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Instability of self-focused optical beams in plasmas

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This paper investigates the stability of intense optical slab beams in infinite plasmas. Profiles depending on the distance x were found in a recent paper by Kurki-Suonio, Morrison, and Tajima [Phys. Rev. A **40**, 3230 (1989)]. The most useful profiles were in the form of solitons. Solutions for which the beam profile was of oscillatory character were also found. Here a stability analysis is performed on all these profiles. The results, in terms of the physical cylindrical variables, imply radial stability but azimuthal instability of the solitons. The wave structures, however, are also unstable in the radial direction, and this explains numerical results of Kurki-Suonio *et al.*

I. INTRODUCTION

In a recent paper, Kurki-Suonio, Morrison, and Tajima[1] discussed the self-focusing of an intense optical beam by a plasma due to nonlinear (ponderomotive) and relativistic effects in what is known as a plasma fiber accelerator. In slab geometry, they obtained an exact steady-state solution in the form of a strongly peaked, localized solitary wave (peaked in the radial direction). Radial peaking has also been considered by other authors.[2–5] This is desirable, as it permits the beam to propagate without losing its intensity.

Using purely numerical methods, Kurki-Suonio, Morrison, and Tajima[1] showed that the soliton solution was stable to one-dimensional parallel perturbations in the slab model (in this model operators in r,θ are replaced by simpler operators in x,y). From this stability property the authors conclude that their solution is a realistic physical candidate for the asymptotic shape of a self-focused laser beam.

In the present paper we show analytically that the solitary-wave profile is indeed stable to parallel perturbations in the slab model. It is also shown in Ref. 1 that in slab geometry there also exist solutions in the form of nonlinear wave structures. Here we find that these structures are unstable. In practical terms this implies that deviations from the exact soliton profile should lead to radial instabilities. This was indeed found numerically in Ref. 1, where a Gaussian profile was investigated and the growth of perturbations was observed.

The present stability analysis thus gives further indication that the solitary wave is the relevant asymptotic state. The relevance of a *transverse* instability found here will of course depend on the relevance of the slab approximation. It should, however, be made known to those interested in practical applications of the original idea of radial peaking of intense laser light in plasmas. In real life, azimuthal instabilities are a possibility. They merit further study.

II. STABILITY ANALYSIS

The system under study is governed by Eqs. (9) and (10) of Ref. 1. To obtain these equations the laser plasma system was described by taking Maxwell's equations and the equation of motion for relativistic electrons. The electron pressure gradient was neglected in comparison with the ponderomotive force. Electrons were assumed cold, and the ions, immobile. The Lorentz gauge should have been taken (*not* the Coulomb gauge, as in fact **A** obtained by the authors of Ref. 1 is not divergence free). In Ref. 1 a trial function for the nonvanishing component of the normalized vector potential **A** was assumed to be

$$a_n(\mathbf{r}\cdot t)\{\exp i[k_0 z-\omega_0 t-\phi(\mathbf{r}\cdot t)]\}(\mathbf{x}+i\mathbf{y}) .$$

Here z is the direction of propagation of the laser beam in the plasma. We now rewrite Eqs. (9) and (10) of Ref 1:

$$\frac{\partial^2 a_n}{\partial t^2}=a_n\left[\omega_0+\frac{\partial\phi}{\partial t}\right]^2+c^2\nabla^2 a_n -c^2 a_n\left[\left[k_0-\frac{\partial\phi}{\partial z}\right]^2+|\nabla_\perp\phi|^2\right] -\omega_p^2 N_e a_n(1+a_n^2)^{-1/2} , \tag{2.1}$$

$$\frac{\partial}{\partial t}\left[a_n^2\frac{\partial\phi}{\partial t}\right]=-\omega_0\frac{\partial a_n^2}{\partial t}-c^2k_0\frac{\partial a_n^2}{\partial z}+c^2(\nabla a_n^2)\cdot(\nabla\phi) +c^2a_n^2\nabla^2\phi . \tag{2.2}$$

Here $\omega_p^2=4\pi ne^2/m$ and $N_e=1+\lambda_c^2\nabla(1+a_n^2)^{1/2}$, $\lambda_c=c/\omega_p$. We will now limit the class of solutions still further.

With $\phi=\rho z+\phi_0(r,\theta,t)$ and $a_n=a(r,\theta,t)$, and rescaling r by λ_c^{-1}, t by ω_p, ω_0 by ω_p, and introducing (ρ is the coefficient of z in ϕ)

$$c_1=\lambda_c^2(k_0-\rho)^2+(\omega_0/\omega_p)^2 ,$$

the above equations take the form

$$\frac{\partial^2 a}{\partial t^2} = -c_1 a + 2\omega_0 a \frac{\partial \phi}{\partial t} + a \left[\frac{\partial \phi}{\partial t} \right]^2 + \nabla_1^2 a - a(1+a^2)^{-1/2}[1+\nabla^2(1+a^2)^{1/2}] - a\nabla_1^2\phi^2 , \quad (2.1')$$

$$\frac{\partial}{\partial t}\left[a^2 \frac{\partial \phi}{\partial t} \right] = -\omega_0 \frac{\partial a^2}{\partial t} + (\nabla_1 a^2)\cdot(\nabla_1 \phi) + a^2 \nabla_1^2 \phi \quad (2.2')$$

The static case of interest in *slab* geometry was such that both a_0 and ϕ_0 were functions of x only. Now (2.2′) reduces to

$$\frac{d}{dx}\left[a_0^2 \frac{d\phi_0}{dx} \right] = 0, \quad \frac{d\phi_0}{dx} = \frac{\text{const}}{a_0^2} . \quad (2.3)$$

Here we will concentrate on the special case of const=0. For nonzero constant only nonlinear waves exist and they are expected to be unstable just as those with zero constant will be seen to be. Now (2.1′) becomes (we drop the zero subscript in a_0)

$$d^2a/dx^2 - a(1+a^2)^{-1/2} d/dx[a(1+a^2)^{-1/2} da/dx] - Q = 0 , \quad (2.4)$$

$$Q = c_1 a + a/(1+a^2)^{1/2} . \quad (2.5)$$

This equation can be integrated once to give

$$(da/dx)^2/(1+a^2) - 2(1+a^2)^{1/2} - c_1 a^2 = 2\epsilon . \quad (2.6)$$

The soliton is obtained for $\epsilon = -1$, all other physically meaningful solutions being in the form of nonlinear waves when $c_1 < 0$. For $\epsilon = -1$, both a and da/dx approach zero as x tends to infinity.

The method of stability analysis used is that introduced by Infeld and Rowlands.[6–10] Here the soliton is considered as a limit of nonlinear wave functions. This procedure gives information about *all* structures and, importantly, eliminates mathematical inconsistencies that direct soliton-perturbation analyses suffer from.[10] Here we will omit the theory behind the calculation, as it can be found in Ref. 10.

We take

$$a(x,y,t) = a_0(x) + \delta a(x) e^{\lambda t + ikx + ily} , \quad (2.7)$$

$$\phi(x,y,t) = \delta\phi(x) e^{\lambda t + ikx + ily} , \quad (2.8)$$

and linearized (2.1′) and (2.2′). We then expand in k and l, considered to be of the same order (for definiteness we expand in k). Here λ is assumed to be at least of order 1 and λ_1^2 positive means instability. We take

$$\delta a = \delta a_0 + k\delta a_1 + k^2 \delta A_2 + \cdots ,$$

$$\delta\phi + \delta\phi_0 + k\delta\phi_1 + \cdots ,$$

$$\lambda = k\lambda_1 + k^2\lambda_2 + \cdots .$$

The linear limit, of little relevance to our problem, but a useful check on the rather heavy calculations for fully nonlinear structures, is investigated by simple algebra. Equation (2.6) is solved by $\epsilon = (c_1 + c_1^{-1})/2$ and $a_0^2 = c_1^{-2} - 1$, $\phi_0 = 0$. When these identities and the forms (2.7) and (2.8) are substituted into the linearized forms of (2.1′) and (2.2′) (neglecting δa^2), we obtain the small k,l dispersion relation

$$\lambda^2 = \frac{|c_1| - |c_1|^3}{4\omega_0^2 - |c_1| + |c_1|^3}(k^2 + l^2) + O(k^4) , \quad (2.9)$$

and this is positive for realistic ω_0 (we will take $\omega_0^2 = 10$, whereas $c_1 < 1$). Thus one root $\lambda > 0$, and we have instability.

When *the full calculation* is performed for arbitrary periodic $a(x)$ from (2.6), we find that $\delta a = da/dx$, $\delta\phi$=const, and, in second order k, the dispersion relation follows in the form of a biquadratic in $\lambda_1(k,l,c_1,\omega_0,\epsilon)$:

$$A\lambda_l^4 + B\lambda_1^2 + C = 0 , \quad (2.10)$$

where

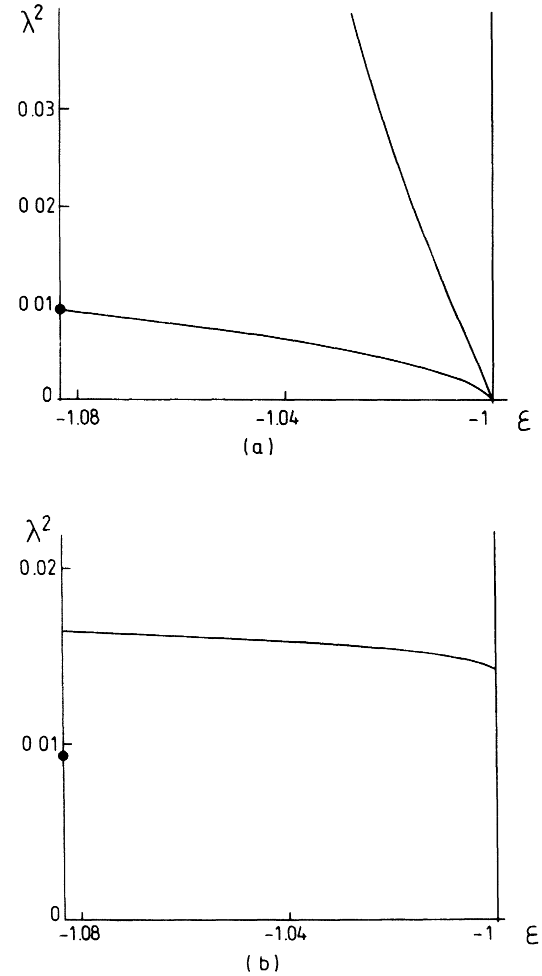

FIG. 1. Value of λ_1^2 for $\omega_0^2 = 10$ and $c_1 = -\frac{2}{3}$. Here the constant ϵ values vary from the linear limit (-1.085) to the soliton limit (-1). In (a) $l=0$, parallel perturbations. In (b) $k=0$, perpendicular perturbations (there is also a stable branch $\lambda^2 < 0$ not shown here). The linear limit (2.9) is indicated by a solid circle in both parts.

$$A = DE, \quad B = DF + EG + H^2, \quad C = FG \ ,$$
$$D = \langle a^2 \rangle + \omega_0^2(\gamma - \beta^2/\alpha) \ ,$$
$$E = \langle a_x^2 \rangle + \omega_0^2(\langle a^2 \rangle - \langle a^{-2} \rangle^{-1}) \ ,$$
$$F = k^2/\alpha + \langle a_x^2 \rangle/(1+a^2)l^2, \quad G = \langle a^{-2} \rangle^{-1}k^2 + \langle a^2 \rangle l^2 \ ,$$
$$H = \omega_0(\beta/\alpha - \langle a^{-2} \rangle^{-1}) \ ,$$

and

$$a_x \int_0^x a^n a_x^{-2} dx = \alpha_n a_x x + \cdots \ ,$$
$$\alpha = \alpha_0 + \alpha_2, \quad \beta = \alpha_2 + \alpha_4, \quad \gamma = \alpha_4 + \alpha_6 \ ,$$
$$\langle f \rangle = \oint f\, dx \Big/ \oint dx \ ,$$

where the dots represent terms periodic in x. Thus the α_n are the secular contributions of the integrals defined above.

The results are presented graphically in Fig. 1. We find by manipulating the discriminant that λ_1^2 is always real. The parallel perturbation case ($l=0$) is unstable for all but the soliton-limit structure. For $k=0$, however, the instability does not go away in the soliton limit. The curious fact that the linear limit is continuous for $l=0$ but not for $k=0$ has been observed in previous calculations.[11]

The most important practical conclusion of this calculation is that azimuthal perturbations to the soliton structure will be expected to grow. However, this is not a certainty, as we used the slab approximation to obtain them.

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