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Tokamak $\beta aB/I$ limit and its dependence on the safety factor

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It is shown that, for a given tokamak cross-sectional shape and arbitrary values of the magnetic axis safety factor q_0, the first stability condition against pressure-driven magnetohydrodynamic modes has the form $40\pi\beta aB_0/I \le C_R(q_0)/q_0$. Moreover, in the limit of large q_0, $C_R(q_0)$ becomes independent of q_0 and independent of the toroidal mode number.

The ratio β of plasma kinetic pressure to magnetic pressure in tokamaks is limited by the onset of pressure gradient driven magnetohydrodynamic (MHD) instabilities (first stability limit), or by the ability of the external fields to maintain toroidal force balance (equilibrium limit). The available experimental evidence as well as widespread numerical simulations support the currently accepted form of the first stability limit as proposed by Troyon and others,[1-4]

$$40\pi\beta aB_0/I \le C_T \simeq 3\,. \tag{1}$$

Here a stands for the plasma minor radius, B_0 is the vacuum toroidal field, I is the total plasma current, and a system of units in which aB_0/I is dimensionless and equal to $(aB_0/\mu_0 I)_{\text{mks}}$ is to be used throughout this paper. Since the expression (1) has been obtained only empirically, a theoretical explanation of it based on first principles is a most desirable goal.

A point worth noting is that Eq. (1) is obtained for the normal tokamak configurations where the safety factor at the magnetic axis q_0 is approximately equal to one, or after maximizing β with respect to variations of q_0 which also results in an optimum value of q_0 about one. However, far more insight can be gained by maximizing β at constant but arbitrary q_0, and then studying the dependence of the β limit on q_0. It is also known that the value of the Troyon ratio C_T depends on the geometrical characteristics (aspect ratio, elongation, triangularity, etc.) of the plasma boundary[5] which shall be denoted in short by Γ. The aim of the present work is to provide a theoretical understanding of the tokamak $\beta aB_0/I$ limit by studying its dependence on q_0 at fixed Γ.

In a recent work,[6] this author has shown that for tokamak configurations with smooth pressure and current profiles and vanishing current density at the plasma edge, there exists an equilibrium limit of the form $40\pi\beta aB_0/I \le C_R^{\text{eq}}(q_0,\Gamma)/q_0$. A numerical investigation of such equilibria with a large aspect ratio and circular cross section shows C_R^{eq} to be virtually independent of q_0 and of the order of 11. In addition, a study[7] of the first stability limit against $n=\infty$ ballooning modes for the large-aspect-ratio circular tokamak equilibrium model of Clarke and Sigmar[8] subject to the above constraints on pressure and current density profiles yields $40\pi\beta aB_0/I \le C_R^{\text{st}}/q_0$ with $C_R^{\text{st}}=3.2$, independent of q_0. The present work shows that for general plasma cross sections Γ, smooth profiles with zero edge-current density, and arbitrary q_0, the first stability limit has the form

$$40\pi\beta aB_0/I \le C_R^{\text{st}}(q_0,\Gamma)/q_0\,, \tag{2}$$

and, in the limit of large q_0, $C_R^{\text{st}}(\infty,\Gamma)\simeq 3$ is independent of q_0 and independent of the toroidal wave number n of the mode under consideration. This result provides an answer to two puzzles posed by the Troyon *et al.* formula (1). The first one is the "unnatural" linear relation between β (which is inversely proportional to the square of the toroidal field) and the normalized current I/aB_0 (which is inversely proportional to the toroidal field); this mismatch is corrected by the q_0^{-1} dependence of the right-hand side of Eq. (2). The second one is the fact that the β limit as expressed by Eq. (1) applies both to low-n external kinks and to high-n ballooning modes with only some small variation in the numerical value of C_T. The only previous theoretical work on this subject[9] does not address these issues as it considers only $n=\infty$ modes in $q_0=1$ configurations, besides being limited to a very special and not fully consistent equilibrium model.

Tokamak equilibria are represented by solutions of the Grad-Shafranov equation

$$\left[R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right)+\frac{\partial^2}{\partial Z^2}\right]\psi = -R^2\frac{dp}{d\psi}-F\frac{dF}{d\psi}\,. \tag{3}$$

This equation is invariant under the following transformation:

$$\begin{aligned}&R\to R,\ Z\to Z\,,\\ &\psi\to\lambda\psi,\ p\to\lambda^2 p\,,\\ &F\to\lambda[F^2+F_a^2(\lambda^{-2}-1)]^{1/2}\,,\end{aligned} \tag{4}$$

where F_a is the vacuum value of the toroidal field stream function $F(\psi)$. This transformation leaves invariant the geometry of the flux surfaces $\psi(R,Z)=$const, hence the magnetic axis $R_0\to R_0$, and the vacuum toroidal field $F_a\to F_a$. It generates a homogeneous scaling of the poloidal field $B_p\to\lambda B_p$, and the toroidal current density $j_t\to\lambda j_t$. Therefore, defining the β and poloidal β parameters as $\beta\equiv 2B_0^{-2}V^{-1}\int p\,dV$ and $\beta_p\equiv 4I^{-2}R_0^{-1}\int p\,dV$, where B_0 is the vacuum field at the magnetic axis ($R_0B_0=F_a$) and V is the plasma volume, we obtain the transformations

$$\beta_p\to\beta_p,\ \beta\to\lambda^2\beta,\ \beta aB_0/I\to\lambda\beta aB_0/I\,. \tag{5}$$

The inverse rotational number $q=(2\pi)^{-1}F\oint dl_p R^{-2}B_p^{-1}$

transforms as

$$q \to \frac{q}{\lambda}\left[1+O\left(\left|\frac{F-F_a}{F_a}\right|\right)\right], \tag{6}$$

where the relative variation of the toroidal field $|(F-F_a)/F_a|$ is of the order of β which for the profiles under consideration is smaller than or of the order of ϵ/q_0^2, ϵ being the inverse aspect ratio. Therefore for $\epsilon/q_0^2 \ll 1$, the homogeneous scaling

$$F \rightleftarrows F,\ q \rightleftarrows q/\lambda \tag{7}$$

approximates very well the actual transformations of F and q. The approximation (7) gets better and better as our transformation is applied iteratively with $\lambda<1$, and becomes exact in the limit $q_0 \to \infty$. Using the scaling of q we can construct an invariant Troyon-like ratio, namely

$$q_0\beta a B_0/I \rightleftarrows q_0\beta a B_0/I\,. \tag{8}$$

Since our transformation is an invariance of the equilibrium equation, any equilibrium limits must be expressed in terms of invariant parameters such as β_p or $q_0\beta aB_0/I$, but not β or $\beta aB_0/I$. The work of Ref. 6 shows the existence of an equilibrium limit for $q_0\beta aB_0/I$, whereas $q_a^x q_0^{(1-x)}\beta aB_0/I$ with $x>0$ need not be bounded, thus I can argue that $q_0\beta aB_0/I$ is indeed the natural invariant Troyon-like ratio that should be used in theoretical studies.

My approach to the problem of the stability β limit is as follows. The above discussed equilibrium scaling is used to generate sequences of equilibria with increasing q values (characterized by increasing q_0) but nearly invariant magnetic shear profiles. Variations of the shear profile should be done separately at fixed q_0. It will then be shown that, in the limit of large q, the ideal MHD stability equations become invariant under such scaling, therefore the stability limit must be expressed in terms of the invariant parameter $q_0\beta aB_0/I$ rather than $\beta aB_0/I$.

Let me consider first $n=\infty$ ballooning modes. I adopt a coordinate system in which the flux coordinate $r(\psi)$ is defined by $r\,dr=R_0qF^{-1}d\psi$ and the poloidal coordinate θ is related to the poloidal arc length through $dl_p=rRR_0^{-1}d\theta$, and define the associated metric functions $D\equiv|\nabla r|^{-1}$ and $u\equiv r|\nabla r|^{-2}\nabla r\cdot\nabla\theta$. In this coordinate system the marginal stability equation for the ballooning eigenfunction $\hat{X}$ reads

$$\frac{d}{d\hat\theta}\left[\frac{DR_0^2}{R^2}(1+\Sigma^2)\frac{d\hat X}{d\hat\theta}\right]+\frac{aD^2R^2}{R_0}(-\kappa_n+\Sigma\kappa_g)\hat X=0\,. \tag{9}$$

Here $\hat\theta$ is the extended variable associated with the poloidal coordinate θ, Σ is the integrated local shear, κ_n and κ_g are the normal and geodesic components of the magnetic curvature, and $a\equiv-2R_0^3q^2F^{-2}dp/dr$. All terms in Eq. (9) are invariant under the transformation (4) except for the functions Σ, κ_n, and κ_g. However, in the limit $q\to\infty$ where the magnitude of the poloidal field is negligible compared to the toroidal field, they become

$$\Sigma=\frac{1}{qD^2}\left[\left(r\frac{\partial}{\partial r}+u\frac{\partial}{\partial\theta}\right)\left(q\int^{\theta}D\,d\theta'\right)\right], \tag{10}$$

$$\kappa_n=-\cos(\theta_n)/R,\quad \kappa_g=\sin(\theta_n)/R\,, \tag{11}$$

where $\cos\theta_n=D\nabla r\cdot\nabla R$. These asymptotic expressions,[10,11] valid in the large-q regime, are invariant under our scaling, which now becomes an invariance of the ballooning equation. In fact the precise condition for the scaling invariance of the $n=\infty$ ballooning equation is $\epsilon/q^2\ll1$ (i.e., the same as that required for equilibrium). This is so because the largest neglected term in Eqs. (10) and (11) is the contribution of the poloidal field to the magnetic curvature whose relative magnitude is of the order of $RB_p^2/rB_t^2\sim\epsilon/q^2$.

Next I turn to the study of low-n modes. Given a plasma displacement ξ, the plasma contribution to the incremental MHD potential energy can be expressed in the previously defined coordinate system as

$$\begin{aligned}W_p[\xi]=\frac{\pi}{2R_0^3}\int rF^2dr\,d\theta\Bigg\{&\frac{R_0^2D}{R^2}\left|\left(inD-\frac{\lambda}{q}\frac{\partial}{\partial\theta}\right)X\right|^2+\frac{R_0^2}{r^2D}\left|\frac{\partial Y}{\partial\theta}-\left(1+\frac{\partial u}{\partial\theta}+r\frac{\partial}{\partial r}+u\frac{\partial}{\partial\theta}\right)(DX)\right|^2\\&+\frac{1}{D^3}\left|inDY+\frac{\lambda}{q}\left(1+\frac{\partial u}{\partial\theta}-r\frac{\partial}{\partial r}-u\frac{\partial}{\partial\theta}\right)(DX)\right|^2\\&+|X|^2\frac{2R_0^2R^2D}{rF^2}\left[\lambda^2\frac{dp}{dr}\left(r\frac{\partial}{\partial r}+u\frac{\partial}{\partial\theta}\right)\ln R+\frac{\lambda^2rFj_t}{R_0Rq}\left(1+\frac{\partial u}{\partial\theta}\right)\right]\Bigg\}.\end{aligned} \tag{12}$$

The variables X and Y are related to the perpendicular components of the plasma displacement through $X=\xi\cdot\nabla r$ and $Y=R_0F^{-1}|\nabla r|^{-2}\xi\cdot(\mathbf{B}\times\nabla r)$; minimization with respect to its parallel component has been carried out by taking $\nabla\cdot\xi=0$. In Eq. (12) I have written explicitly the λ factors that arise when applying the scaling of Eqs. (4) and (7). The stability in the large-q regime can now be investigated by formally letting λ tend to zero. In this limit I observe that, unless the derivatives of the displacement are of order λ^{-1}, the leading terms are the positive definite contributions of the stable Alfvén and magnetosonic waves. This situation is entirely analogous to that found in the high-n ballooning theory when one takes the $n\to\infty$ limit. In complete analogy with the high-n ballooning theory, in order to obtain an instability when $\lambda\to0$ one must construct a long parallel, short perpendicular wavelength perturbation of the form $X=\hat X\times\exp(inq\lambda^{-1}\int^{\theta}Dd\theta')$ and

$$inq\lambda^{-1}DY-(r\partial/\partial r+u\partial/\partial\theta)(DX)=\hat U\exp\left(inq\lambda^{-1}\int^{\theta}Dd\theta'\right)$$

with $\partial\hat{X}/\partial\theta \sim \partial\hat{U}/\partial\theta \sim 1$. An algebraic minimization of W_p with respect to $\hat{U}$ can now be carried out perturbatively in powers of λ. Finally, a minimization with respect to $\hat{X}$ yields an Euler equation which is identical to the large-q form of the ballooning equation given in Eqs. (9)-(11). Since the latter is invariant under the λ scaling, the instability threshold against low-n modes in the $q_0 \rightarrow \infty$ regime is also invariant under such scaling, and identical to the threshold against $n=\infty$ modes. Any explicit dependence on the λ parameter disappears as $\lambda \rightarrow 0$ in the same manner that any explicit dependence on n drops out in the $n=\infty$ ballooning theory. Moreover, since in my analysis the parameter $n/\lambda \gg 1$ plays the same role that $n \gg 1$ does in the conventional ballooning theory, I conclude that finite-q_0 (finite λ) corrections to the asymptotic stability limit for $q_0 = \infty$, ($\lambda = 0$) correspond to the finite-n corrections to the $n=\infty$ limit in ballooning theory, which are known to be stabilizing.[10] Therefore, at large q_0, the $q_0\beta aB_0/I$ limit against low-n modes should approach its $q_0 = \infty$ asymptotic value (which coincides with the corresponding one for $n=\infty$ modes) from above. To summarize, the asymptotic conditions for the validity of my scaling are $q^2/\epsilon \gg 1$ for invariance of the equilibrium and $n=\infty$ ballooning stability equations, plus $nq \gg 1$ for equivalence of low-n and high-n stability. This result *does not* require a large-aspect-ratio assumption as my asymptotic regime can always be reached with sufficiently large q irrespective of ϵ.

These theoretical predictions have been verified in a numerical study of tokamak stability using the PEST code.[11] I consider a large aspect ratio ($A=10$) tokamak with circular cross section. The choice of large aspect ratio is deliberate because this allows us to reach the relevant large-q regime as far as *equilibrium and $n=\infty$ ballooning stability* is concerned with moderate q values. The two flux functions that specify an equilibrium are chosen to be the pressure $p(\psi)$ and the Ohmic current $j_{\rm Oh}(\psi) \equiv \langle \mathbf{j}\cdot\mathbf{B}\rangle / \langle R_0\nabla\phi\cdot\mathbf{B}\rangle$, where $\langle\,\rangle$ stands for the conventional flux surface average and ϕ is the toroidal angle. By taking these to be analytic functions of ψ with the appropriate behavior at the plasma boundary, the desired constraints on pressure and current density profiles are always satisfied. Specifically, I consider profiles of the form $p = p_0(1-\hat{\psi}^{\alpha_{1p}})^{\alpha_{2p}}$ and $j_{\rm Oh} = j_0(1-\hat{\psi}^{\alpha_{1j}})^{\alpha_{2j}}$ where $\hat{\psi}$ is the normalized poloidal flux ($0 \le \hat{\psi} \le 1$). Three different choices of such profiles are studied here:

$$\alpha_{1p} = \alpha_{2p} = 2,\ \alpha_{1j} = \alpha_{2j} = 1\,, \tag{13a}$$

$$\alpha_{1p} = 2,\ \alpha_{2p} = 3,\ \alpha_{1j} = 1,\ \alpha_{2j} = 2\,, \tag{13b}$$

$$\alpha_{1p} = 4,\ \alpha_{2p} = 2,\ \alpha_{1j} = \alpha_{2j} = 2\,. \tag{13c}$$

For each of these, three sequences of equilibria with increasing β and constant q_0 are generated by increasing p_0 while adjusting j_0 in such a way that q_0 remains equal to 1.0, 2.0, and 3.0, respectively. These nine sequences are tested for ideal MHD stability against $n=\infty$ ballooning and $n=1$ external modes. The resulting β limits are plotted in Figs. 1 and 2. It is clearly seen there that the critical value of the parameter $q_0\beta aB_0/I$ becomes independent of q_0 at large q_0, thus confirming the form of the β limit as expressed in my Eq. (2). It is also seen that, at large q_0, the instability threshold for $n=1$ is higher than for $n=\infty$ as predicted theoretically. Finally, I observe that the instability thresholds for $n=1$ modes approach their large-q asymptotic values more slowly than for $n=\infty$, since, as discussed previously, the convergence rate for low n is not affected by the smallness of ϵ. In any case, even for $n=1$, $C_R(q_0) = q_0 C_T(q_0)$ is found to be a rather weak function of q_0 down to $q_0 = 1$. Its numerical value would become

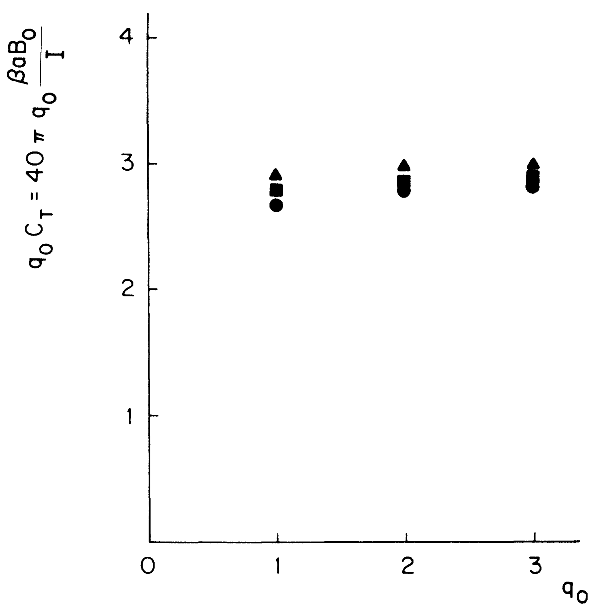

FIG. 1. Marginal stability points for $n=\infty$ ballooning modes. The critical value of the parameter $C_R = q_0 C_T$ is plotted vs q_0 for nine sequences of equilibria characterized by constant q_0 and different pressure and current density profiles. The circles, squares, and triangles correspond to the profiles given by Eqs. (13a), (13b), and (13c), respectively.

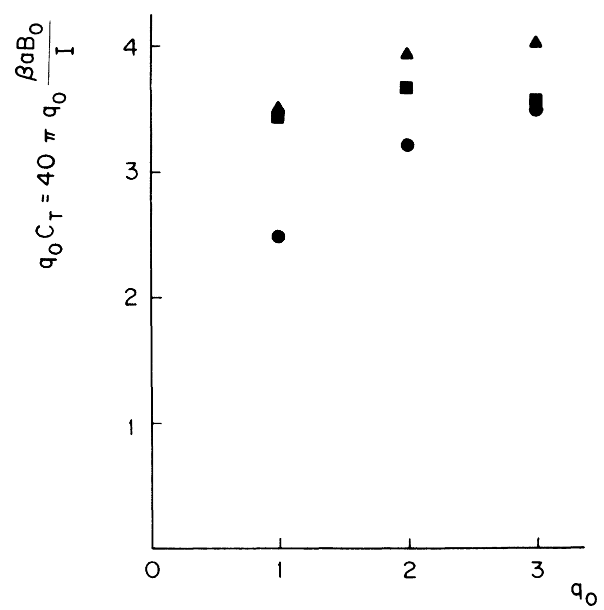

FIG. 2. Marginal stability points against $n=1$ external modes for the sequences of equilibria described in Fig. 1.

somewhat higher following a more thorough profile optimization. For example, the quoted $n=\infty$ ballooning limits correspond to the point where a single flux surface becomes marginally stable, but higher β values would be obtained by allowing for profiles that are marginally stable over most of the plasma domain. However, the purpose of this numerical example is not to obtain the maximum possible $\beta aB/I$, but to show the general characteristics of its dependence on q_0, according to theory.

As a final comment I note that the obtained thresholds correspond to the upper boundary of the first stability region for which the q_0-modified limit $40\pi q_0 \beta a B_0/I \lesssim 3$ holds. Because of the large aspect ratio, circular geometry and the types of profiles under consideration, these equilibria never access the second stability regime for any value of q_0. This allows a clear demonstration of my form of the β limit (2) as it makes it possible to investigate the first stability limit at large q_0 without having to contend with the effects of a possible access to the second stability region.

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