_v$) and then the determination of the $\Delta^n F_s^{(N)}$ in terms of the $k_u^{(N)}$ and $\Delta^i w_u^{(N)}$.

The results are valid for both classes of factorization: for class-I problems, i.e., when the constant $a$ involved in the unperturbed potential $U^{(0)}(x,m)$ is a real constant, $\epsilon=+1$, $\bar{j}=j+1$, while for class-II problems, i.e., when $a$ is a pure imaginary constant, $\epsilon=-1$, $\bar{j}=j$. Problems involving spherical harmonics or symmetric-top functions are relevant to class-I factorization (see Appendix D), while the use of perturbed type-A factorization for finding, for instance, an analytical approximate solution of the Schrödinger equation with a Gaussian potential requires class-II factorization.[23]

Let us remark that, as a byproduct of the computation, compact expressions of the diagonal $\langle jm|[\tan(ax/2)]^k|jm\rangle$ matrix elements ($k=2,4$) between the unperturbed type-A eigenfunctions (see Table II) have been obtained. As already pointed out,[23,6] such integrals, involving products of Jacobi polynomials, are not so easy to calculate by brute termwise integration: a possible way (valid for diagonal and off-diagonal integrals) is to use the orthonormality property of the eigenfunctions after expanding the Jacobi polynomials $P_v^{(\alpha,\beta)}(\cos ax)$ on the finite basis of the $P_k^{(a,b)}(\cos ax)$ by means of Miller's formula[24]

$$P_v^{(\alpha,\beta)}(\cos ax)=\sum_{k=0}^{v}C_k^{(v)}P_k^{(a,b)}(\cos ax)\ , \tag{4.13}$$

where

$$C_k^{(v)}=\frac{\Gamma(\alpha+\beta+v+k+1)\Gamma(a+b+k+1)\Gamma(\alpha+v+1)}{(v-k)!\Gamma(\alpha+\beta+v+1)\Gamma(a+b+2k+1)\Gamma(\alpha+k+1)}$$
$$\times{}_3F_2(k-v,\alpha+\beta+v+k+1,a+1;\alpha+k+1,a+b+2k+2;1)\ .$$

${}_3F_2(\ )$ is a hypergeometric function. When working out analytical expressions of the $\Lambda_j^{(N)}(m)$ within the classical Rayleigh-Schrödinger perturbation scheme, the final expressions of the $\Lambda_j^{(N)}(m)$ involve intricate and hardly reducible summations.

### B. Perturbed type-C factorization with the associated basis functions $y_s(x)=x^{2s}$

Let us assume that the perturbations can be written

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1}b_s^{(N)}x^{2s}\ . \tag{4.14}$$

The associated perturbed factorization and ladder functions are (see Table IV)

$$L^{(N)}(m;\mu)=-(4m-2\mu-1)k_0^{(N)}-\sum_{n=1}^{S_N}\lambda_n\Delta^nF_0^{(N)}\ ,$$
$$K^{(N)}(x,m;\mu)=\sum_{s=0}^{S_N}x^{2s+1}\left[k_s^{(N)}+\sum_{n=1}^{S_N-s}\begin{bmatrix}m-\mu\\n\end{bmatrix}\Delta^nF_s^{(N)}\right]\ , \tag{4.15}$$

where

$$\lambda_n=(2\mu+2n+1)\begin{bmatrix}m-\mu\\n\end{bmatrix}+(2n+4)\begin{bmatrix}m-\mu\\n+1\end{bmatrix}\ .$$

Since $Q_s(m)=1$, the reduced relation (3.17) holds for the determination of the $\Delta^n F_s^{(N)}$ coefficients in terms of the $k_s^{(N)}=\Delta^0 F_s^{(N)}$. One gets

$$\Delta^n F_s^{(N)}=\sum_{u=s+1}^{S_N-n+1}\left[-\frac{1}{b}\right]^{u-s}(\mu+u+n-1/2)_{u-s-1}\times[(2u+n-1)\Delta^{n-1}F_u^{(N)}+\tfrac{1}{2}\Delta^n w_u^{(N)}] . \quad (4.16)$$

As an illustrative application, let us apply these results to the determination of perturbed ladder functions and eigenvalues up to the fourth order ($N=4$) of the perturbation and consider the octic anharmonic perturbation

$$V(x)=\eta b_1^{(1)}x^2+\eta^2(b_1^{(2)}x^2+b_2^{(2)}x^4)+\eta^3(b_1^{(3)}x^2+b_2^{(3)}x^4+b_3^{(3)}x^6)+\eta^4(b_1^{(4)}x^2+b_2^{(4)}x^4+b_3^{(4)}x^6+b_4^{(4)}x^8) . \quad (4.17)$$

In order to avoid reproducing too many results, the low values $S_1=0$ and (see Table IV) $S_2=2S_1+1=1$, $S_3=3S_1+2=2$, and $S_4=4S_1+3=3$ have been chosen.

One writes down, once for all, the required expressions of the $\Delta^n F_s^{(N)}$ to be used at the successive orders of the perturbation. Using Eq. (4.16), one gets

$$\Delta F_{S_N-1}=-\frac{1}{b}\left[(2S_N+2)k_{S_N}+\tfrac{1}{2}\Delta w_{S_N}\right] ,$$

$$\Delta F_{S_N-2}=-\frac{1}{b}P(S_N-1,1)+\frac{1}{b^2}(\mu+S_N+1/2)P(S_N,1) ,$$

$$\Delta F_{S_N-3}=-\frac{1}{b}P(S_N-2,1)+\frac{1}{b^2}(\mu+S_N-1/2)P(S_N-1,1)-\frac{1}{b^3}(\mu+S_N+1/2)P(S_N,1) , \quad (4.18)$$

$$\Delta^2 F_{S_N-2}=-\frac{1}{b}P(S_N-1,2) ,$$

$$\Delta^2 F_{S_N-3}=-\frac{1}{b}P(S_N-2,2)+\frac{1}{b^2}(\mu+S_N+1/2)P(S_N-1,2) ,$$

$$\Delta^3 F_{S_N-3}=-\frac{1}{b}P(S_N-2,3) .$$

The shortened notation $k_s=k_s^{(N)}=\Delta^0 F_s^{(N)}$, $\Delta^n F_s=\Delta^n F_s^{(N)}$, and $P(s,n)=(2s+n+1)\Delta^{n-1}F_s^{(N)}+\frac{1}{2}\Delta^n w_s^{(N)}$ has been used.

*1. First order $N=1$ of the perturbation: $S_1=0$*

The perturbed factorization and ladder functions are

$$L^{(1)}(m;\mu)=-(4m-2\mu-1)k_0^{(1)} ,\qquad K^{(1)}(x,m;\mu)=k_0^{(1)}x . \quad (4.19)$$

*2. Second order $N=2$ of the perturbation: $S_2=1$*

Let us first calculate the contributions $\Delta^i w_u^{(2)}$ originating from the first order ($N=1$) of the perturbation. Since, on the one hand, $(K^{(1)})^2=(k_0^{(1)})^2x^2$, and, on the other hand, $(K^{(1)})^2=w_1^{(2)}(m)x^2$, one gets, for any $i$,

$$w_1^{(2)}=(k_0^{(1)})^2,\quad \Delta^i w_1^{(2)}=0 . \quad (4.20)$$

The perturbed factorization and ladder functions are

$$L^{(2)}(m;\mu)=-(4m-2\mu-1)k_0^{(2)}+\left[(2\mu+3)\binom{m-\mu}{1}+6\binom{m-\mu}{2}\right]\Delta F_0^{(1)} , \quad (4.21)$$

$$K^{(2)}(x,m;\mu)=\left[k_0^{(2)}+\binom{m-\mu}{1}\Delta F_0^{(2)}\right]x+k_1^{(2)}x^3 ,$$

where [see Eq. (4.18) with $S_N=1$]

$$\Delta F_0^{(2)}=-\frac{4}{b}k_1^{(2)} .$$

*3. Third order $N=3$ of the perturbation: $S_3=2$*

Since, on the one hand, $2K^{(1)}K^{(2)}=w_1^{(3)}(m)x^2+w_2^{(3)}(m)x^4$ [see the definition (3.4) of the $w_u^{(N)}(m)$], and, on the other hand, $2K^{(1)}K^{(2)}=2k_0^{(1)}[k_0^{(2)}+\binom{m-\mu}{1}\Delta F_0^{(2)}]x^2+2k_0^{(1)}k_1^{(2)}x^4$ [see Eqs. (4.19) and (4.21)], one gets the following nonvanishing increments originating from the preceding orders of the perturbation:

$$w_1^{(3)}=2k_0^{(1)}k_0^{(2)} ,\qquad \Delta w_1^{(3)}=2k_0^{(1)}\Delta F_0^{(2)} ,\qquad w_2^{(3)}=2k_0^{(1)}k_1^{(2)} . \quad (4.22)$$

The third-order perturbed factorization and ladder functions are

$$L^{(3)}(m;\mu)=-(4m-2\mu-1)k_0^{(3)}+\left[(2\mu+3)\binom{m-\mu}{1}+6\binom{m-\mu}{2}\right]\Delta F_0^{(3)}-\left[(2\mu+5)\binom{m-\mu}{2}+8\binom{m-\mu}{3}\right]\Delta^2 F_0^{(3)} , \quad (4.23)$$

$$K^{(3)}(x,m;\mu)=\left[k_0^{(3)}+\binom{m-\mu}{1}\Delta F_0^{(3)}+\binom{m-\mu}{2}\Delta^2 F_0^{(3)}\right]x+\left[k_1^{(3)}+\binom{m-\mu}{1}\Delta F_1^{(3)}\right]x^3+k_2^{(3)}x^5 ,$$

where [see Eq. (4.18) with $S_N=2$]

$$\Delta F_1^{(3)}=-\frac{6}{b}k_2^{(3)} ,$$

$$\Delta F_0^{(3)}=\frac{6}{b^2}(\mu+5/2)k_2^{(3)}-\frac{4}{b}k_1^{(3)}-\frac{1}{2b}\Delta w_1 ,$$

$$\Delta^2 F_0^{(3)}=\frac{30}{b^2}k_2^{(3)} .$$

### 4. Fourth order $N=4$ of the perturbation

Using, on the one hand, the already known expression of $2K^{(1)}K^{(3)}+(K^{(2)})^2$ [see Eqs. (4.19), (4.21), and (4.23)] and, on the other hand, their definition (3.4), the $w_u^{(N)}(m)$ are obtained as series of $\binom{m-\mu}{u}$. Such series can be viewed as their Newton's expansion and one gets

$$\begin{aligned} w_3^{(4)}&=2k_0^{(1)}k_2^{(3)}+(k_1^{(2)})^2 ,\\ w_2^{(4)}&=2k_0^{(1)}k_1^{(3)}+2k_1^{(2)}k_0^{(2)} ,\\ \Delta w_2^{(4)}&=2k_0^{(1)}\Delta F_1^{(3)}+2k_1^{(2)}\Delta F_0^{(2)} ,\\ w_1^{(4)}&=2k_0^{(1)}k_0^{(3)}+(k_0^{(2)})^2 ,\\ \Delta w_1^{(4)}&=2k_0^{(1)}\Delta F_0^{(3)}+2k_0^{(2)}\Delta F_0^{(2)}+(\Delta F_0^{(2)})^2 ,\\ \Delta^2 w_1^{(4)}&=2k_0^{(1)}\Delta^2 F_0^{(3)}+2(\Delta F_0^{(2)})^2 . \end{aligned} \tag{4.24}$$

The expressions of the factorization and ladder functions follow from Eq. (4.15), with $N=4$ and $S_N=3$, and, for brevity, are not reproduced. From Eq. (4.18) with $S_N=3$, one obtains, after a few algebraic manipulations,

$$\begin{aligned} \Delta F_2^{(4)}&=-\frac{8}{b}k_3^{(4)} ,\\ \Delta F_1^{(4)}&=\frac{8}{b^2}(\mu+7/2)k_3^{(4)}-\frac{6}{b}(k_2^{(4)}+\tfrac{1}{12}\Delta w_2^{(4)}) ,\\ \Delta F_0^{(4)}&=-\frac{8}{b^3}(\mu+7/2)_2k_3^{(4)}\\ &\quad+\frac{6}{b^2}(\mu+5/2)(k_2^{(4)}+\tfrac{1}{12}\Delta w_2^{(4)})\\ &\quad-\frac{4}{b}(k_1^{(4)}+\tfrac{1}{8}\Delta w_1^{(4)}) , \end{aligned} \tag{4.25}$$

$$\begin{aligned} \Delta^2 F_1^{(4)}&=\frac{54}{b^2}k_3^{(4)} ,\\ \Delta^2 F_0^{(4)}&=-\frac{96}{b^3}(\mu+7/2)k_3^{(4)}+\frac{30}{b^2}\left[k_2^{(4)}+\tfrac{1}{12}\Delta w_2^{(4)}\right]\\ &\quad-\frac{1}{2b}\Delta^2 w_1^{(4)} ,\\ \Delta^3 F_0^{(4)}&=-\frac{336}{b^3}k_3^{(4)} . \end{aligned}$$

### 5. Expressions of the total eigenvalue and ladder operator

The last step of the computation is the determination of the arbitrary constants $k_u^{(v)}$ in terms of the potential expansion coefficients $b_i^{(v)}$. One picks up from the first column of Table IV the expression of $C_{us}(\mu)$, and using Eq. (3.24), one obtains the following required expressions to be used successively for $N=1$ and $S_1=0$, $N=2$ and $S_2=1$, $N=3$ and $S_3=2$, and $N=4$ and $S_4=3$:

$$\begin{aligned} k_{S_N}&=\frac{1}{2b}b_{S_N+1} ,\\ k_{S_N-1}&=-\frac{1}{2b^2}(\mu-S_N-1/2)b_{S_N+1}+\frac{1}{2b}b_{S_N} ,\\ k_{S_N-2}&=\frac{1}{2b^3}(\mu-S_N+1/2)_2b_{S_N+1}\\ &\quad-\frac{1}{2b^2}(\mu-S_N+1/2)b_{S_N}+\frac{1}{2b}b_{S_N-1} ,\\ k_{S_N-3}&=-\frac{1}{2b^4}(\mu-S_N+3/2)_3b_{S_N+1}\\ &\quad+\frac{1}{2b^3}(\mu-S_N+3/2)_2b_{S_N}\\ &\quad-\frac{1}{2b^2}(\mu-S_N+3/2)b_{S_N-1}+\frac{1}{2b}b_{S_N-2} , \end{aligned} \tag{4.26}$$

where $k_s=k_s^{(N)}$ and $b_s=b_s^{(N)}+w_s^{(N)}$. Using these expressions together with the above expressions of the factorization functions $L^{(N)}(m;\mu)$, substituting $m$ with $\bar{j}=j+\epsilon/2+\frac{1}{2}$, $\mu$ with $m$ and rearranging the terms, one obtains

$$\begin{aligned} \Lambda_j^{(1)}(m)&=\langle x^2\rangle b_1^{(1)} ,\\ \Lambda_j^{(2)}(m)&=\langle x^2\rangle(b_1^{(2)}+w_1^{(2)})+\langle x^4\rangle b_2^{(2)} ,\\ \Lambda_j^{(3)}(m)&=\langle x^2\rangle(b_1^{(3)}+w_1^{(3)})+\langle x^4\rangle(b_2^{(3)}+w_2^{(3)})+\langle x^6\rangle b_3^{(3)}+\frac{\bar{\lambda}_1}{b}\Delta w_1^{(3)} ,\\ \Lambda_j^{(4)}(m)&=\langle x^2\rangle(b_1^{(4)}+w_1^{(4)})+\langle x^4\rangle(b_2^{(4)}+w_2^{(4)})+\langle x^6\rangle(b_3^{(4)}+w_3^{(4)})+\langle x^8\rangle b_4^{(4)}\\ &\quad+\frac{\bar{\lambda}_1}{2b^2}[b\Delta w_1^{(4)}-(m+5/2)\Delta w_2^{(4)}]+\frac{\bar{\lambda}_2}{2b^2}(\Delta^2 w_1^{(4)}-5\Delta^2 w_2^{(4)}) , \end{aligned} \tag{4.27}$$

where $\bar{\lambda}_n=(2m+2n+1)\binom{\bar{j}-m}{n}+(2n+4)\binom{\bar{j}-m}{n+1}$, the $\Delta^i w_u^{(N)}$ are given by Eqs. (4.20), (4.22), and (4.24), and

$$\langle x^2\rangle = -\frac{1}{b}(2\bar{j}-m-1/2)\ ,$$

$$\langle x^4\rangle = \frac{1}{b^2}\left[(m-1/2)_2+(6m+3)\begin{bmatrix}\bar{j}-m\\1\end{bmatrix}+12\begin{bmatrix}\bar{j}-m\\2\end{bmatrix}\right],$$

$$\langle x^6\rangle = -\frac{1}{b^3}\left[(m-5/2)_3+3(4m^2+4m+5)\begin{bmatrix}\bar{j}-m\\1\end{bmatrix}+30(2m+3)\begin{bmatrix}\bar{j}-m\\2\end{bmatrix}+120\begin{bmatrix}\bar{j}-m\\3\end{bmatrix}\right],$$

$$\langle x^8\rangle = \frac{1}{b^4}\left[(m-7/2)_4+5(4m^3+6m^2+23m+21/2)\begin{bmatrix}\bar{j}-m\\1\end{bmatrix}+15(12m^2+40m+49)\begin{bmatrix}\bar{j}-m\\2\end{bmatrix}\right.$$
$$\left.+420(2m+5)\begin{bmatrix}\bar{j}-m\\3\end{bmatrix}+1680\begin{bmatrix}\bar{j}-m\\4\end{bmatrix}\right].$$

For class-I problems $\binom{\bar{j}-m}{n}=\binom{v+1}{n}$, while for class-II problems

$$\begin{bmatrix}j-m\\n\end{bmatrix}=\begin{bmatrix}-v\\n\end{bmatrix}=(-1)^n\begin{bmatrix}v+n+1\\n\end{bmatrix},$$

where $v=0,1,2,\ldots$.

Finally, since $L^{(0)}(m)=-4bm$ (see Table I), for both classes of factorization ($\epsilon=+1$ or $-1$), the total eigenvalue of the eigenequation (4.12) with the perturbing potential (4.13) is

$$\Lambda_j(m)=-4b\bar{j}+\eta\Lambda_j^{(1)}(m)+\eta^2\Lambda_j^{(2)}(m)+\eta^3\Lambda_j^{(3)}(m)$$
$$+\eta^4\Lambda_j^{(4)}(m)\ . \qquad (4.28)$$

The expression of the associated ladder function follows from Eqs. (2.8) and (4.15) (with $N=4$ and $S_N=3$) and, for brevity, has not been reproduced.

### C. Perturbed type-D factorization with the associated basis functions $y_s=x^{2s}$

The perturbations and the associated ladder function are still given by Eqs. (4.14) and (4.15), while the associated factorization function is (see Table IV)

$$L^{(N)}(m;\mu)=-(2m-2\mu-1)k_0^{(N)}-\sum_{n=1}^{S_N}\lambda_n\Delta^nF_0^{(N)}\ , \qquad (4.29)$$

where

$$\lambda_n=\begin{bmatrix}m-\mu\\n\end{bmatrix}+2\begin{bmatrix}m-\mu\\n+1\end{bmatrix}. \qquad (4.30)$$

One substitutes $(m-\mu)$ with $(v+1)$ or $(m-\mu)$ with $(-v)$ for class-I problems ($b<0,\epsilon=+1$) or for class-II problems ($b>0,\epsilon=-1$) and one gets the following expression of the perturbed eigenvalues:

$$\Lambda_v^{(N)}=-2\epsilon b(v+1/2)k_0^{(N)}-\sum_{n=1}^{S_N}\bar{\lambda}_n\Delta^nF_0^{(N)}\ , \qquad (4.31)$$

where

$$\bar{\lambda}_n=\begin{bmatrix}v+1\\n\end{bmatrix}+2\begin{bmatrix}v+1\\n+1\end{bmatrix}\quad\text{for class I}$$

$$\bar{\lambda}_n=(-1)^n\left[\begin{bmatrix}v+n-1\\n\end{bmatrix}-2\begin{bmatrix}v+n\\n+1\end{bmatrix}\right]\quad\text{for class II}\ .$$

Since $Q_s(m)=1$, the reduced relation (3.17) holds for the determination of the $\Delta^nF_s^{(N)}$ coefficients in terms of the $k_s^{(N)}=\Delta^0F_s^{(N)}$. One gets

$$\Delta^nF_s^{(N)}=\sum_{u=s+1}^{S_N-n+1}\left[-\frac{1}{b}\right]^{u-s}(u+1/2)_{u-s}$$
$$\times\left[\Delta^{n-1}F_u^{(N)}+\frac{\Delta^nw_u^{(N)}}{4(u+1/2)}\right]. \qquad (4.32)$$

As a short illustrative application, let us consider the solution of a perturbed type-D eigenequation (3.30) with the sextic perturbating potential

$$V(x)=\eta(b_1^{(1)}x^2+b_2^{(1)}x^4)$$
$$+\eta^2(b_1^{(2)}x^2+b_2^{(2)}x^4+b_3^{(2)}x^6)\ . \qquad (4.33)$$

The maximum value of $S_N$ is $S_2=2S_1+1=3$. The required expressions of the $\Delta^nF_s^{(N)}$ to be used at the successive orders of the perturbation are easily obtained by means of Eq. (4.32) and a few algebraic manipulations,

$$\Delta F_{S_N-1}=-\frac{2}{b}(S_N+1/2)\bar{k}_{S_N}\ ,$$

$$\Delta F_{S_N-2}=\frac{2}{b^2}(S_N+1/2)_2\bar{k}_{S_N}-\frac{2}{b}(S_N-1/2)\bar{k}_{S_N-1}\ ,$$

$$\Delta F_{S_N-3}=-\frac{2}{b^3}(S_N+1/2)_3\bar{k}_{S_N}+\frac{2}{b^2}(S_N-1/2)_2\bar{k}_{S_N-1}$$
$$-\frac{2}{b}(S_N-3/2)\bar{k}_{S_N-2}\ , \qquad (4.34)$$

$$\Delta^2F_{S_N-2}=\frac{4}{b^2}(S_N+1/2)_2\bar{k}_{S_N}-\frac{1}{2b}\Delta^2w_{S_N-1}\ ,$$

$$\Delta^2 F_{S_N-3} = -\frac{8}{b^3}(S_N+1/2)_3\bar{k}_{S_N} + \frac{4}{b^2}(S_N-1/2)_2\bar{k}_{S_N-1} + \frac{1}{2b^2}(S_N-3/2)\Delta^2 w_{S_N-1} - \frac{1}{2b}\Delta^2 w_{S_N-2} ,$$

$$\Delta^3 F_{S_N-3} = \bar{k} - \frac{8}{b^3}(S_N+1/2)_3\bar{k}_{S_N} + \frac{1}{b^2}(S_N-3/2)\Delta^2 w_{S_N-1} - \frac{1}{2b}\Delta^3 w_{S_N-2} ,$$

where the shortened notation $\bar{k}_u = k_u^{(n)} + [\Delta w_u/4(u+1/2)]$ is used.

Picking up the expression of $C_{us}(\mu)$ from Table IV and using Eq. (3.24), one gets

$$k_s^{(N)} = -\tfrac{1}{2}\sum_{u=s+1}^{S_N+1}\left[\frac{1}{b}\right]^{u-s}(u-1/2)_{u-s-1} \times (b_u^{(N)} + w_u^{(N)}) . \qquad (4.35)$$

Particularly, at any order $N$ of the perturbation

$$k_{S_N} = -\frac{1}{2b}b_{S_N+1} ,$$
$$k_{S_N-1} = -\frac{1}{2b^2}(S_N+1/2)b_{S_N+1} - \frac{1}{2b}b_{S_N} ,$$
$$k_{S_N-2} = -\frac{1}{2b^3}(S_N+1/2)_2 b_{S_N+1} - \frac{1}{2b^2}(S_N-1/2)b_{S_N} - \frac{1}{2b}b_{S_N-1} , \qquad (4.36)$$
$$k_{S_N-3} = -\frac{1}{2b^4}(S_N+1/2)_3 b_{S_N+1} - \frac{1}{2b^3}(S_N-1/2)_2 b_{S_N} - \frac{1}{2b^2}(S_N-3/2)b_{S_N-1} - \frac{1}{2b}b_{S_N-2} ,$$

where $k_u = k_u^{(N)}$ and $b_u = b_u^{(N)} + w_u^{(N)}$.

*1. First order $N=1$ of the perturbation: $S_1=1$*

The perturbed ladder function and eigenvalues are [see Eqs. (4.15) and (4.29)]

$$K^{(1)}(x,m;\mu) = \left[k_0^{(1)} + \begin{bmatrix} m-\mu \\ 1 \end{bmatrix}\Delta F_0^{(1)}\right]x + k_1^{(1)}x^3 ,$$
$$\Lambda_v^{(1)} = k_0^{(1)} - (2k_0^{(1)} + \Delta F_0^{(1)})\begin{bmatrix} v+1 \\ 1 \end{bmatrix} - 2\Delta F_0^{(1)}\begin{bmatrix} v+1 \\ 2 \end{bmatrix} \quad \text{for class I} , \qquad (4.37)$$
$$\Lambda_v^{(1)} = k_0^{(1)} + (2k_0^{(1)} + \Delta F_0^{(1)})\begin{bmatrix} v \\ 1 \end{bmatrix} - 2\Delta F_0^{(1)}\begin{bmatrix} v+1 \\ 2 \end{bmatrix} \quad \text{for class II} ,$$

where [set $S_N=1$ in Eqs. (4.36) and (4.34)]

$$k_1^{(1)} = -\frac{1}{2b}b_2^{(1)} ,$$
$$k_0^{(1)} = -\frac{3}{4b^2}b_2^{(1)} - \frac{1}{2b}b_1^{(1)} ,$$
$$\Delta F_0^{(1)} = -\frac{3}{b}k_1^{(1)} = \frac{3}{2b^2}b_2^{(1)} .$$

*2. Second order $N=2$ of the perturbation: $S_N=3$*

Since, on the one hand, $(K^{(1)})^2 = w_1^{(2)}(m)x^2 + w_2^{(2)}(m)x^4 + w_3^{(2)}(m)x^6$, and on the other hand [see Eq. (4.36) and use $\binom{m}{1}\binom{m}{1} = \binom{m}{1} + 2\binom{m}{1}$].

$$(K^{(1)})^2 = \left[k_0^2 + (2k_0\Delta F_0 + \Delta F_0^2)\begin{bmatrix} m-\mu \\ 1 \end{bmatrix} + 2\Delta F_0^2\right]x^2 + 2k_1\left[k_0 + \begin{bmatrix} m-\mu \\ 1 \end{bmatrix}\Delta F_0\right]x^4 + k_1^2 x^6 ,$$

one gets the following expressions of the $w_u^{(2)}(m)$.

$$w_3^{(2)}(m) = (k_1^{(1)})^2 ,$$
$$w_2^{(2)}(m) = 2k_1^{(1)}k_0^{(1)} + 2k_1^{(1)}\Delta F_0^{(1)}\begin{bmatrix} m-\mu \\ 1 \end{bmatrix} , \qquad (4.38)$$
$$w_1^{(2)}(m) = (k_0^{(1)})^2 + [2k_0^{(1)}\Delta F_0^{(1)} + (\Delta F_0^{(1)})^2]\begin{bmatrix} m-\mu \\ 1 \end{bmatrix} + 2(\Delta F_0^{(1)})^2\begin{bmatrix} m-\mu \\ 2 \end{bmatrix} .$$

The required nonvanishing expressions of the $\Delta^n w_u^{(2)} = \Delta^n w_u^{(2)}(m=\mu)$ directly follow from the comparison of these expressions with their Newton expansion [see Eq. (A9)].

The expression of the perturbed eigenvalue $\Lambda_v^{(2)}$ in terms of the $k_u^{(2)}$ and nonvanishing $\Delta^t w_u^{(2)}$ is given by Eq. (4.32) where [see Eq. (4.34) with $N=2$, $S_2=3$]

$$\Delta F_0^{(2)} = -\frac{105}{4b^3}k_3 + \frac{15}{2b^2}k_2 - \frac{3}{b}k_1 + \frac{3}{4b^2}\Delta w_2 - \frac{1}{2b}\Delta w_1 ,$$
$$\Delta^3 F_0^{(2)} = -\frac{105}{b^3}k_3 + \frac{15}{b^2}k_2 + \frac{3}{2b^2}\Delta w_2 - \frac{1}{2b}\Delta^2 w_1 , \qquad (4.39)$$
$$\Delta^3 F_0^{(2)} = -\frac{105}{b^3}k_3 ,$$

and, finally, in terms of the potential expansion coefficients, one gets

$$k_0^{(2)} = -\frac{105}{16b^2}b^4 - \frac{15}{8b^3}(b_3+w_3) - \frac{3}{4b^2}(b_2+w_2) - \frac{1}{2b}(b_1+w_1) ,$$
$$\Delta F_0^{(2)} = \frac{105}{8b^4}b_4 - \frac{15}{2b^3}b_3 + \frac{3}{2b^2}b_2 - \frac{15}{2b^3}w_3 + \frac{3}{2b^2}w_2 + \frac{3}{4b^2}\Delta w_2 - \frac{1}{2b}\Delta w_1 ,$$
$$\qquad (4.40)$$
$$\Delta^2 F_0^{(2)} = \frac{105}{4b^4}b_4 - \frac{15}{2b^3}b_3 - \frac{15}{2b^3}w_3 + \frac{3}{2b^2}\Delta w_2 - \frac{1}{2b}\Delta w_1 ,$$
$$\Delta^3 F_0^{(2)} = \frac{105}{2b^4}b_4 .$$

The associated perturbed ladder function $K^{(2)}(x,m;\mu)$ directly follows from Eq. (4.15) where the required $(u\neq 0)$ $\Delta^n F_u^{(2)}$ are given by Eq. (4.34) (set $N=2$ and $S_N=3$).

Finally, since $L^{(0)}(m) = -2bm$ (see Table I), it follows that $2E_v^{(0)} = \Lambda^{(0)} + b(2m+1) = -2b(j+\epsilon/2+1/2) + b(2m+1) = -2b\epsilon(v+1/2)$ and the total eigenvalue of the perturbed type-D eigenequation, with the perturbating potential (4.33), is

$$E_v=-b\epsilon(v+1/2)+\tfrac{1}{2}\Lambda_v^{(1)}+\tfrac{1}{2}\Lambda_v^{(2)} .$$

As a particular case, let us consider the $x^4$-perturbed harmonic-oscillator eigenequation

$$\left[\frac{d^2}{dx^2}-x^2-2qx^4+2E\right]\Psi(x)=0 .$$

Setting $b_2^{(1)}=-2q$ and otherwise $b_u^{(v)}=0$ in the above results, one gets, at the first order of the perturbation

$$k_1^{(1)}=\frac{1}{b}q,\quad k_0^{(1)}=\frac{3}{2b^2}q,\quad \Delta F_0^{(1)}=-\frac{3}{b^2}q ,$$

and, for both classes ($b=-1$, $\epsilon=+1$ or $b=1$, $\epsilon=-1$) one finds

$$E_v^{(1)}=\tfrac{1}{2}\Lambda_v^{(1)}=3q\begin{bmatrix} v+1\\ 2\end{bmatrix}+\tfrac{3}{4}q .$$

At the second order $N=2$ of the perturbation, the expressions (4.40) reduce to $k_0^{(2)}=-21q^2/4b^5$, $\Delta F_0^{(2)}=0$, $\Delta^2F_0^{(2)}=-51q^2/2b^5$, and one finds

$$E_v^{(2)}=\left[-\frac{51}{2}\begin{bmatrix} v+1\\ 3\end{bmatrix}-\frac{51}{4}\begin{bmatrix} v+1\\ 2\end{bmatrix}-\frac{21}{4}\begin{bmatrix} v+1\\ 1\end{bmatrix}+\frac{21}{8}\right]q^2 \text{ for class I },$$

$$E_v^{(2)}=\left[-\frac{51}{2}\begin{bmatrix} v+2\\ 3\end{bmatrix}+\frac{51}{4}\begin{bmatrix} v+1\\ 2\end{bmatrix}-\frac{21}{4}\begin{bmatrix} v\\ 1\end{bmatrix}-\frac{21}{8}\right]q^2 \text{ for class II }.$$

It is easily checked that these two expressions of $E_v^{(2)}$ are quite equivalent and, as well as $E_v^{(1)}$, give again previous results.[25]

Of course, the algebraic recursive determination of the $\Lambda_v^{(N)}$ can be pursued up to any order $N$ of the perturbation without special difficulty: the expression of the $k_u^{(v)}$ in terms of the potential coefficients is straightforward [see Eq. (4.36)] while the determination of the $\Delta^nF_u^{(v)}$ and of the $\Delta^iw_u^{(v)}$ only requires some few algebraic manipulations.

### D. Perturbed type-D factorization with the associated basis functions $y_s(x)=\chi_sH_{2s}(b^{1/2}x)$

Let us use the expansion of the perturbating potential $V(x)$ on the Hermite basis[26] and set

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1} b_s^{(N)}\chi_sH_{2s}(b^{1/2}x) , \tag{4.41}$$

where the factor $\chi_s=b(s-1)!/2(2s-1)!$ is introduced for computational convenience and we have limited ourselves to class-II problems ($b>0,\epsilon=-1,m-j=v=0,1,2,\ldots$).

The associated perturbed factorization and ladder functions are

$$\begin{aligned} L^{(N)}(m;\mu)&=-2(m-\mu)k_0^{(N)}-\sum_{n=1}^{S_N}\lambda_n\Delta^nF_0^{(N)} ,\\ K^{(N)}(x,m;\mu)&=b^{-1/2}\sum_{s=0}^{S_N}\chi_{s+1}H_{2s+1}(b^{1/2}x)\left[k_s^{(N)}+\sum_{n=1}^{S_N-s}\begin{bmatrix} m-\mu\\ n\end{bmatrix}\Delta^nF_s^{(N)}\right] ,\end{aligned} \tag{4.42}$$

where

$$\lambda_n=2\begin{bmatrix} m-\mu\\ n\end{bmatrix}+2\begin{bmatrix} m-\mu\\ n+1\end{bmatrix} .$$

Replacing $m$ by $j$ and $\mu$ by $m$ in the above expression of $L^{(N)}(m;\mu)$, one obtains the following expression of the perturbed harmonic-oscillator eigenvalue ($b>0,v=m-j$):

$$\Lambda_v^{(N)}=2vk_0^{(N)}+2\sum_{n=1}^{S_N}(-1)^n\begin{bmatrix} v+n-1\\ n+1\end{bmatrix}\Delta^nF_0^{(N)} . \tag{4.43}$$

Since $Q_s(m)=1$, the reduced relation (3.17) holds for the determination of the $\Delta^nF_s^{(N)}$ coefficients in terms of the $k_s^{(N)}=\Delta^0F_s^{(N)}$ constants and one gets (see Table IV)

$$\Delta^nF_s^{(N)}=\sum_{u=s+1}^{S_N}(-2)^{u-s}(\Delta^{n-1}F_u^{(N)}+\tfrac{1}{2}\Delta^nw_u^{(N)}) . \tag{4.44}$$

Owing to the simplicity of this relation, a closed-form expression of the $\Delta^nF_s^{(N)}$ in terms of the $k_s^{(N)}=\Delta^0F_s^{(N)}$ and of the $\Delta^iw_u^{(N)}$ can be derived. One obtains (see Appendix B)

$$\Delta^nF_{S_N-\sigma}^{(N)}=\sum_{t=0}^{\sigma-n}\left[(-2)^{\sigma-t}\begin{bmatrix} \sigma-t-1\\ n-1\end{bmatrix}k_{S_N-t}^{(N)}+\tfrac{1}{2}\sum_{i=0}^{n-1}(-1)^i\begin{bmatrix} \sigma-t-i-1\\ n-i-1\end{bmatrix}\Delta^{i+1}w_{S_N-t-i}^{(N)}\right] . \tag{4.45}$$

Finally, after substituting for $\Delta^nF_0^{(N)}$ this last expression into Eq. (4.43), one finds, after rearranging the summations and using (A11), the following closed-form expression of the perturbed harmonic-oscillator eigenvalues:

$$\Lambda_v^{(N)}=\sum_{u=0}^{S_N} 2^{u+1}\begin{bmatrix} v \\ u+1\end{bmatrix} k_u^{(N)}+\sum_{u=1}^{S_N} 2^u \sum_{i=0}^{u-1}(-1)^i \begin{bmatrix} v+i \\ u+1\end{bmatrix}\Delta^{i+1} w_{u-i}^{(N)} , \quad (4.46)$$

where

$$k_u^{(N)}=b_{u+1}^{(N)}+w_{u+1}^{(N)} ,$$

$$w_u^{(N)}(m=\mu)=\sum_{v=1}^{N-1}\sum_{s=0}^{S_v}\sum_{t=0}^{S_N-v} h(s,t;u)k_s^{(v)}k_t^{(N-v)} ,$$

$$\Delta^i w_u^{(N)}=\sum_{v=1}^{N-1}\sum_{s=0}^{S_v}\sum_{t=0}^{S_N-v} h(s,t;u)\sum_{n=0}^{S_v-s}\sum_{j=0}^{S_N-v-t}\begin{bmatrix} i \\ n\end{bmatrix}\begin{bmatrix} n \\ i-j\end{bmatrix}\Delta^n F_s^{(v)}\Delta^j F_t^{(N-v)} ,$$

$$h(s,t;u)=\frac{2^{s+t-u}(2u-1)!s!t!}{(u-1)!(s+u-t)!(t+u-s)!(s+t+1-u)!} .$$

The expression of $k_u^{(N)}$ is obtained by picking up the expression of $C_{us}(\mu)$ from Table IV and using Eq. (3.24), while the expressions of the $w_u^{(N)}$ and of the $\Delta^i w_u^{(N)}$ directly follow from the general results of Appendix C [see Eqs. (C1)–(C3)].

Since $L^{(0)}(m)=-2bm$ (see Table I), one obtains $2E_v^{(0)}=L^{(0)}(j)+b(2m+1)=2b(v+1/2)$, and the total perturbed harmonic-oscillator energy is

$$E_v=b(v+1/2)+\tfrac{1}{2}\Lambda_v^{(1)}+\tfrac{1}{2}\Lambda_v^{(2)}+\cdots . \quad (4.47)$$

Of course, keeping in mind that $x^4=\frac{3}{4}+\frac{3}{4}H_2(x)+\frac{1}{16}H_4(x)$, formula (4.46) should be used for computing the $x^4$-perturbed harmonic-oscillator energies up to high orders of the perturbation. The eventual interest of such a computation is under consideration.

Note that the use of an Hermite polynomial $x$ basis instead of the familiar $x^k$ basis greatly simplifies the computational algorithm. Indeed, when using the perturbative series

$$V(x)=\sum_{s=1}^{S_1+1} b_s\chi_s H_{2s}(b^{1/2}x) ,$$

instead of

$$V(x)=\sum_{s=1}^{S_1+1} B_s x^{2s} ,$$

one obtains the following compact expression of the first-order eigenenergy involving only one summation:

$$\Lambda_v^{(1)}=\sum_{s=1}^{S_1+1} 2s\begin{bmatrix} v \\ s\end{bmatrix} b_s \quad (4.48)$$

instead of[27]

$$\Lambda_v^{(1)}=\sum_{s=1}^{S_1+1}\frac{(2s)!}{s!(4b)^s}B_s\sum_{u=0}^{v} 2u\begin{bmatrix} s \\ u\end{bmatrix}\begin{bmatrix} v \\ u\end{bmatrix} . \quad (4.49)$$

As already pointed out,[15] this simplification is not a matter of algebraic manipulation, it is a telescopic effect: indeed, the two expressions of $V(x)$ are not equivalent unless the mathematical assumption $S_1\to\infty$ holds. This simplification still persists at the higher orders of the perturbation.

## V. CONCLUSION

The perturbed ladder operator method has been rendered hopefully operational for solving analytically wave equations which can be viewed as "perturbed factorizable" equations. This can be done by extracting from the given physical model potential $V(x,m)$ an unperturbed kernel $U^{(0)}(x,m)$ leading (directly or eventually via the artificial device) to a factorizable eigenequation. Of course, this unperturbed potential $U^{(0)}(x,m)$ has to be a fair approximation of the given potential and has to involve the same kinds of singularities. In many cases, the accuracy of the results can be significantly improved by incorporating in $U^{(0)}(x,m)$ a part of the perturbation in a homogeneous way, i.e., in terms of functions already appearing in $U^{(0)}(x,m)$; this is equivalent to a scaling device.[23,28,29]

Once the kernel $U^{(0)}(x,m)$ has been so chosen, the main ingredients for the computation are fixed. Indeed, to each factorization type correspond specific $x$-basis functions $y_s(x)$ satisfying the required selective conditions (3.2) and (3.3). The perturbing potential $V(x)$ has to be expanded on such a basis. Then, the general formulas, which are given in the present paper, allow an analytical computation of the perturbed ladder $K^{(N)}(x,m;\mu)$ and factorization $L^{(n)}(m;\mu)$ functions, at any order $N$ of the perturbation solely by means of algebraic manipulations. Once these "factorization instruments" have been obtained, the perturbed eigenequation may be handled in the same way as an exact factorizable (unperturbed) equation. Its solution is readily obtained without prior knowledge of the unperturbed spectrum and without having to calculate any matrix element.

The efficiency of the present perturbed ladder operator method follows from the consideration of basis functions $y_s(x)$ which satisfy selective ladderlike properties and the use of Newton's binomial $\binom{m-\mu}{n}$ expansions of the ladder and factorization functions. A subsequent computational advantage of the method is that many intermediate functions do not depend on the order $N$ of the perturbation and thus can be calculated once for all; the $N$th order of the perturbation is not significantly more difficult to handle than the first order.

Although the ladderlike conditions (3.2) and (3.3) imply a rather selective choice of the specific $x$-basis functions $y_s(x)$ associated with each factorization type, the particular perturbed (A to D) factorizations which have been examined in the present paper are not at all exhaustive. For instance, one can also apply perturbed type-A factorization with an associated potential basis set $y_s(x)=a^2[\cos(ax/2)]^{2s}$ or $y_s(x)=a^2[\sin(ax/2)]^{2s}$ (see Table V) and both bases $y_s(x)=a^2e^{asx}$ or $y_s(x)=a^2e^{-asx}$ can be associated with perturbed type-B factorization (see Tables IV and V). It is interesting to note that, when dealing with problems involving perturbed spherical harmonics $Y_{lm}(\theta,\varphi)$, one can use not only the abovementioned potential basis sets, but also the basis functions $y_s(x)=(\cos\theta)^{2s}$ (see Table V). Indeed, the spherical harmonics are directly related to the solutions of a reduced type-A factorizable eigenequation with $d=0$ (see Appendix D).

Of course, since the procedure worked out in the present paper is valid for all factorization types, it can also be applied to the solution of perturbed type-E or -F eigenequations. However, it may be worthwhile to examine separately these cases and, possibly, take advantage of the fact that the following $m$-parity relationships hold:[13] $K^{(N)}(x,-m)=-K^{(N)}(x,m)$, $L^{(N)}(-m)=L^{(N)}(m)$, and that, as a consequence, $U^{(N)}(x,m)$ is a function of $m(m+1)$. In another way, since the factorization types are interrelated,[1] by an adequate transformation of variable and function, one can obtain an alternative factorizable equation corresponding to the same problem. Consequently, perturbed type-E (or perturbed type-F) problems can also be worked out by making use of the connection between type-E and -A factorizations (or between type-F and type-B, -C, or -D factorizations).

The solution of perturbed type-E and -F eigenequations is under investigation and special attention is paid to perturbed type-F factorization with the potential basis functions $y_s(x)=x^s$ [with associated ladder basis functions and coefficients: $Y_s(x)=x^{s+1}$, $A_s(m)=2m$, $B_s(m)=2q/m$, $\alpha_s=s+1$, and $\beta_s=0$]. This choice is particularly interesting to show the capabilities of the method for an elaborate treatment of the generalized central field problem and for studying the Stark effect up to a high order of the perturbation.

## APPENDIX A: SOME RESULTS FROM FINITE DIFFERENCE CALCULUS

### 1. Generalized factorials, reciprocal factorials, and binomial coefficients

These functions are,[30] respectively,

$$(m)_u=m(m-1)\cdots(m-u+1)\ , \tag{A1}$$

$$(m)_{-u}=1/(m+u)_u\ , \tag{A2}$$

$$\begin{bmatrix} m \\ u \end{bmatrix}=\frac{\Gamma(m+1)}{\Gamma(u+1)\Gamma(m-u+1)}\ , \tag{A3}$$

where $\Gamma(m)$ is the gamma function which satisfies the functional relation $\Gamma(m+1)=m\Gamma(m)$.

These functions play the same central role in finite difference calculus as $x^u$ in differential calculus. Their finite differences $\Delta^n$ are

$$\Delta^n(m)_u=(u)_n(m)_{u-n}\ , \tag{A4}$$

$$\Delta^n(m)_{-u}=(-1)^n(u+n-1)_n(m)_{-u-n}\ , \tag{A5}$$

$$\Delta^n\begin{bmatrix} m \\ u \end{bmatrix}=\begin{bmatrix} m \\ u-n \end{bmatrix}\ . \tag{A6}$$

For negative arguments, the following relations hold:

$$(-m)_u=(-1)^u(m+u-1)_u\ , \tag{A7}$$

$$\begin{bmatrix} -m \\ u \end{bmatrix}=(-1)^u\begin{bmatrix} m+u-1 \\ u \end{bmatrix}\ . \tag{A8}$$

### 2. Expansion of a function $F(m)$ into a series of binomial coefficients

We have

$$F(m)=F(m=\mu)+\begin{bmatrix} m-\mu \\ 1 \end{bmatrix}\Delta F(m=\mu)+\begin{bmatrix} m-\mu \\ 2 \end{bmatrix}\Delta^2F(m=\mu)+\cdots\ . \tag{A9}$$

This is Newton's formula. Conversely, the following relation holds:

TABLE V. Some other perturbed factorizations. Suitable basis sets and coefficients.

| Type | A | | Reduced A | B |
|---|---|---|---|---|
| $y_s(x)$ | $a^2\left[\cos\frac{ax}{2}\right]^{2s}$ | $a^2\left[\sin\frac{ax}{2}\right]^{2s}$ | $a^2(\cos ax)^{2s}$ | $a^2e^{-asx}$ |
| $Y_s(x)$ | $a\sin\frac{ax}{2}\left[\cos\left[\frac{ax}{2}\right]^{2s+1}\right]$ | $a\cos\frac{ax}{2}\left[\sin\frac{ax}{2}\right]^{2s+1}$ | $a\sin ax(\cos ax)^{2s+1}$ | $ae^{-a(s+1)x}$ |
| $A_s(m)$ | $d-m$ | $d+m$ | 0 | $2d$ |
| $B_s(m)$ | $2m$ | $-2m$ | $2m$ | $-2m$ |
| $\alpha_s$ | $-(s+1/2)$ | $s+1/2$ | $-(2s+1)$ | 0 |
| $\beta_s$ | $s+1$ | $-(s+1)$ | $2s+2$ | $-(s+1)$ |
| $Y_sY_t$ | | $y_{s+t+1}-y_{s+t+2}$ | | $y_{s+t+2}$ |

$$\sum_{u=0}^{n}(-1)^u\begin{bmatrix}n\\u\end{bmatrix}F(u)=(-1)^n\Delta^nF(u=0)\ . \tag{A10}$$

Particularly,

$$\sum_{u=0}^{n}(-1)^u\begin{bmatrix}n\\u\end{bmatrix}\begin{bmatrix}u+a\\m\end{bmatrix}=(-1)^n\begin{bmatrix}a\\m-n\end{bmatrix}\ . \tag{A11}$$

### 3. Finite differences of a product

We have

$$\Delta^n(FG)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}\Delta^iF(m)\sum_{t=0}^{i}\begin{bmatrix}i\\t\end{bmatrix}\Delta^{n-i+t}G(m)\ . \tag{A12}$$

Particularly,

$$[\Delta^n(FG)]_{m=\mu}=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}\Delta^iF(m=\mu)\Delta^{n-i}G(m=\mu+i)\ . \tag{A13}$$

This last relation directly follows from (A12). One notes that when introducing the discrete difference with regard to the $i$ argument, namely $\Delta_iG(m+i)=G(m+i+1)-G(m+i)$, one obtains $\Delta^tG(m)=[\Delta_i^tG(m+i)]_{i=0}$. Hence the last summation, in (A12), can be viewed as the Newton's expansion of $\Delta^{n-i}G(m+i)$ in a series of the $\binom{i}{t}$.

### 4. Expansion of the product of binomials into a series of binomial coefficients

We have

$$\begin{bmatrix}m\\t\end{bmatrix}\begin{bmatrix}m\\u\end{bmatrix}=\sum_{k=0}^{u}\begin{bmatrix}u\\k\end{bmatrix}\begin{bmatrix}k+t\\u\end{bmatrix}\begin{bmatrix}m\\k+t\end{bmatrix}\ . \tag{A14}$$

Particularly,

$$\begin{bmatrix}m\\1\end{bmatrix}\begin{bmatrix}m\\u\end{bmatrix}=u\begin{bmatrix}m\\u\end{bmatrix}+(u+1)\begin{bmatrix}m\\u+1\end{bmatrix}\ . \tag{A15}$$

## APPENDIX B: DETERMINATION OF THE $F_s^{(N)}(m)$ FUNCTION

Let us start with Eq. (3.11) which can be written again $\Delta F_{s-1}(m)=R_s(m)/Q_{s-1}(m+1)$. Since $\gamma_s(m)=Q_s(m)[k_s+F_s(m)]$ and $\gamma_s(m+1)=Q_s(m+1)[k_s+F_s(m)+\Delta F_s(m)]$, one obtains the following two-terms algebraic recursive relation for the determination of $\Delta F_s(m)$:

$$\Delta F_{s-1}^{(N)}=G_s^{(N)}(m)+a_s(m)\Delta F_s^{(N)}\ , \tag{B1}$$

where

$$G_s^{(N)}(m)=\frac{G_1}{G_2}\ ,$$

$$a_s(m)=-Q_s(m)[A_s(m+1)+\alpha_s]/G_2\ ,$$

$$G_1=[k_s^{(N)}+F_s^{(N)}(m)]\{[A_s(m)-\alpha_s]Q_s(m)-[A_s(m+1)+\alpha_s]Q_s(m+1)\}-\Delta w_s^{(N)}(m)\ ,$$

$$G_2=Q_{s-1}(m+1)[B_{s-1}(m+1)+\beta_{s-1}]\ .$$

Applying $(S_N-s)$ times relation (B1) and keeping in mind that, for $s=S_N$, $\Delta F_{S_N}^{(N)}=0$, one finds

$$\Delta F_s^{(N)}=\sum_{u=s+1}^{S_N}G_u^{(N)}(m)\prod_{t=s+1}^{u-1}a_t(m) \tag{B2}$$

or

$$\Delta F_s^{(N)}(m)=\sum_{u=s+1}^{S_N}\{f_{us}(m)[k_u^{(N)}+F_u^{(N)}(m)]+g_{us}(m)\Delta w_u^{(N)}(m)\}\ , \tag{B3}$$

where $f_{us}(m)$ and $g_{us}(m)$ are given by (3.16). Expression (A12) can be used to compute the value, for $m=\mu$, of the $(n-1)$th discrete derivative of $\Delta F_s^{(N)}(m)$ and, keeping in mind that $F_s^{(N)}(m=\mu)=0$, one obtains the required relation (3.14).

As a consequence of (3.14), for any factorization type and at any order $N$ of the perturbation, one obtains the following expressions of the $\Delta^nF_s^{(N)}(m=\mu)=\Delta_nF_s^{(N)}$ coefficients:

$$\Delta^nF_{S_N}=0\quad\text{for any } n\ ,$$

$$\Delta F_{S_N-1}=I(S_N,S_N-1;0,1)k_{S_N}+J(S_N,S_N-1;0,1)\Delta w_{S_N}\ ,$$

$$\Delta F_{S_N-2}=I(S_N,S_N-2;0,1)k_{S_N}+I(S_N-1,S_N-2;0,1)k_{S_N-1}+(\ )\ ,$$

and so on, up to the required $\Delta F_{S_N-\sigma}$,

$$\Delta^2F_{S_N-1}=I(S_N,S_N-1;0,2)k_{S_N}+J(S_N,S_N-1;0,2)\Delta w_{S_N}+J(S_N,S_N-1;1,2)\Delta^2w_{S_N}\ ,$$

$$\Delta^2F_{S_N-2}=I(S_N,S_N-2;0,2)k_{S_N}+I(S_N-1,S_N-2;0,2)k_{S_N-1}+I(S_N-1,S_N-2;1,2)\Delta F_{S_N-1}+(\ )\ ,$$

and so on, up to the required $\Delta^2F_{S_N-\sigma}$,

$$\Delta^3F_{S_N-1}=I(S_N,S_N-1;0,3)k_{S_N}+(\ )\ ,$$

$$\Delta^3F_{S_N-2}=I(S_N,S_N-2;0,3)k_{S_N}+I(S_N-1,S_N-2;0,3)k_{S_N-1}+2I(S_N-1,S_N-2;1,3)\Delta F_{S_N-1}+I(S_N-1,S_N-2;2,3)\Delta^2F_{S_N-1}+(\ )\ ,$$

and so on, up to the required $\Delta^3F_{S_N-\sigma}$,

$$\Delta^4 F_{S_N-1}=I(S_N,S_N-1;0,4)k_{S_N}+(\;) ,$$

$$\Delta^4 F_{S_N-2}=I(S_N,S_N-2;0,4)k_{S_N}+I(S_N-1,S_N-2;0,4)k_{S_N-1}+3I(S_N-1,S_N-2;1,4)\Delta F_{S_N-1}$$
$$+3I(S_N-1,S_N-2;2,4)\Delta^2 F_{S_N-1}+I(S_N-1,S_N-2;3,4)\Delta^3 F_{S_N-1}+(\;) , \quad \text{(B4)}$$

and so on, up to the required $\Delta^n F_{S_N-\sigma}$. The shortened notation ( ) stands for the $J(u,s;i,n)\Delta^{i+1}w_u$ terms which are easily obtainable from their $I(u,s;i,n)\Delta^i F_u$ (already written) counterpart, by means of the substitutions

$$I(S_N,S_N-\sigma;0,n)k_{S_N}\to \sum_{i=0}^{n-1} J(S_N,S_N-\sigma;i,n)\Delta^{i+1}w_{S_N} ,$$

for $u=S_N$, and, otherwise,

$$I(u,s;i,n)\Delta^i F_u \to J(u,s;i,n)\Delta^{i+1}w_u .$$

Note that, for any $s$ and $u$,

$$\begin{aligned}
&I(u,s;0,1)=f_{us}(m=\mu) ,\\
&I(u,s;0,2)=\Delta f_{us}(m=\mu) ,\\
&I(u,s;1,2)=f_{us}(m=\mu+1) ,\\
&I(u,s;0,3)=\Delta^2 f_{us}(m=\mu) ,\\
&I(u,s;1,3)=\Delta f_{us}(m=\mu+1) ,\\
&I(u,s;2,3)=f_{us}(m=\mu+2) ,
\end{aligned} \quad \text{(B5)}$$

and so on.

When $A_s(m)$ is a linear function of $m$ and, also $Q_s(m)=1$ [with $B_s(m)=B_0$ and $\beta_s=0$], one can write [see Eq. A9)]

$$\gamma_s(m)=\sum_{n=0}^{n_s} \begin{bmatrix} m-\mu \\ n \end{bmatrix} \Delta^n\gamma_s(m=\mu) ,$$

where $\Delta^0\gamma_s=k_s$ and $\gamma_{S_N}=k_{S_N}$. Noting that

$$\Delta\gamma_s=\sum_{n=1}^{n_s}\begin{bmatrix} m-\mu \\ n-1 \end{bmatrix}\Delta^n\gamma_s=\sum_{n=0}^{n_s-1}\begin{bmatrix} m-\mu \\ n \end{bmatrix}\Delta^{n+1}\gamma_s \quad \text{(B6)}$$

and that

$$m\begin{bmatrix} m-\mu \\ n \end{bmatrix}=\begin{bmatrix} m-\mu \\ 1 \end{bmatrix}\begin{bmatrix} m-\mu \\ n \end{bmatrix}+\mu\begin{bmatrix} m-\mu \\ n \end{bmatrix}$$
$$=(n+\mu)\begin{bmatrix} m-\mu \\ n \end{bmatrix}+(n+1)\begin{bmatrix} m-\mu \\ n+1 \end{bmatrix}$$

[see Eq. (A15)], one obtains, after some rearrangements,

$$R_s(m)=-B_0^{-1}\sum_{n=0}^{n_s}\begin{bmatrix} m-\mu \\ n \end{bmatrix}\{[(n+1)\Delta A_s+2\alpha_s]\Delta^n\gamma_s+[n+\mu+1+A_s(m=0)+\alpha_s]\Delta^{n+1}g_s+\Delta^{n+1}w_s\} . \quad \text{(B7)}$$

This last expression can be viewed as Newton's expansion of $R_s(m)$ and, since $\Delta\gamma_{s-1}=R_s(m)$, one finds

$$\Delta^n\gamma_{s-1}=-B_0^{-1}\{[(n+\mu)\Delta A_s+A_s(m=0)+\alpha_s]\Delta^n\gamma_s+(n\Delta A_s+2\alpha_s)\Delta^{n-1}\gamma_s+\Delta^n w_s\} . \quad \text{(B8)}$$

Keeping in mind that $\Delta^n\gamma_s=0$ for $s>S_N-n+1$, and applying $S_N-s$ times this relation, one obtains the required recursive relation (3.17).

For the particular case of perturbed type-D factorization with associated basis functions $y_s(x)=\chi_s H_{2s}(b^{1/2}x)$, the recursive relation (3.17) reduces to Eq. (4.44). Let us set $\Delta^n F_s=(-2)^s(\Delta^n F_s+\frac{1}{2}\Delta^{n+1}w_s)$, $\Delta^{n+1}w_s=\frac{1}{2}(-2)^s\Delta^{n+1}w_s$, and $k_s=\Delta^0 F_s$ and let us remark that $F_u^{(N)}(m)$ and $w_u^{(N)}(m)$ are both[31] of degree $S_N-u$ in $m$. Then, relation (4.44) reduces to

$$\Delta^n F_s^{(N)}=\Delta^{n+1}w_s^{(N)}+\sum_{u=s+1}^{S_N-n+1}\Delta^{n-1}F_u^{(N)} . \quad \text{(B9)}$$

Using this last relation, the following closed-form expression is found:

$$\Delta^n F_{S_N-\sigma}^{(N)}=\Delta^{n+1}w_{S_N-\sigma}+\sum_{t=0}^{\sigma-n}\left[\begin{bmatrix} \sigma-t-1 \\ n-1 \end{bmatrix}k_{S_N-t}^{(N)}+\sum_{i=1}^{n-1}\begin{bmatrix} \sigma-t-i-1 \\ n-i-1 \end{bmatrix}\Delta^{i+1}w_{S_N-t-i}^{(N)}\right] , \quad \text{(B10)}$$

and, finally, one obtains the closed-form expression (4.45).

## APPENDIX C: DETERMINATION OF THE $\Delta^{i+1}w_u^{(N)}$ COEFFICIENTS

The $w_u^{(N)}(m)$ functions are defined by relation (3.4). Using the expression (3.1) of the perturbed ladder function $K^{(N)}(x,m;\mu)$ together with expression (3.3) of the product $Y_sY_t$, both expanded in finite series of the $y_s(x)$ basis functions, one gets

$$w_u^{(N)}(m)=\sum_{v=1}^{N-1}\sum_{s=0}^{S_v}\sum_{t=0}^{S_N-v}h(s,t;u)Q_s(m)Q_t(m)\sum_{n=0}^{n_s^{(v)}}\sum_{j=0}^{n_t^{(N-v)}}\begin{bmatrix} m-\mu \\ n \end{bmatrix}\begin{bmatrix} m-\mu \\ j \end{bmatrix}\Delta^n F_s^{(v)}\Delta^j F_t^{(N-v)} , \quad \text{(C1)}$$

$$\Delta^i w_u^{(N)} = \sum_{v=1}^{N-1} \sum_{s=0}^{S_v} \sum_{t=0}^{S_N-v} h(s,t;u) \sum_{v=0}^{n_s^{(v)}} \sum_{j=0}^{n_t^{(N-v)}} W_i(s,t;n,j) \Delta^n F_s^{(v)} \Delta^j F_t^{(N-v)} , \tag{C2}$$

where

$$W_i(s,t;n,j) = \Delta^i \left[ Q_s(m) Q_t(m) \begin{bmatrix} m-\mu \\ n \end{bmatrix} \begin{bmatrix} m-\mu \\ j \end{bmatrix} \right]_{m=\mu} .$$

It is easily found that

$$\left[ \Delta^k \begin{bmatrix} m-\mu \\ n \end{bmatrix} \begin{bmatrix} m-\mu \\ j \end{bmatrix} \right]_{m=\mu} = \begin{bmatrix} k \\ n \end{bmatrix} \begin{bmatrix} n \\ k-j \end{bmatrix} ,$$

and one gets, after some rearrangements,

$$W_i(s,t;n,j) = \sum_{k=0}^{i} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix} \begin{bmatrix} n \\ k-j \end{bmatrix} (\Delta^{i-k} Q_s Q_t)_{m=\mu+k} . \tag{C3}$$

It is worthwhile to note that, when $Q_s(m) = Q_s$ does not depend on $m$, this last expression reduces to

$$W_i(s,t;n,j) = Q_s Q_t \begin{bmatrix} i \\ n \end{bmatrix} \begin{bmatrix} n \\ i-j \end{bmatrix} . \tag{C4}$$

Particularly, when $Y_s Y_t = y_{s+t+\sigma}$, one gets

$$\Delta^{i+1} w_u^{(N)} = \sum_{v=1}^{N-1} \sum_{s=0}^{S_v} Q_s Q_{u-s-\sigma} \sum_{t=0}^{i+1} \sum_{r=0}^{i+1} \begin{bmatrix} i+1 \\ r \end{bmatrix} \begin{bmatrix} r \\ i+1-t \end{bmatrix} \Delta^t F_s^{(v)} \Delta^r F_{\mu-s-\sigma}^{(N-v)} . \tag{C5}$$

## APPENDIX D: SOME PARTICULAR EIGENFUNCTIONS OF FACTORIZABLE EIGENEQUATIONS

### 1. Associated spherical harmonics

We have

$$Y_{LM}(\theta,\varphi) = (2\pi)^{-1/2} \exp(iM\varphi)(\sin\theta)^{-1/2} \Psi_{LM}(\theta) .$$

$\Psi_{LM}(\theta)$ is a class-I ($\epsilon = +1$) solution of a type-A factorizable eigenequation with $x=\theta$, $0 \le \theta \le \pi$; $a=1$; $d=0$; $j=L-1/2$; $m=M-1/2$ and $L-M=j-m$ equals a positive integer or zero.

### 2. Symmetric-top functions

We have

$$D_{MM'}^{(L)}(\varphi,\theta,\phi) = \exp(iM\varphi) d_{MM'}^{(L)}(\theta) \exp(iM'\phi) ,$$

where $\varphi,\theta,\phi$ are the three Euler angles: $0 \le \varphi \le 2\pi$; $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$,

$$d_{MM'}^{(L)}(\theta) = [2/(2L+1)]^{1/2} (\sin\theta)^{-1/2} \Psi_{LM}(\theta) .$$

$\Psi_{LM}(\theta)$ is a class-I ($\epsilon = +1$) solution of a type-A factorizable eigenfunction with $x=\theta$, $a=1$, $j=L-1/2$. Since both differences $L-M$ and $L-M'$ are positive integers or zero, $\Psi_{LM}(\theta)$ is a type-A eigenfunction either when setting $M=m+1/2$ and $M'=d$, or when setting $M=d$ and $M'=m+1/2$.

### 3. Gauss hypergeometric functions

The differential equation satisfied by the Gauss hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ is[32]

$$\left[ z(1-z)\frac{d^2}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{d}{dz} - \alpha\beta \right] F(z) = 0 .$$

Setting $z=\sin^2 x$ and

$$F = (\sin x)^{-\gamma+1/2} (\cos x)^{-\alpha-\beta+\gamma-1/2} \Psi(x) ,$$

one obtains

$$\left[ \frac{d^2}{dx^2} - \frac{4(m^2+d^2-1/4+2md\cos 2x)}{\sin^2 2x} + (\alpha-\beta)^2 \right] \Psi(x) = 0 ,$$

where $m=(\alpha+\beta-1)/2$ and $d=(2\gamma-\alpha-\beta-1)/2$.

$\Psi(x)$ is a class-I ($\epsilon = +1$) solution of a type-A factorizable equation with $a=2$, $\Lambda_j = (\alpha-\beta)^2$. On the other hand, the quantization condition requires $j-m=v$ equal to a positive integer and $\Lambda_j = L(j+1) = L(m+v+1) = 4[v+(\alpha+\beta)/2]^2$. As a consequence, either $\alpha=-v$ or $\beta=-v$, i.e., one finds again the well-known condition for finite hypergeometric series.

### 4. Morse oscillator

The Morse potential is $U_m(r) = D(1-e^{-ar})^2$, where $D$ is the well depth and $a$ is a scale parameter. Setting $x=ar$, $\mathcal{K} = (2MD)^{1/2}/a\hbar$, $\mathcal{E} = 2ME/\hbar^2 a^2$, one obtains the Morse oscillator wave equation $[-\infty < x < +\infty$, within the Ter Haar extension[33]]

$$\left[\frac{d^2}{dx^2}-\mathcal{K}^2(e^{-2x}-2e^{-x})+\mathcal{E}-\mathcal{K}^2\right]\Psi(x)=0 \ .$$

$\Psi(x)$ is a class-II ($\epsilon=-1$) solution of a type-B eigenequation (see Table I).

Note that the change of variable $X=(2\mathcal{K})^{1/2}\exp(-x/2)$ transforms the above eigenequation into the radial equation of a two-dimensional harmonic oscillator with unit angular frequency and unit mass (type-C factorizable equation).

### 5. *D*-dimensional harmonic oscillator

The radial eigenfunction $\psi(x)$ of the $D$-dimensional simple harmonic oscillator of mass $M$ and angular frequency $w$ is a class-I ($\epsilon=+1$) solution of a type-C factorizable eigenequation with $m=L+\frac{1}{2}(D-1)$, $b=-Mw/\hbar$, $v=j-m$ which is equal to positive integer or zero. One finds again the already known expression[34] of the energies $E_N=\frac{1}{2}\hbar w(2N+D)$ where $N=2v+L$.


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$$C_s=(1/2^n n!\pi^{1/2})\int_{-\infty}^{+\infty}V(x)\exp(-x^2)H_s(x)dx \ .$$

Furthermore, it is well known that the $n$th-degree least-squares polynomial approximation to a function $V(x)$ over $(-\infty,+\infty)$, relevant to the weighting function $w(x)=\exp(-x^2)$, is defined as a series of Hermite polynomials $H_s(x)$ with $s=0$ to $n$. Thus the expansion of the perturbation on the Hermite polynomials basis constitutes a rearrangement of the $x^s$ which obviates the difficulties of convergence [see F. B. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill, New York, 1956)].
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