Strong-coupling analog of the Born series in terms of a modified WKB approach

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In this paper we incorporate different strengths of the centrifugal potential into a single partial wave function of the WKB approach in different spatial regions. By a suitable choice of these regions and strengths, the WKB wave functions can be made bounded and smooth everywhere as well as compatible with quantum mechanics both close to and far from the origin. This modified WKB approximation becomes exact in the strong-coupling limit. Moreover, by iteration, it develops an absolute convergent series expansion of the correct scattering wave function at fixed coupling constants.

I. INTRODUCTION

The standard WKB approach (WKBA) relies on pairs of analytically introduced wave functions, the character of which changes from the exponential to the trigonometric type and vice versa each time we cross a classical turning point (TP). In particular, such a pair of independent wave functions reads, inside the nearest TP for scattering of a spinless particle of energy k^2 by the potential $g^2V(r)$ in the lth partial wave[1],

$$w_\lambda^\pm(r) \equiv \frac{k^{1/2}}{K_\lambda^{1/2}(r)} \exp\left[\pm \int_{R_\lambda}^{r} d\rho\, K_\lambda(\rho)\right], \quad r < R_\lambda \tag{1.1}$$

where

$$K_\lambda^2(r) \equiv \left| k^2 - g^2V(r) - \frac{\lambda^2}{r^2} \right| \tag{1.2}$$

and, at one's disposal,

$$\lambda^2 = \lambda_a^2 \equiv (l+\tfrac{1}{2})^2$$

or (1.3)

$$\lambda^2 = \lambda_b^2 \equiv l(l+1)$$

(yet the former alternative is preferred), while R_λ is the smallest positive real root of the equation

$$K_\lambda(R_\lambda) = 0 \ . \tag{1.4}$$

Within the region indicated, the functions $w_\lambda^\pm(r)$ are considered to reproduce a pair of solutions of quantum mechanics (QM) either exactly or approximately, depending on the fulfillment of certain conditions.

The conditions under which WKBA is expected to work best can easily be extracted from the Schrödinger-type differential equation solved by both $w_\lambda^+(r)$ and $w_\lambda^-(r)$ of Eq. (1.1) and governed by the WKBA potential[2]

$$W_\lambda(r) = g^2V(r) + \frac{5}{16}\frac{1}{K_\lambda^4}\left[\frac{dK_\lambda^2}{dr}\right]^2 - \frac{1}{4}\frac{1}{K_\lambda^2}\frac{d^2K_\lambda^2}{dr^2} + \frac{\lambda^2 - l(l+1)}{r^2} \ , \tag{1.5}$$

consisting of the physical potential plus an $O(1)$, $(g \to \infty)$ contribution. Therefore WKBA should become exact in strong coupling, at least *inside* the nearest TP.

How to get $w_\lambda^\pm(r)$ *across* the singularity of $W_\lambda(r)$ at the TP [see Eq. (1.4)] is called the connection problem. By insisting on the form (1.5) of the WKBA potential beyond the TP, solutions of the trigonometric type can be introduced there. Certain superpositions of them correctly match the inner solutions $w_\lambda^\pm(r)$ of Eq. (1.1). The only reasonable interpretation of the term "correctly" is that the overall $(r = 0 \to \infty)$ solutions should reproduce exactly (e.g., for $g \to \infty$) or approximately (e.g., for large g) inside and beyond the TP *one and the same* QM wave function. In QM, in turn, the self-identity of a solution is ensured by its overall smoothness, a property that is automatic whenever $V(r)$ is continuous and can be required if the potential is discontinuous but bounded. In WKBA, however, the potential $W_\lambda(r)$ of Eq. (1.5) is singular at $r = R_\lambda$. Hence the smoothness postulate here is necessarily frustrated and ought to be substituted for by some virtually equivalent claim. How this issue has been dealt with by different semiclassical approaches is briefly outlined just below.

The standard treatment of the connection problem, the linear-turning-point approach (LTPA) combines the WKBA formalism near the TP with QM, whereby one realizes smoothness at two matching points (MP's), one taken inside, the other beyond the singularity. Yet, proper selection of the MP's is rather problematic. The nearer we put them to the TP the less realistic values of the WKBA enter the argument. The farther we set them the less reliable will be our knowledge of the exact solution. What is more, however sophisticated our choice may be, it still cannot heal an additional weakness of LTPA, namely, its failing to *simultaneously* reproduce the physical behavior for $r \to 0$ and $r \to \infty$. Whichever of the alternatives offered by Eq. (1.3) is chosen for λ^2, $(l+\frac{1}{2})^2$ or $l(l+1)$, the potentials of QM and WKBA *differ by the dominant term* $1/(4r^2)$ either for $r \to \infty$ or $r \to 0$, respectively, as is straightforward to extract from Eq. (1.5).

The recently proposed[3] double-centrifugal-strength approach (DCSA) goes beyond the standard WKBA and

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solves the connection problem in such a way that the relevant wave functions do become compatible with QM for $r \to 0$ and $r \to \infty$, simultaneously. The clue to doing this rests in incorporating into the formalism *both* alternatives of the centrifugal strength λ_a^2 and λ_b^2 of Eq. (1.3) and working thus with a pair of TP's, R_a and R_b. The crucial point is that for decreasing (repulsive) potentials always $R_b < R_a$. In consequence, any point R taken between R_b and R_a divides the space into the regions I_a and I_b so that I_a contains R_b (observe the subscripts) while I_b contains R_a. Therefore the inner solutions $w_a^{\pm}(r)$ corresponding to the strength λ_a^2 do not develop singularities in I_a nor do the external solutions $w_b^{(j)}(r)$, $(j=1,2)$, of the strength λ_b^2 in the interval I_b. Smooth matching at $r=R$ of the solutions $w_a^{\pm}(r)$ to proper superpositions of $w_b^{(j)}(r)$ yields smooth WKBA wave functions of correct physical behavior at both small and large distances.

The present paper is meant to be a generalization of DCSA of Ref. 3. What is common in both methods is the presence of more than one centrifugal strength. What will be new here is, on the one hand, the inclusion of potentials which, while being still repulsive near the origin, yet develop an *attractive tail.* On the other hand, while Ref. 3 has been restricted to first-order WKBA, the present considerations introduce higher-order approximations to finally generate for the exact QM scattering wave function an infinite *series expansion,* the convergence and truncation problem of which will also be examined in detail.

II. REDEFINING CONCEPTS OF STANDARD WKBA

The LTPA of the WKBA divides the space into domains in terms of the classical turning points and uses the same centrifugal term everywhere. The decomposition of the space in the present approach is done, in turn, by means of the zeros of the physical potential and certain unperturbed turning points. Each of the new intervals will then be supplied by an adequately chosen centrifugal strength. The new system of intervals will be specified in this section.

The conditions imposed on the physical potential are, besides the ones usual in scattering theory, the following:

$$rV(r) \to 0+, \quad r \to 0 \tag{2.1}$$

and

$$r^2 V(r) \to 0-, \quad r \to \infty \tag{2.2}$$

as well as

$$V(z)=0, \quad V(r) \neq 0, \quad r \neq z \ . \tag{2.3}$$

Thus the potential is repulsive in the region inside the unique zero and attractive beyond. Unperturbed TP's are defined in $r=a$, $r=b$ via Eqs. (1.3) by setting

$$a = \frac{\lambda_a}{k}, \quad b \equiv \frac{\lambda_b}{k} \tag{2.4}$$

whence

$$b < a \ . \tag{2.5}$$

We distinguish three types of scattering problems depending on whether (i),

$$b < z < a \ ,$$

(ii),

$$b < a < z \ , \tag{2.6}$$

(iii),

$$z < b < a \ .$$

The axis $r=(0,\infty)$ is divided into intervals I_m, $(m=1,2,M)$, with running coordinates r_m,

$$d_{m-1} \le r_m < d_m \ , \tag{2.7}$$

and with end points d_m specified in Table I for the types (i)–(iii) of problems, separately.

Owing to Eqs. (2.3), the potential has within each of the intervals a constant sign,

$$\sigma_m \equiv \operatorname{sgn} V(r_m) \ , \tag{2.8}$$

by virtue of the single zero z of $V(r)$ being included in each list of the end points; see Table I.

The character of the basis functions should depend on the local sign of the potential as follows:

$$w_{mj}(r) \equiv \frac{k^{1/2}}{K_m^{1/2}(r)} \exp\left[(3-2j)\int_{d_m}^{r} d\rho\, K_m(\rho)\right] , \qquad \sigma_m = 1, \quad j=1,2 \tag{2.9a}$$

and

$$w_{mj}(r) \equiv (-1)^j 2^{1/2} \frac{k^{1/2}}{K_m^{1/2}} \sin\left[\int_{d_{m-1}}^{r} d\rho\, K_m(\rho) + (3-2j)\pi/4\right] , \qquad \sigma_m = -1, \quad j=1,2 \ . \tag{2.9b}$$

The functions $K_m(r)$ involved are unspecified for the time being. It is straightforward to calculate the Wronskian determinants

$$\mathcal{W}(w_{m1}(r); w_{m2}(r)) = -2k, \quad \sigma_m = \pm 1 \ , \tag{2.10}$$

the r independence of which already suggests that the functions (2.9) of a given m may both solve the same second-order differential equation. Indeed,

TABLE I. End points d_m of the intervals $I_m=(d_{m-1},d_m)$, which the r axis is decomposed into, for the different types of scattering problems [see Eqs. (2.3)–(2.6)].

Type	M	d_0	d_1	d_2	d_3
(i)	2	0	z	∞	
(ii)	3	0	a	z	∞
(iii)	3	0	z	a	∞

$$\left[\frac{d^2}{dr^2}-\sigma_m K_m^2(r)-\mathcal{D}_m(r)\right]w_{mj}(r)=0\ ,$$

$$\sigma_m=\pm 1,\quad j=1,2 \qquad (2.11)$$

where

$$\mathcal{D}_m(r)\equiv\frac{5}{16}\frac{1}{K_m^4(r)}\left[\frac{dK_m^2}{dr}\right]^2-\frac{1}{4}\frac{1}{K_m^2(r)}\frac{d^2K_m^2}{dr^2}\ . \qquad (2.12)$$

Notice that Eqs. (2.11) work independently of the definition of $K_m^2(r)$. Yet, henceforward we shall work with the choice

$$K_m^2(r)\equiv\sigma_m\left[-k^2+g^2V(r)+\frac{\lambda_m^2}{r^2}\right]\ , \qquad (2.13)$$

involving still the unspecified parameters λ_m^2. Each of the equations

$$K_m^2(r)=0,\quad r=R_{m\mu},\quad \mu=1,2,M \qquad (2.14)$$

has, in general, a set of real roots some of which may fall onto the actual interval I_m. The points $R_{m\mu}$ are, in fact, the TP's, candidates for causing singularities of the relevant basis functions of Eq. (2.9). Yet, appearance of active singularities can be avoided if we work with the functions $w_{mj}(r)$ exclusively in the interval I_m and, this is the point, this I_m contains none of the TP's $R_{m\mu}$, ($\mu=1,2,M$). Appropriate choice of the free parameters λ_m^2 will satisfy this claim.

Incorporate Eq. (2.13) into Eq. (2.11) and obtain after rearrangement the Schrödinger-type differential equation

$$\left[\frac{d^2}{dr^2}+k^2-W_m(r)-\frac{l(l+1)}{r^2}\right]w_{mj}(r)=0\ , \qquad (2.15)$$

with

$$W_m(r)\equiv g^2V(r)+\Delta_m(r)\ , \qquad (2.16)$$

where $\Delta_m(r)$ is the residual interaction in I_m, expressed via Eq. (2.12) as

$$\Delta_m(r)=\mathcal{D}_m(r)+\frac{\lambda_m^2-l(l+1)}{r^2}\ . \qquad (2.17)$$

Overall WKBA potential and residual interaction is introduced by

$$\begin{aligned}&W(r)\equiv g^2V(r)+\Delta(r),\quad 0\le r\\ &\Delta(r)\equiv\Delta_m(r),\quad d_{m-1}\le r<d_m\ .\end{aligned} \qquad (2.18)$$

Observe that both $W(r)$ and $\Delta(r)$ develop jumps at $r=d_{m-1}$ whenever our future choice of λ_μ^2 will be different in the intervals I_{m-1} and I_m [see Eq. (2.13)].

In terms of $W(r)$, an overall WKBA differential equation can also be set up as follows:

$$\left[\frac{d^2}{dr^2}+k^2-W(r)-\frac{l(l+1)}{r^2}\right]w(r)=0\ , \qquad (2.19)$$

each solution of which can conveniently be labeled by the parameters that specify the solution in the interval I_1, and can be constructed in the rest of the intervals as superpositions of the relevant basis $w_{mj}(r)$, $j=1,2$. An example of the notation is given here

$$\begin{aligned}&w^+(r)\equiv w_{11}(r)\ ,\\ &w^-(r)\equiv w_{12}(r)\ ,\\ &\text{where } 0\le r<d_1;\ \text{and} \qquad (2.20)\\ &w^\pm(r)\equiv A_{m1}^\pm w_{m1}(r)+A_{m2}^\pm w_{m2}(r),\quad d_{m-1}\le r<d_m\ .\end{aligned}$$

Notice that the propagation of the constants $A_{mj}^\pm$ from interval to interval should be prescribed by some appropriate principle.

We are also interested in the behavior of the residual interaction under extreme conditions. It is straightforward to extract from the definitions (2.18), (2.17), and (2.12), that

$$\Delta(r)\rightarrow\begin{cases}[\lambda_1^2-l(l+1)-\frac{1}{4}]r^{-2}+O(1), & r\rightarrow 0\ ,\\ \frac{5}{16}(r-R_{m\mu})^{-2}, & r\rightarrow R_{m\mu}\ ,\\ [\lambda_M^2-l(l+1)]r^{-2}+O(r^{-4}), & r\rightarrow\infty\ ,\\ [\lambda_m^2-l(l+1)]r^{-2}+\mathcal{D}^\infty(r), & g\rightarrow\infty,\ r=r_m\ ,\end{cases} \qquad (2.21)$$

where

$$\mathcal{D}^\infty(r)\equiv\frac{5}{16}\frac{1}{V(r)^2}\left[\frac{dV}{dr}\right]^2-\frac{1}{4}\frac{1}{V(r)}\frac{d^2V}{dr^2}\ . \qquad (2.22)$$

This concludes the list of definitions and notations.

III. PHYSICS POSTULATES SPECIFY FREE PARAMETERS

In exact QM the scattering problem under discussion is governed by the Schrödinger equation

$$\left[\frac{d^2}{dr^2}+k^2-g^2V(r)-\frac{l(l+1)}{r^2}\right]u_l(r)=0\ . \qquad (3.1)$$

Recall that after Sec. II we are left with the sets λ_m^2 and $A_{mj}^\pm$ of unspecified parameters. We seek the optimum choice of them so as to bring the differential equations of WKBA and QM, Eqs. (2.19) and (3.1), as close to each other as possible. To this end, we raise, by physics considerations, a set of postulates listed below. Henceforward the present "variable-centrifugal-strength" approach to WKBA will be referred to by the abbreviation VCSA. The postulates (1) through (6) read as follows.

(1) VCSA should reproduce the QM's small-distance behavior

$$u_l(r)\rightarrow\text{const}\times(kr)^{1/2}(kr)^{\pm[l+(1/2)]},\quad kr\rightarrow 0\ ,$$

of the physical and nonphysical solutions, respectively.

(2) VCSA should reproduce the QM's large-distance behavior

$$u_l(r)\rightarrow a_l\hat{j}_l(kr)+b_l\hat{n}_l(kr),\quad kr\rightarrow\infty$$

(observe the same subscripts on both sides).

(3) VCSA should be governed by a potential $W(g^2,r)$

that remains free of singularities at any fixed value of g^2, just as QM does.

(4) The VCSA's potential should reproduce the QM's potential at least to $O(g^2)$, $(g\to\infty)$.

(5) The residual interaction between QM and VCSA should be minimized at each fixed r so far as is still compatible with above postulates, e.g., (3).

(6) The VCSA wave functions should be everywhere smooth.

The propositions for selecting the appropriate values of the free parameters to meet postulates (1)–(6) are collected in Table II and Eqs. (3.2) and (3.3).

We extract from Eqs. (2.20) that

$$A_{11}^{+}=A_{12}^{-}=1,\quad A_{12}^{+}=A_{11}^{-}=0\ . \tag{3.2}$$

Our proposition for the coefficients $A_{mj}^{\pm}$ is implied in the iteration scheme

$$A_{m+1,j}^{\pm}=\frac{1}{k}[\mathcal{W}_{d_m}(w_{m1};w_{m+1,2/j})A_{m1}^{\pm}+\mathcal{W}_{d_m}(w_{m2};w_{m+1,2/j})A_{m2}^{\pm}]\ , \tag{3.3}$$

where the wave functions contained in the arguments of the Wronskian determinants are to be taken from Eqs. (2.9).

We are now going to check performance of our propositions in satisfying the postulates listed above.

Postulate (1). Table II suggests $\lambda_1^2=\lambda_a^2$ for all the types (i)–(iii) of scattering, a choice that yields by Eqs. (2.18), (2.1), (2.21), and (1.3), that

$$rW(r)\to 0,\quad r\to 0\ . \tag{3.4}$$

It is obvious that this property ensures, indeed, QM-compatible small-distance behavior and should be compared to its QM analog (2.1). This statement is reinforced by incorporating the above value of λ_1^2 into Eq. (2.9a). After some algebra one obtains that

$$w^{\pm}(r)\to \text{const}\times(kr)^{1/2}(kr)^{\pm[l+(1/2)]},\quad r\to 0\ . \tag{3.5}$$

For the details of the derivation we refer to Ref. 2.

Postulate (2). Insertion of $\lambda_M^2=\lambda_b^2$ of Table II (notice the subscript on the left-hand side) for problems of types (i)–(iii) furnishes by Eqs. (2.18), (2.2), (2.21), and (1.3), that

$$r^2W(r)\to 0,\quad r\to\infty\ . \tag{3.6}$$

This property is, just as its QM analog (2.2), sufficient for the QM-compatible large-distance behavior to hold. Moreover, the above value of λ_M^2 provides by Eqs. (2.20) and (2.9b) the phase shift of the VCSA as

$$\delta_{\text{VCSA}}^{+}=(l+\tfrac{1}{2})\pi/2-\int_{d_{M-1}}^{\infty}d\rho\,\rho\frac{d}{d\rho}K_b(\rho)+[\sigma_M-d_{M-1}K_b(d_{M-1})]\ , \tag{3.7}$$

where

$$\sigma_M\equiv\arccos\frac{A_{M1}^{+}}{[(A_{M1}^{+})^2+(A_{M2}^{+})^2]^{1/2}}\ . \tag{3.8}$$

For comparison, we reproduce here the analogous formula of the LTPA as

$$\delta_{\text{LTPA}}^{+}=(l+\tfrac{1}{2})\pi/2-\int_{R_a}^{\infty}d\rho\,\rho\frac{d}{d\rho}K_a(\rho)\ . \tag{3.9}$$

Observe that $K_a(\rho)$ and R_a involved here depend on the centrifugal strength $\lambda_a^2=(l+\frac{1}{2})^2$ while the VCSA's phase shift of Eq. (3.7) involves $\lambda_b^2=l(l+1)$.

Postulate (3). The entire formalism is, by Eqs. (2.9) and (2.12), singularity free whenever

$$K_m^2(r_m)>0,\quad m=1,2,M \tag{3.10}$$

i.e., if the TP's lie off the respective intervals I_m. We introduce the notation

$$y_m(r)\equiv k^2-\frac{\lambda_m^2}{r^2}\ ,\qquad K_m^2(r)=\sigma_m[y_m(r)-g^2V(r)]\ . \tag{3.11}$$

Thus Postulate (3) is fulfilled if none of the curves $y_m(r)$ crosses $g^2V(r)$ in I_m, $(m=1,2,M)$. Therefore we conclude from Figs. 1–3 that condition (3.10) is indeed real-

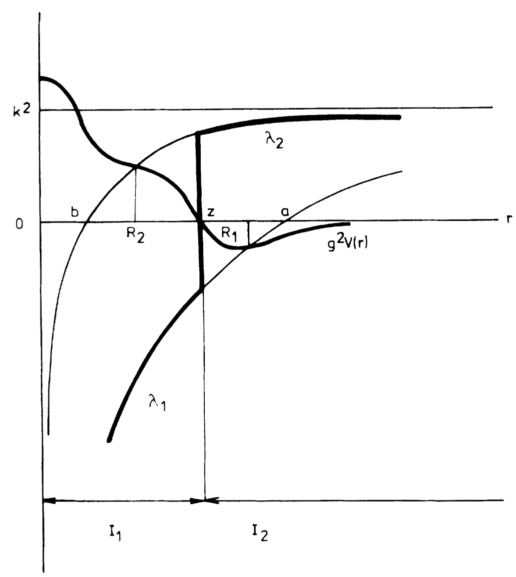

FIG. 1. Type-(i) scattering problems: $b<z<a$; $M=2$ [see Eqs. (2.4)–(2.7)]. The parameters λ_m $(m=1,2)$, the numerical values of which are taken from Table II label the curves $y_m(r)$ of Eq. (3.11) which cross the curve $g^2V(r)$ at the turning points R_m. Observe that each of the points R_m lies outside the respective interval I_m, however large g is chosen. The functions $K_m^2(r_m)$ of Eqs. (2.13) and (3.11) are seen to be positive, proving thereby the inequality (3.10).

TABLE II. The strengths λ_m^2 of the centrifugal terms vary from interval to interval [see Eqs. (2.3)–(2.6) and (2.13)].

Type	λ_1^2	λ_2^2	λ_3^2
(i)	λ_a^2	λ_b^2	
(ii)	λ_a^2	$k^2z^2+\epsilon^2$	λ_b^2
(iii)	λ_a^2	$k^2z^2-\epsilon^2$	λ_b^2

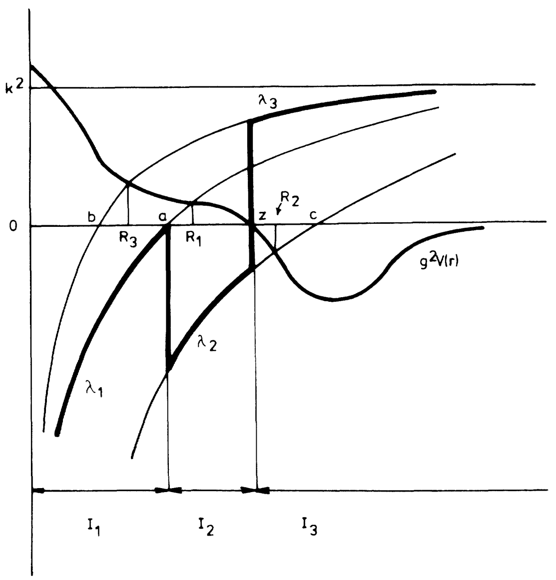

FIG. 2. Type-(ii) scattering problems: $b<a<z$; $M=3$. Upon increasing g from zero to infinity the turning points R_1, R_2, and R_3 start at the respective points a, c, and b to get asymptotically to $r=z$, alike. Observe that meanwhile none of the points R_m, $m=1,2,3$ enters the respective interval I_m.

ized by the choice of λ_m^2 contained in Table II. Observe also that, in each case, a single, g-independent value of λ_m^2 works for all possible values of the coupling constant. This experience is, in fact, a crucial point of our argument to come.

Postulate (4). A simple comparison of Eqs. (2.18) and (2.21) shows that but for two isolated points, the origin

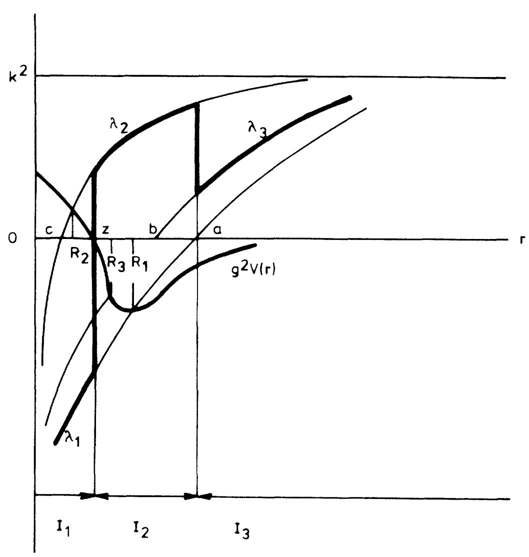

FIG. 3. Type-(iii) scattering problems: $z<b<a$; $M=3$. While g varies from 0 to ∞ the turning points R_1, and R_3 move exclusively in I_2 but R_2 moves in the interval I_1.

$r=0$ and $r=z$, i.e., the zero of the potential, the postulate of strong-coupling exactness is everywhere fulfilled. At the origin this property breaks down by the dominance of the centrifugal term. At $r=z$, the physical and the WKBA potentials differ also by an $O(g^2)$, $(g\to\infty)$ term.

Postulate (5). The values of λ_1^2 and λ_M^2 that are exclusively acceptable from the physical point of view have already been uniquely fixed by Postulates (1) and (2). The postulate of optimum choice of λ_m^2 concerns therefore the case $m=2$, types (ii) and (iii) of scattering. In these cases the single free parameter in finding the optimum is, by Table II, ϵ^2. Also, since we are primarily interested in strong coupling it is convenient to consider the limiting formula obtained by Eqs. (2.21) and (2.22) as

$$|\Delta_2(r_2)|\to\left|\mathcal{D}^\infty(r_2)+\frac{\lambda_2^2-\lambda_b^2}{r_2^2}\right|,\quad g\to\infty\ . \tag{3.12}$$

Particularly simple is optimization if, throughout the interval I_2, the centrifual contribution dominates the $\mathcal{D}^\infty$ term [see Eq. (2.22)], i.e., if

$$r_2^2|\mathcal{D}^\infty(r_2)|<|\lambda_2^2-\lambda_b^2|\ . \tag{3.13}$$

If so then one can write by Eq. (3.12) that

$$|\Delta_2(r_2)|=\operatorname{sgn}(\lambda_2^2-\lambda_b^2)\Delta_2(r_2)\ . \tag{3.14}$$

On the other hand, Eqs. (2.4)–(2.6), together with Table II, imply that for the two types of scattering considered here, we have for (ii),

$$\begin{gathered}\lambda_a<kz,\quad \lambda_b^2<\lambda_2^2=k^2z^2+\epsilon^2\ ,\\ \text{and for (iii)}\ ,\\ kz<\lambda_a,\quad \lambda_2^2=k^2z^2-\epsilon^2<\lambda_b^2\ .\end{gathered} \tag{3.15}$$

Since $\mathcal{D}^\infty(r_2)$ of Eq. (2.22) is independent of ϵ^2, Eqs. (3.12), (3.14), and (3.15) combine to yield

$$\frac{d|\Delta_2(\epsilon^2,r_2)|}{d\epsilon^2}\xrightarrow[g\to\infty]{}\frac{d}{d\epsilon^2}\frac{k^2z^2+\epsilon^2-l(l+1)}{r_2^2}=\frac{1}{r_2^2},\quad \text{case (ii)}\ ;$$

$$\to-\frac{d}{d\epsilon^2}\frac{k^2z^2-\epsilon^2-l(l+1)}{r_2^2}=\frac{1}{r_2^2},\quad \text{case (iii)}\ . \tag{3.16}$$

Thus, whenever condition (3.13) holds throughout I_2, the optimum of λ_2^2 is attained by infinitesimal values of ϵ^2 for both types (ii) and (iii) of scattering problems. Consider also Figs. 2 and 3.

Postulate (6). It is straightforward to check that Eqs. (3.3) are just the formulas of smooth matching of $w^\pm(r_m)$ and $w^\pm(r_{m+1})$ of Eqs. (2.20) at $r_m=d_m=r_{m+1}$. Thus the set $\{m\}$ of Eqs. (3.3) ensures, indeed, overall smoothness of the solutions $w^\pm(r)$ along the entire r axis.

Note that Eqs. (3.3) have been derived with due regard to Eq. (2.10). Recall that the overall VCSA potential $W(r)$ is discontinuous at the end points of the intervals I_m.

Nevertheless, smoothness of the solutions $w^{\pm}(r)$ is sufficient for their Wronskian determinants to remain constant along the entire r axis:

$$\mathcal{W}(w^+(r);w^-(r))=-2k, \quad 0\le r \tag{3.17}$$

the numerical value of which has been taken from applying Eq. (2.10) for the case $m=1$.

IV. SERIES EXPANSION OF THE EXACT SCATTERING STATE

In this section we shall construct an integral equation for the exact scattering wave function in terms of the VCSA scattering problem as a reference system. By iteration, the integral equation generates a series whose convergence will also be studied.

Owing to Eqs. (2.18), (3.1), and (2.19) the Volterra equation is given by

$$u^+(r)=w^+(r)+\int_0^r dr' G(r,r')\Delta(r')u^+(r') , \tag{4.1}$$

with the resolvent normalized by Eq. (3.17) as

$$G(r,r')\equiv\frac{1}{2k}[w^+(r)w^-(r')-w^+(r')w^-(r)] . \tag{4.2}$$

That the possible solution of Eq. (4.1) will, indeed, supply the *physical* scattering state is warranted by the small-distance behavior of the inhomogeneous term as contained in Eq. (3.5). The iteration scheme for solving Eq. (4.1) is obviously

$$\begin{aligned} &u^{(0)}(r)\equiv w^+(r) , \\ &u^{(s)}(r)\equiv\int_0^r dr' G(r,r')\Delta(r')u^{(s-1)}(r'), \quad s\ge 1 . \end{aligned} \tag{4.3}$$

The right-hand side can be decomposed into contributions of the intervals I_m as

$$\begin{aligned} u^{(s)}(r_m)=&\sum_{\mu=1}^{m-1}\int_{d_{\mu-1}}^{d_\mu} dr_\mu g_{m\mu}(r_m,r_\mu)\Delta(r_\mu)u^{(s-1)}(r_\mu) \\ &+\int_{d_{m-1}}^{r_m} dr'_m g_{mm}(r_m,r'_m)\Delta(r'_m)u^{(s-1)}(r'_m) , \end{aligned} \tag{4.4}$$

with the shorthand notation

$$g_{m\mu}(r_m,r'_\mu)\equiv\frac{1}{2k}\sum_{i,j=1}^{2} B_{\mu j}^{mi} w_{mi}(r_m)w_{\mu j}(r'_\mu) . \tag{4.5}$$

The coefficients involved are obtained by means of Eq. (2.20) as

$$B_{\mu j}^{mi}=A_{mi}^+A_{\mu j}^- - A_{\mu j}^+A_{mi}^- . \tag{4.6}$$

We are looking for majorizing of the series to be developed by iteration of Eq. (4.1). We first rewrite the definition (2.13) by Eq. (3.10) as

$$K_m^2(r_m)=\left|\left[k^2-\frac{\lambda_m^2}{r_m^2}\right]-g^2V(r_m)\right| . \tag{4.7}$$

Hence we shall extract a number of inequalities, which, however, can also be checked by Figs. 1–3. The notation to be used below rests on Eqs. (2.7) and Table I. Accordingly, we find for problems of type (i) that

$$\begin{aligned} &0<\frac{\lambda_1^2}{z^2}-k^2<K_1^2(r_1)<\frac{C_\alpha^2}{r_1^2} , \\ &0<k^2-\frac{\lambda_2^2}{z^2}<K_z^2(r_2)<k^2+g^2|V(r_2)| , \end{aligned} \tag{4.8}$$

with C_α^2 being a suitably chosen constant. In a similar way, one obtains for case (ii) that

$$\begin{aligned} &0<\min V(r_1)<K_1^2(r_1)<\frac{C_\beta^2}{r_1^2} , \\ &0<k^2-\frac{\lambda_2^2}{z^2}<K_2^2(r_2)<\frac{\lambda_2^2}{a^2}+g^2\max V(r_2) , \\ &0<k^2-\frac{\lambda_3^2}{z^2}<K_3^2(r_3)<k^2+g^2\max|V(r_3)| , \end{aligned} \tag{4.9}$$

where C_β^2 is again an adequate constant. The corresponding relations for case (iii) are the following:

$$\begin{aligned} &0<\frac{\lambda_1^2}{z^2}-k^2<K_1^2(r_1)<\frac{C_\alpha^2}{r_1^2} , \\ &0<k^2-\frac{\lambda_2^2}{z^2}<K_2^2(r_2)<k^2+g^2\max|V(r_2)| , \\ &0<k^2-\frac{\lambda_3^2}{a^2}<K_3^2(r_3)<k^2+g^2\max|V(r_3)| . \end{aligned} \tag{4.10}$$

Notice the arguments r_m of the functions $K_m(r)$ throughout the set of the above inequalities. Hence we conclude that each of the functions $K_m(r)$ remains bounded and nonzero in its "eigeninterval" I_m. Consequently, for $m\ge 2$ both the trigonometric and exponential-type basis functions $w_{mj}(r)$ of Eq. (2.9) can be majorized in I_m by suitable constants as

$$|w_{mj}(r_m)|<\beta_{mj}^2, \quad \sigma_m=\pm 1, \quad j=1,2 . \tag{4.11}$$

As to interval I_1, we refer to the small-distance formula (3.5), which can also be slightly modified as

$$w_{1j}(r_1)\rightarrow \text{const}\times\eta_j(r_1), \quad r_1\rightarrow 0 , \tag{4.12}$$

where

$$\eta_j(r)\equiv\left[\frac{kr}{1+kr}\right]^{(1/2)+(3-2j)[l+(1/2)]}, \quad j=1,2 . \tag{4.13}$$

Notice that the definition (2.20) has also been incorporated in Eq. (4.12). The relations (4.11) and (4.12) can be combined into the set of inequalities

$$|w_{mj}(r_m)|<\text{const}\times\eta_j(r_m), \quad j=1,2, \quad m=1,2,M . \tag{4.14}$$

By monotonicity, one extracts from Eq. (4.13) that

$$\eta_2(r_m)\eta_1(r_\mu)\le\eta_2(r_\mu)\eta_1(r_m), \quad r_\mu<r_m . \tag{4.15}$$

We infer from the last two statements that

$$|w_{mi}(r_m)w_{\mu j}(r_\mu)| < \text{const}\times\eta_2(r_\mu)\eta_1(r_m)\ , \quad r_\mu < r_m \tag{4.16}$$

for all possible pairs of i and j. Definition (4.5) leads then to the majorization of the partial resolvent as

$$|g_{m\mu}(r_m,r'_\mu)| \le \text{const}\times\eta_2(r'_\mu)\eta_1(r_m),\quad r'_\mu \le r_m\ . \tag{4.17}$$

The number M of the intervals is finite, thus it permits working with an overall constant in the set $\{m\}$ of Eqs. (4.17) throughout the r axis. The insertion of inequality (4.17) into Eq. (4.4) yields

$$|u^{(s)}(r)| < \text{const}\times\eta_1(r)\int_0^r dr'\eta_2(r')|\Delta(r')||u^{(s-1)}(r')|\ , \tag{4.18}$$

or, identically,

$$\eta_1(r)^{-1}|u^{(s)}(r)| \le c\int_0^r dr'\frac{r'}{1+kr'}|\Delta(r')|\eta_1(r')^{-1}|u^{(s-1)}(r')|\ . \tag{4.19}$$

Owing to the definitions (2.20) and (4.3) as well as the inequality (4.14) we can write

$$\eta_1(r)^{-1}|u^{(0)}(r)| < c_0 = \text{const}\ . \tag{4.20}$$

Iteration of the inequality (4.19) after some algebra leads therefore to

$$\eta_1(r)^{-1}|u^{(s)}(r)| < c_0\frac{1}{s!}\left[c\int_0^r dr'\frac{r'}{1+kr'}|\Delta(r')|\right]^s . \tag{4.21}$$

On the other hand, the centrifugal strength is given in the external region for all types of scattering problems by

$$\lambda_M^2 = l(l+1)\ . \tag{4.22}$$

Hence we see by the relevant asymptotical formula of Eqs. (2.21) that

$$\Delta(r)\to\frac{\text{const}}{r^4},\quad r\to\infty \tag{4.23}$$

on account of which the integral contained in Eq. (4.21) is bounded in the variable r along the entire r axis. We conclude then from Eq. (4.21) that

$$\sum_{s=0}^{\infty}|u^{(s)}(r)| < c_0\left[\frac{kr}{1+kr}\right]^{l+1}\exp\left[c\int_0^r dr'\frac{r'}{1+kr'}|\Delta(r')|\right] . \tag{4.24}$$

This inequality is equivalent to the statement that the series

$$u(r) \equiv \sum_{s=0}^{\infty} u^{(s)}(r) \tag{4.25}$$

is absolutely convergent. Remember that the set of functions $u^{(s)}(r)$ has been introduced by iteration in terms of Eqs. (4.3). Boundedness of the argument of the exponential function in the formula (4.24) suggests also uniform convergence. The sum $u(r)$ of Eq. (4.25) is by analysis the unique solution of the integral equation (4.1) and provides thus the exact scattering state of QM.

V. INCREASING THE COUPLING CONSTANT

Up to this point we have been interested in producing an expansion of the scattering solution of the Schrödinger equation at fixed strength of the physical potential. The present section is devoted to the study of how the single terms of this expansion behave in the case of strong coupling.

As we have already pointed out, the increase of the coupling constant does not, in fact, induce active TP's, i.e., such ones that would give rise to singularities of the VCSA potential or wave functions. The second step of the present argument is to find the strong-coupling form of the iteration scheme (4.3). From Eq. (4.7) we extract the limits, where $g\to\infty$,

$$K_m^2(r_m)\to g^2V(r_m),\quad r_m\ne z\ ,$$
$$\to\left|k^2-\frac{\lambda_m^2}{r_m^2}\right|,\quad r_m = z\ , \tag{5.1}$$

by means of which we consider the higher-order contributions $u^{(s)}(r)$ to Eq. (4.25) in the interval I_1. Equations (4.3) read for $s=1$

$$u^{(1)}(r_1) = \frac{1}{2k}\int_0^{r_1}dr'[w^+(r_1)w^-(r') - w^+(r')w^-(r_1)]\Delta(r')w^+(r')\ , \tag{5.2}$$

where Eq. (4.2) has also been incorporated. If we insert here Eq. (5.1) integration by parts furnishes the strong-coupling leading term, by substituting $d_0=0$, as

$$u^{(1)}(r_1)\to\left[-\tfrac{1}{2}\int_{d_0}^{r_1}dr'\frac{\Delta(r')}{K_1(r')}\right]u^{(0)}(r_1),\quad g\to\infty\ . \tag{5.3}$$

Owing to Eqs. (4.3) and (5.3), one finds the strong-coupling form of $u^{(2)}(r_1)$ by simultaneously replacing in Eq. (5.2) the functions

$$u^{(1)}(r_1)\ \text{by}\ u^{(2)}(r_1)\ \ \text{and}\ \Delta(r')\ \text{by}\ \left[-\tfrac{1}{2}\Delta(r')\int_{d_0}^{r'}dr''\frac{\Delta(r'')}{K_1(r'')}\right] \tag{5.4}$$

to obtain

$$u^{(2)}(r_1)\rightarrow\left[(-1/2)^2\int_{d_0}^{r_1}dr'\frac{\Delta(r')}{K_1(r')}\int_{d_0}^{r'}dr''\frac{\Delta(r'')}{K_1(r'')}\right]u^{(0)}(r_1),\quad g\rightarrow\infty\ . \tag{5.5}$$

Further iteration procedure is straightforward and provides eventually by analysis

$$u^{(s)}(r_1)\rightarrow\frac{1}{s!}\left[-\tfrac{1}{2}\int_{d_0}^{r_1}dr'\frac{\Delta(r')}{K_1(r')}\right]^s u^{(0)}(r_1),\quad g\rightarrow\infty \tag{5.6}$$

whence

$$\begin{aligned}\frac{u^{(s)}(r_1)}{u^{(s-1)}(r_1)}&\rightarrow\frac{1}{s}\left[-\tfrac{1}{2}\int_{d_0}^{r_1}dr'\frac{\Delta(r')}{K_1(r')}\right]\\&=O(g^{-1}),\quad g\rightarrow\infty\ .\end{aligned} \tag{5.7}$$

We have employed here the limit (5.1) as well as the estimation

$$\Delta(r_1)=O(1),\quad g\rightarrow\infty \tag{5.8}$$

obtained from Eqs. (2.21).

The treatment for $m\geq 2$ becomes the more complicated the farther we get from the origin. To avoid extensive calculations the subsequent discussion will be limited to the type-(i) scattering problems, where there are only two intervals involved, i.e., $M=2$. Using the notations (4.3) and (2.20), the first approximation to the scattering wave function reads in the interval I_2

$$u^{(0)}(r_2)=A_{21}^{+}w_{21}(r_2)+A_{22}^{+}w_{22}(r_2)\ . \tag{5.9}$$

Remember that the basis $w_{2j}(r_2)$ is in case (i) of the trigonometric type. The first iteration of Eq. (5.9) is done in terms of Eqs. (4.4)–(4.6) with the result

$$\begin{aligned}u^{(1)}(r_2)=u^{(1)}(z)+\frac{1}{2k}\Big[&A_{21}^{+}\left[B_{22}^{21}w_{21}(r_2)\int_z^{r_2}dr'w_{22}(r')\Delta(r')w_{21}(r')+B_{21}^{22}w_{22}(r_2)\int_z^{r_2}dr'w_{21}(r')\Delta(r')w_{21}(r')\right]\\&+A_{22}^{+}\left[B_{22}^{21}w_{21}(r_2)\int_z^{r_2}dr'w_{22}(r')\Delta(r')w_{22}(r')+B_{21}^{22}w_{22}(r_2)\int_z^{r_2}dr'w_{21}(r')\Delta(r')w_{22}(r')\right]\Big]\ .\end{aligned} \tag{5.10}$$

In strong coupling, Eqs. (2.9b) and (5.1) combine to

$$w_{2j}(g^2,r_2)\rightarrow\frac{k^{1/2}}{K_2(r_2)^{1/2}}\sin\left[g\int_z^{r_2}dr'V(r')^{1/2}+(3-2j)\pi/4\right],\quad g\rightarrow\infty\ . \tag{5.11}$$

Hence we conclude that, for $g\rightarrow\infty$, the contribution of the integrands containing squares of $w_{2j}(r_2)$ is the respective contribution of the products $w_{21}(r_2)w_{22}(r_2)$ times a factor of $O(g)$, $(g\rightarrow\infty)$. Retaining the dominant terms we can write

$$u^{(1)}(r_2)\rightarrow u^{(1)}(z)+\tfrac{1}{4}[A_{22}^{+}B_{22}^{21}w_{21}(r_2)+A_{21}^{+}B_{21}^{22}w_{22}(r_2)]\int_z^{r_2}dr'\frac{\Delta(r')}{K_2(r')},\quad g\rightarrow\infty\ . \tag{5.12}$$

On the other hand, Eqs. (3.2) and (3.3) furnish for $m=1$

$$\begin{aligned}A_{21}^{+}&=\frac{1}{2k}\mathcal{W}_z(w_{11};w_{22})\ ,\\A_{21}^{-}&=\frac{1}{2k}\mathcal{W}_z(w_{12};w_{22})\ ,\\A_{22}^{+}&=\frac{1}{2k}\mathcal{W}_z(w_{11};w_{21})\ ,\\A_{22}^{-}&=\frac{1}{2k}\mathcal{W}_z(w_{12};w_{21})\ .\end{aligned} \tag{5.13}$$

Furthermore, the definitions (2.9) inserted in Eqs. (5.13) supply

$$\begin{aligned}2kA_{21}^{+}&=\alpha_{12}-\beta_{12}-\beta_{21}\ ,\\2kA_{21}^{-}&=\alpha_{12}-\beta_{21}+\beta_{12}\ ,\\2kA_{22}^{+}&=\alpha_{12}+\beta_{21}-\beta_{12}\ ,\\2kA_{22}^{-}&=\alpha_{12}+\beta_{21}+\beta_{12}\ ,\end{aligned} \tag{5.14}$$

where

$$\begin{aligned}\beta_{\mu\nu}&\equiv\left[\frac{K_\mu(z)}{K_\nu(z)}\right]^{1/2},\\\alpha_{12}&\equiv\frac{1}{2}\left[\frac{K_1^{(\prime)}(z)}{K_1(z)^{3/2}K_2(z)^{1/2}}-\frac{K_2^{(\prime)}(z)}{K_1(z)^{1/2}K_2(z)^{3/2}}\right]\ .\end{aligned} \tag{5.15}$$

Observe that

$$\alpha_{12}=O(g^2),\quad \beta_{\mu\nu}=O(1),\quad g\to\infty \tag{5.16}$$

whence

$$B_{22}^{21}(g)\xrightarrow[g\to\infty]{} -4\beta_{12}\beta_{21} = -4 . \tag{5.17}$$

One has by the definition (4.6) that

$$B_{21}^{22}=-B_{22}^{21} . \tag{5.18}$$

Incorporation of Eqs. (5.17) and (5.18) into Eq. (5.12) yields

$$u^{(1)}(r_2)\to u^{(1)}(z)-[A_{22}^{+}w_{21}(r_2)-A_{21}^{+}w_{22}(r_2)] \times\int_z^{r_2}dr'\frac{\Delta(r')}{K_2(r')} . \tag{5.19}$$

On the other hand, Eqs. (5.14) and (5.16) imply that, where $g^2\to\infty$,

$$A_{21}^{+}=\frac{1}{2k}\alpha_{12}+O(1) ,\qquad A_{22}^{+}=-\frac{1}{2k}\alpha_{12}+O(1) , \tag{5.20}$$

on account of which Eq. (5.19) can be rewritten as

$$u^{(1)}(r_2)\to[A_{21}^{+}w_{21}(r_2)+A_{22}^{+}w_{22}(r_2)]\int_z^{r_2}dr'\frac{\Delta(r')}{K_2(r')} = O(g^{1/2}),\quad g\to\infty \tag{5.21}$$

since the trigonometric-type basis set $w_{2j}(r_2)$, $(j=1,2)$, is $O(g^{-1/2})$, $g\to\infty$. Omission of the contribution $u^{(1)}(z)$ from Eq. (5.21) is justified by Eqs. (5.3), (4.3), and (2.9a) since

$$u^{(1)}(z)=\frac{k^{1/2}}{K_1(z)^{1/2}}\int_0^z dr'\frac{\Delta(r')}{K_1(r')} = O(g^{-3/2}),\quad g\to\infty . \tag{5.22}$$

A comparison of Eqs. (5.9) and (5.21) leads, after setting $d_1=z$, to

$$u^{(1)}(r_2)\to\left[\int_{d_1}^{r_2}dr'\frac{\Delta(r')}{K_2(r')}\right]u^{(0)}(r_2),\quad g\to\infty . \tag{5.23}$$

This relationship is for the interval I_2 the counterpart of Eq. (5.3) working in I_1. Just as Eq. (5.3) had led to Eq. (5.7) so does Eq. (5.23) lead to

$$\frac{u^{(s)}(r_2)}{u^{(s-1)}(r_2)}\longrightarrow\frac{1}{s}\int_{d_1}^{r_2}dr'\frac{\Delta(r')}{K_2(r')} = O(g^{-1}),\quad g\to\infty . \tag{5.24}$$

Equations (5.7) and (5.24) can be combined to the overall statement that

$$\frac{u^{(s)}(r)}{u^{(s-1)}(r)}=O(g^{-1}),\quad g\to\infty . \tag{5.25}$$

This result can be reworded by saying that the truncation error caused by cutting off the series (4.24) decreases by a factor of $O(g^{-1})$ once one includes one additional term in the expansion.

VI. DISCUSSION

We have expanded the exact wave function of a particle scattered by a short- (but not finite-) range central potential having just one zero, via iterating a radial integral equation. This equation implies as reference system a *modified* WKB problem, which we call the variable-centrifugal-strength approach. The modification is aimed primarily at preventing the classical turning points from causing singularities of the semiclassical wave functions (see Figs. 1–3). Additional characteristic features of VCSA include that, from the viewpoint of quantum mechanics, (i) *both the small- and large-distance* behaviors are, apart from constant factors, simultaneously correct; (ii) the expansion obtained by iteration is a convergent strong-coupling series of the QM wave function. This performance is due to the residual (physical minus reference) interaction $\Delta(g,r)$ remaining bounded *both* in g *and* r. In particular [see Eqs. (2.21), (2.22), and (5.8)],

$$\Delta_{\rm VCS}(g,r)=O(1),\quad g\to\infty . \tag{6.1}$$

Just by this property, the nth term of the iteration series becomes for large values of g^2 the product of the reference wave function times a factor of $O(g^{-n})$; see Eq. (5.24). By virtue of this structure, the expansion (4.25) is, indeed, a large-g^2 counterpart of the Born series, which is known to be a small-g^2 expansion. Once one knows the physical solution correct to any prescribed order in the inverse coupling constant it is then easy to extract, besides the two-body scattering phase shift, the relevant nonphysical solutions and the perturbed resolvent as well as the off the energy shell transition matrix, i.e., the two-body input to the three-body dynamics.

The first question that immediately arises is whether the above properties, (i) and (ii), are realized within LTPA, the linear-turning-point approach of WKBA. As regards point (i), correct small- and large-distance behavior mutually exclude each other in LTPA, as we have seen in Sec. I. As for point (ii), it is worth going into some details. The singularity in r is eliminated from LTPA by replacing WKBA in a narrow interval (ρ_1,ρ_2) around the turning point by the physical problem itself yet having the potential straightened. As is obvious for the scattering problems represented by Figs. 1–3,

$$R_1(g)\to z\pm 0,\quad g\to\infty \tag{6.2}$$

where z is the unique zero of the potential. Hence the residual interaction reads in this region

$$\Delta_{\rm LTP}(g,r)\to\frac{g^2}{2!}\left[\frac{d^2V}{dr^2}\right]_z(z-r)^2,\quad g\to\infty,\quad \rho_1<r<\rho_2 \tag{6.3}$$

which is, however, at any fixed value of r, of $O(g^2)$. To cure the large-g^2 divergency, we can choose g-dependent matching points $\rho_j(g)$, e.g., by setting

$$\rho_j(g) \equiv R_1(g) + (-1)^j \frac{c^2}{g}, \quad j=1,2 \ . \tag{6.4}$$

Indeed, we obtain by Eqs. (6.2) and (6.3) that

$$\Delta_{\rm LTP}(g,r) = O(1), \quad g \to \infty, \quad \rho_1(g) < r < \rho_2(g) \ . \tag{6.5}$$

The residual interaction remains thus bounded throughout the linearity region at the cost, however, of diverging before and beyond it in the proper WKBA regions. Namely, the matching points of Eqs. (6.4) penetrate in the strong-coupling limit so close to the singularity point $R_1(g)$ that one can rely on the relevant statement of Eqs. (2.21) and write

$$\Delta_{\rm WKB}(g,r) \approx \frac{5}{16} \frac{1}{[R_1(g)-r]^2}, \quad g \to \infty, \quad r \approx \rho_j(g) \ . \tag{6.6}$$

If so, then Eqs. (6.4) and (6.6) combine to

$$\Delta_{\rm WKB}(g,r) = O(g^2), \quad g \to \infty, \quad r \approx \rho_j(g) \ . \tag{6.7}$$

Therefore the requested simultaneous boundedness in g and r fails to be materialized in LTPA. Hence the existence of an LTPA-based strong-coupling expansion seems to be rather problematic. At this point see also the comment following Eq. (6.1).

There are, however, scattering problems for which the VCSA breaks down while the LTPA does work. Thus, singular potentials of the type r^{-p} $(p>1)$, are excluded by condition (2.1) from the present VCSA. The standard WKBA, in turn, exactly reproduces quantum mechanics in the high-energy limit just for these potentials. In particular, this possibility has been suggested by Bertocchi, Fubini, and Furlan[4] for s-wave scattering by inverse power potentials. Soon afterwards, Paliov and Rosendorff[5] extended the WKB approach to cover $r^{-p}\exp(-\mu v)$ type of potentials, too, and coupling constants that depend on energy. These authors have developed distinct series expansions of the phase shift for the small and the large orbital angular momenta preparing thus the summation of the partial wave contributions. The early development of the singular potentials treated by WKBA has been well reviewed by Frank, Land and Spector.[6] The first numerical check of the performance of the WKBA in this subject has been done by Dolinszky[7] for the particular case $p=4$, $l=0$. The first general analytic proof of the high-energy exactness of the WKBA has been given by Fröman and Thylwe.[8] They have reinforced the correctness of the first and second l-dependent terms in the high-energy series of the phase shift as given earlier by Paliov and Rosendorff. The variable phase approach[9] in a generalized form supplied means for Dolinszky[10] to develop the asymptotical form $(k^2 \to \infty)$ of the exact phase shift for scattering by singular potentials to get a formula that is recognized as the WKBA expression itself.

It should also be remarked that the phase shift in itself may be correct even if the wave functions themselves fail. Thus the LTPA that involves $(l+\frac{1}{2})^2$ as the centrifugal strength furnishes exact phase shifts at any energy in all partial waves for the case of the physical potential r^{-2} and reasonable ones for the Coulomb problem. Observe that in the inverse square case the residual interaction itself becomes asymptotically $\frac{1}{4}r^{-2}$ $(r \to \infty)$.

Finally, mention should be made of the force-free problem, the vanishing phase shift of which is exactly reproduced by the $(l+\frac{1}{2})^2$ LTPA. As regards the VCSA, the unperturbed case can be considered from two points of view. First, as a single problem with a potential that vanishes everywhere and violates thereby our condition (2.3) postulating just one zero. Second, as a set $(g \to 0)$ of problems with freely chosen potential shape (with a single zero). If so, the VCSA yields, indeed, the relevant exact solution (the Riccati-Bessel function) in a rather complicated representation by an infinite series. This is, in fact, the cost for treating a weak-coupling scattering problem by a strong-coupling method.

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