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Higher-n squeezed states and factorization of the squeezing operator

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It is shown that $n>0$ simple-harmonic-oscillator squeezed states have the same time-evolution behavior as the fundamental ($n=0$) squeezed state. That is, the probability density $|\psi(x,t)|^2$ retains its original shape as time progresses, except that the scales of the ordinate and abscissa oscillate in a reciprocal manner with frequency 2ω. This allows the time-evolved squeezing operator $\hat{S}_H(\lambda,t)$ to be written as a product of simple factors. Moreover, if a thermal distribution with density matrix $\hat{\rho}$ is squeezed, then the mixed squeezed state with minimum free energy has a density matrix given by a unitary transformation $\hat{\rho}_s=\hat{S}(\lambda)\hat{\rho}\hat{S}^{\dagger}(\lambda)$, where λ is a suitable squeezing parameter.

I. INTRODUCTION

Squeezed quantum states of the harmonic oscillator have generated much interest in recent years due to their relevance to various experimental programs of a fundamental nature. For example, in gravity wave detectors they would permit operation at a sensitivity beyond the standard quantum limit,[1] and in optics they have been observed experimentally both for visible light,[2] and for a microwave cavity.[3] Any quantum state can be squeezed, but in usual parlance the term "squeezed state" refers to the minimum-uncertainty $n=0$ (or "vacuum") state, where n represents the principle quantum number of a harmonic oscillator. However, since the work of Ref. 3 was done at elevated temperatures, the higher-n states of the cavity were thermally excited, and therefore one should not arbitrarily exclude higher-n states from the discussion at hand.

It is only recently that squeezed higher-n states have been analyzed. Kim, de Oliveira, and Knight[4,5] have studied the photon statistics (that is, distributions in n) of these states, and have found oscillations which have analogs in the squeezed $n=0$ case. In this article, I wish to consider another aspect of the problem, i.e., the time evolution of a squeezed arbitrary-n state, and show that the squeezed higher-n states have the same time evolution characteristics as the squeezed $n=0$ state. In particular, if $\hat{x}$ is the canonical coordinate of the problem, and $\hat{p}$ its conjugate momentum, then for fixed n the probability distributions $|\psi(x,t)|^2$ and $|\psi(p,t)|^2$ retain their shape as time progresses except that the scales of their ordinates and abscissas (i.e., their "heights" and "widths") oscillate in a reciprocal manner with frequency 2ω, where ω is the characteristic frequency of the oscillator. While it is a general property of simple-harmonic-oscillator wave functions that their probability distributions are periodic in time, this represents a special case where the higher-n members of a family of wave functions have the same behavior as the $n=0$ member. Similar results have been seen before in the case of displaced states, and the underlying reason is the same. Namely, the relevant transformation operator (the squeezing operator in this case), when allowed to evolve in time, can be written as a product of simple factors of which one is the same operator with a time-dependent argument. This factorization will be given explicitly in Sec. II.

Kim and co-workers have also studied squeezed thermal distributions by applying a single squeezing factor to the thermal equilibrium density matrix. In Sec. III, I consider the justification for such a step.

II. TIME EVOLUTION OF A SQUEEZED HIGHER-n STATE

The Hamiltonian for a harmonic oscillator is $\hat{H}=(\hat{n}+1/2)\hbar\omega$, with eigenstates $|n\rangle$, where $\hat{n}|n\rangle=n|n\rangle$ and $n=0,1,2,3,\ldots$. In the coordinate representation these eigenstates are represented by eigenfunctions $u_n(x)=\langle x|n\rangle$, where I follow the notation of Schiff:[6]

$$u_n(x)=\left[\frac{\alpha}{\sqrt{\pi}2^n n!}\right]^{1/2} e^{-(\alpha x)^2/2}H_n(\alpha x)\ , \tag{1a}$$

where the H_n's are Hermite polynomials, and α is a measure of quantum fluctuations. (For a mass m connected to a spring, $\alpha=\sqrt{m\omega/\hbar}$.) In the momentum representation these functions are

$$u_n(p)=\langle p|n\rangle=(-i)^n\left[\frac{\beta}{\sqrt{\pi}2^n n!}\right]^{1/2} e^{-(\beta p)^2/2}H_n(\beta p)\ , \tag{1b}$$

where $\beta=1/\hbar\alpha$. I have used δ-function normalization for both $|x\rangle$ and $|p\rangle$ so that $\langle x|x'\rangle=\delta(x-x')$, $\langle p|p'\rangle=\delta(p-p')$, and $\langle x|p\rangle=(2\pi\hbar)^{-1/2}e^{ipx/\hbar}$.

It is well known that if a wave function at time $t=0$ is a displaced ground-state eigenfunction, i.e., $\psi_0^{(a)}(x,0)=u_0(x-a)$, then as time evolves $|\psi_0^{(a)}(x,t)|^2$ simply oscillates about the origin with unchanging shape at frequency ω and amplitude a. This result was first shown by Schrödinger.[7] Moreover, the momentum probability distribution $|\psi_0^{(a)}(p,t)|^2$ behaves similarly except that its phase of oscillation differs by 90°. What is not so widely appreciated is that the same holds true for *any* displaced eigenstate $\psi_n^{(a)}(x,0)=u_n(x-a)$. Thus

$$|\psi_n^{(a)}(x,t)|^2=|u_n(x-a\cos\omega t)|^2 ,$$

and

$$|\psi_n^{(a)}(p,t)|^2=|u_n(p+\hbar\alpha^2 a\sin\omega t)|^2$$

for any n. (This was demonstrated by Senitzky and others.[8])

The displaced ground state has achieved such a prominent status in the literature that it has been given its own name: *the coherent state.* It is also an eigenstate of the annihilation operator $\hat{a}$, which helps to define its formal coherence properties.[9] However, in an informal sense (i.e., in terms of maintaining the shape of probability distributions), we see that the displaced higher-n eigenstates are just as "coherent" as the coherent state, only that their uncertainty products $\Delta x\,\Delta p$ are larger. A similar result will be found for squeezed states, for which the properties of the $n=0$ case have already been widely discussed in the literature.[10–13]

A quantum harmonic oscillator can mimic a classical one in the following way. In classical mechanics, the oscillator is represented as a single point in phase space. If the phase-space Cartesian axes are taken as x and $p/\hbar\alpha^2$, then the system point follows a circular orbit about the origin with a fixed radius and frequency ω. (Note that because α is a measure of quantum fluctuations, it is related to Planck's constant and varies as $\hbar^{-1/2}$. In fact, the product $\hbar\alpha^2$ equals $m\omega$, where m is the equivalent inertia of the oscillator system, and being independent of $\hbar$, is therefore a perfectly good classical quantity.) In quantum mechanics the point is expanded to an error region whose size is governed by the uncertainty principle.[1,10] Nevertheless, this error region still orbits the phase-space origin at frequency ω with its center at a fixed radius equal to the amplitude of the sinusoidal displacement.

One can now transform to a rotating coordinate system in phase space by defining operators $\hat{X}_1$ and $\hat{X}_2$ in the Schrödinger picture to be[1,10]

$$\hat{X}_1=\hat{x}\cos\omega t-\frac{\hat{p}}{\hbar\alpha^2}\sin\omega t , \tag{2a}$$

$$\hat{X}_2=\hat{x}\sin\omega t+\frac{\hat{p}}{\hbar\alpha^2}\cos\omega t , \tag{2b}$$

$$[\hat{X}_1,\hat{X}_2]=i/\alpha^2 . \tag{2c}$$

The uncertainty principle dictates that $\Delta X_1\Delta X_2\geq 1/2\alpha^2$, and in the case of the displaced nth eigenstate, $\Delta X_1=\Delta X_2=(n+1/2)^{1/2}/\alpha$. With X_1 and X_2 as new phase-space axes, the error region merely sits at a particular point in this new coordinate frame.

To generate squeezed states I will make use of the squeezing operator $\hat{s}(\lambda)$ which is defined by[10–13]

$$\hat{S}(\lambda)\equiv\exp[i\lambda(\hat{x}\hat{p}+\hat{p}\hat{x})/2] \tag{3}$$

and has the properties

$$\hat{S}(\lambda)f(\hat{x})\hat{S}^{\dagger}(\lambda)=f(e^{\lambda}\hat{x}), \tag{4a}$$

and

$$\hat{S}(\lambda)f(\hat{p})\hat{S}^{\dagger}(\lambda)=f(e^{-\lambda}\hat{p}) . \tag{4b}$$

In a particular representation wave functions are given by

$$\psi^{(\lambda)}(x)=\langle x|\psi^{(\lambda)}\rangle\equiv\langle x|\hat{S}(\lambda)|\psi\rangle=e^{\lambda/2}\psi(xe^{\lambda}) , \tag{5a}$$

$$\psi^{(\lambda)}(p)=\langle p|\psi^{(\lambda)}\rangle\equiv\langle p|\hat{S}(\lambda)|\psi\rangle=e^{-\lambda/2}\psi(pe^{-\lambda}) . \tag{5b}$$

Notice that $\hat{S}(\lambda)$ is unitary so that $\hat{S}^{-1}(\lambda)=\hat{S}^{\dagger}(\lambda)=\hat{S}(-\lambda)$. The squeezing parameter λ is taken as a real number, is related to the squeezing parameter Σ of Ref. 3 by $\Sigma=e^{2\lambda}$, and equals the parameter r of Ref. 4. The phase-space error region remains constant in area, though not in shape, during the squeezing operation.

We start at time $t=0$ with a squeezed state $\hat{S}(\lambda)|n\rangle$ and allow it to evolve with time:

$$|\psi_n^{(\lambda)}(0)\rangle=\hat{S}(\lambda)|n\rangle \tag{6a}$$

or

$$\psi_n^{(\lambda)}(x,0)=\left[\frac{\alpha e^{\lambda}}{\sqrt{\pi}2^n n!}\right]^{1/2}e^{-(\alpha e^{\lambda}x)^2/2}H_n(\alpha e^{\lambda}x) . \tag{6b}$$

We have

$$|\psi_n^{(\lambda)}(t)\rangle=e^{-i\hat{H}t/\hbar}\hat{S}(\lambda)|n\rangle \tag{7a}$$

$$=\hat{S}_H(\lambda,-t)|n\rangle e^{-i(n+1/2)\omega t} , \tag{7b}$$

where

$$\hat{S}_H(\lambda,-t)=e^{-i\hat{H}t/\hbar}\hat{S}(\lambda)e^{i\hat{H}t/\hbar}$$

$$=\exp[i\alpha^2\lambda(\hat{X}_1\hat{X}_2+\hat{X}_2\hat{X}_1)/2] . \tag{8}$$

(Although $\hat{S}_H$ has the form of an operator in the Heisenberg picture, we are still using the Schrödinger picture; hence the minus sign for t.)

In terms of Δx and Δp, the uncertainty product is time dependent for $|\psi_n^{(\lambda)}(t)\rangle$:

$$\Delta x\,\Delta p=(n+1/2)\{1+[\sinh(2\lambda)\sin(2\omega t)]^2\} . \tag{9}$$

However, the only time dependence in Eq. (8) is the implicit time dependence of $\hat{X}_1$ and $\hat{X}_2$. Therefore, in terms of ΔX_1 and ΔX_2 the uncertainty product is constant:

$$\Delta X_1=e^{-\lambda}\sqrt{n+1/2}/\alpha , \tag{10a}$$

$$\Delta X_2=e^{\lambda}\sqrt{n+1/2}/\alpha , \tag{10b}$$

$$\Delta X_1\Delta X_2=(n+1/2)/\alpha^2 . \tag{10c}$$

Given the commutation relation in Eq. (2c), the similarity

in form between Eqs. (8) and (3) shows that what we have done is to squeeze along the X_1 axis and stretch along the X_2 axis preserving the area of the error region.

It remains to get the explicit time dependence in either the coordinate or momentum representation. I choose to start with Eq. (6b) and use the Green's function in the coordinate representation:

$$\psi_n^{(\lambda)}(x,t)=\int_{-\infty}^{\infty} G(x,x',t)\psi_n^{(\lambda)}(x',0)dx' , \tag{11a}$$

where[14]

$$G(x,x',t)=\left[\frac{\alpha^2}{2\pi i\sin\omega t}\right]^{1/2}\exp\left[\frac{-\alpha^2\{[x^2+x'^2](\cos\omega t)-2xx'\}}{2i\sin\omega t}\right] . \tag{11b}$$

The integration in Eq. (11) can be performed in a straightforward manner by making use of the series expansion for H_n.[15] The result is

$$\psi_n^{(\lambda)}(x,t)=\exp[i(\alpha' x)^2\sinh(2\lambda)\sin(2\omega t)/2]\exp[-i(n+1/2)\phi_\lambda(t)]\left[\frac{\alpha'}{\sqrt{\pi}2^n n!}\right]^{1/2} e^{-(\alpha' x)^2/2}H_n(\alpha' x) . \tag{12a}$$

Moreover, we can acquire the momentum representation merely by making the substitution $x\rightarrow p$, $\alpha\rightarrow\beta$, and $\lambda\rightarrow-\lambda$. We get

$$\psi_n^{(\lambda)}(p,t)=\exp[-i(\beta' p)^2\sinh(2\lambda)\sin(2\omega t)/2]\exp[-i(n+1/2)\phi_{-\lambda}(t)](-i)^n\left[\frac{\beta'}{\sqrt{\pi}2^n n!}\right]^{1/2} e^{-(\beta' p)^2/2}H_n(\beta' p) , \tag{12b}$$

where

$$\alpha'=\alpha e^{\lambda_-(t)} , \tag{13a}$$

$$\beta'=\beta e^{-\lambda_+(t)} , \tag{13b}$$

$$\phi_\lambda(t)=\tan^{-1}(e^{2\lambda}\tan\omega t) , \tag{13c}$$

$$\lambda_\pm(t)=\pm\tfrac{1}{2}\ln(e^{\pm 2\lambda}\cos^2\omega t+e^{\mp 2\lambda}\sin^2\omega t) . \tag{13d}$$

This generalizes the $n=0$ case[14] and gives the desired result. We see that

$$|\psi_n^{(\lambda)}(x,t)|^2=e^{\lambda_-(t)}|u_n(xe^{\lambda_-(t)})|^2 , \tag{14a}$$

$$|\psi_n^{(\lambda)}(p,t)|^2=e^{-\lambda_+(t)}|u_n(pe^{-\lambda_+(t)})|^2 , \tag{14b}$$

where the only time dependence in Eq. (14) is found in the scale factors which define the ordinates and abscissas of the probability distributions.

The time-dependent squeezing operator can be written in a number of equivalent ways, including the normal ordered form of Ref. 11. Which way is best depends on its convenience to the problem at hand. In that respect, Eqs. (12) and (13) give rise to a product of simple factors as follows:

$$e^{-i\hat{H}t/\hbar}\hat{S}(\lambda)=\hat{S}\left[\lambda_-(t)\right]\exp[i(\alpha\hat{x})^2\sinh(2\lambda)\sin(2\omega t)/2]\times\exp[-i(\hat{n}+1/2)\phi_\lambda(t)] , \tag{15a}$$

$$=\hat{S}\left[\lambda_+(t)\right]\exp[-i(\beta\hat{p})^2\sinh(2\lambda)\sin(2\omega t)/2]\times\exp[-i(\hat{n}+1/2)\phi_{-\lambda}(t)] . \tag{15b}$$

[Here I have used Eqs. (4a) and (4b) to switch the order of $\hat{S}(\lambda_\pm)$ and functions of $\hat{x}$ or $\hat{p}$.] Alternately, we can change the sign of t in Eq. (15) to obtain a factored expression for $\hat{S}_H$ in the Heisenberg picture:

$$\hat{S}_H(\lambda,t)=\hat{S}\left[\lambda_-(t)\right]\exp[-i(\alpha\hat{x})^2\sinh(2\lambda)\sin(2\omega t)/2]\times\exp[i(\hat{n}+1/2)(\phi_\lambda(t)-\omega t)] , \tag{16a}$$

$$=\hat{S}\left[\lambda_+(t)\right]\exp[i(\beta\hat{p})^2\sinh(2\lambda)\sin(2\omega t)/2] ,$$
$$\times\exp[i(\hat{n}+1/2)(\phi_{-\lambda}(t)-\omega t)] . \tag{16b}$$

It should be noted that the free-particle limit can be obtained by letting $\omega\rightarrow 0$ and $\alpha\rightarrow 0$ such that $\alpha^2/\omega=m/\hbar=$const. Either form of Eq. (15) gives

$$e^{-i\hat{p}^2t/2m\hbar}\hat{S}(\lambda)=\hat{S}(\lambda)e^{-i(\hat{p}e^{\lambda})^2t/2m\hbar} , \tag{17}$$

in agreement with Eq. (4b).

III. SQUEEZED THERMAL DISTRIBUTIONS

Since experimental realizations of squeezed oscillator systems already exist[2,3] or are being planned,[1] we must come to grips with the fact that in the laboratory, an unsqueezed oscillator will never be found in its pure ground state $|0\rangle$. At best, it will be in thermal equilibrium at some temperature $T>0$, which in many cases will be much greater than $\hbar\omega/k$ where k is the Boltzmann's constant. Accordingly, many of the higher-n states will be thermally excited, so the idealized analyses in the literature must be extended to include higher-n squeezed states as well. It should be noted, in fact, that the authors of Ref. 3 made no reference to quantum mechanics in their own data analysis, and presented what amounts to a classical Maxwell-Boltzmann description of their squeezed oscillator. It was to help bridge this gap between the theoretical idealization of an unsqueezed oscillator in a pure state $|0\rangle$ (or perhaps a displaced ground state) and one represented by a thermal distribution that the development of Sec. II was undertaken.

In a thermal distribution, the system is described by a density matrix (unsqueezed) given in operator form as

$$\hat{\rho}=\sum_n \rho_n|n\rangle\langle n| , \tag{18}$$

where $\rho_n=e^{-n\hbar\omega/kT}(1-e^{-\hbar\omega/kT})$. In fact, we can generalize this treatment and make it hold for any density operator which is diagonal in the n representation. In such a case, the expectation value of an operator $\hat{A}$ is given by

$$\langle A\rangle=\mathrm{Tr}(\hat{\rho}\hat{A})=\sum_n \rho_n\langle n|\hat{A}|n\rangle \text{ (unsqueezed)} . \tag{19}$$

Kim, de Oliviera, and Knight[4,5] have considered

squeezed thermal distributions by applying a single squeezing factor to the distribution in Eq. (18). That is, $\hat{\rho} \rightarrow \hat{S}\hat{\rho}\hat{S}^{\dagger}$. I now wish to examine the justification for this approach, since, in squeezing such a distribution, we must allow for the fact that because of a lack of detailed control over the squeezing agent, the various unsqueezed states $|n\rangle$ will likely be squeezed by differing amounts. Thus $|n\rangle \rightarrow |\psi_n^{(\lambda_n)}\rangle = \hat{S}(\lambda_n)|n\rangle$; $|n'\rangle \rightarrow |\psi_{n'}^{(\lambda_{n'})}\rangle = \hat{S}(\lambda_{n'})|n'\rangle$, and $\lambda_n \neq \lambda_{n'}$, in general. The squeezed density operator is now

$$\hat{\rho}_s = \sum_n \rho_n |\psi_n^{(\lambda_n)}\rangle\langle\psi_n^{(\lambda_n)}| \tag{20a}$$

$$= \sum_n \rho_n \hat{S}(\lambda_n)|n\rangle\langle n|\hat{S}^{\dagger}(\lambda_n) \quad \text{(squeezed)} \tag{20b}$$

and the expectation value for $\hat{A}$ becomes

$$\langle A\rangle = \mathrm{Tr}(\hat{\rho}_s \hat{A}) = \sum_n \rho_n \langle\psi_n^{(\lambda_n)}|\hat{A}|\psi_n^{(\lambda_n)}\rangle \quad \text{(squeezed)} . \tag{21}$$

Since we are considering (as yet) undisplaced states, we have $\langle X_1\rangle = \langle X_2\rangle = 0$, $\Delta X_1^2 = \langle X_1^2\rangle$, and $\Delta X_2^2 = \langle X_2^2\rangle$. From Eqs. (10) and (21) we get

$$\Delta X_{1,2}^2 = \alpha^{-2} \sum_n \rho_n (n+1/2) e^{\mp 2\lambda_n} . \tag{22}$$

In terms of $\hat{X}_1$ and $\hat{X}_2$ the Hamiltonian is

$$\hat{H} = (\alpha^2 \hbar\omega/2)(\hat{X}_1^2 + \hat{X}_2^2) , \tag{23}$$

giving an energy of

$$E = \mathrm{Tr}\left[\hat{\rho}_s \hat{H}\right] = \hbar\omega \sum_n \rho_n (n+1/2)\cosh(2\lambda_n) \tag{24a}$$

$$= (\alpha^2 \hbar\omega/2)\left[\Delta X_1^2 + \Delta X_2^2\right] . \tag{24b}$$

A weaker form of the uncertainty relation now follows from Eq. (22). We have

$$\alpha^4 \Delta X_1^2 \Delta X_2^2 = \sum_{n,n'} \rho_n \rho_{n'} (n+1/2)(n'+1/2) e^{2(\lambda_n - \lambda_{n'})} \tag{25a}$$

$$= \sum_n \left[\rho_n^2 (n+1/2)^2 + 2 \sum_{n'(<n)} \rho_n \rho_{n'} (n+1/2)(n'+1/2)\cosh 2(\lambda_n - \lambda_{n'})\right] , \tag{25b}$$

or

$$\Delta X_1 \Delta X_2 \geq (\bar{n}+1/2)/\alpha^2 , \tag{26}$$

where

$$\bar{n} = \sum_n n\rho_n , \tag{27}$$

in agreement with Refs. 4 and 5. Note that $\bar{n}$ and $\langle n\rangle$ are different quantities. $\langle n\rangle$, i.e., the expectation value of $\hat{n}$, is given by Eq. (21) with $\hat{A} = \hat{n}$.

Since the experimental determination of an infinite set $\{\lambda_n\}$ would be a tedious enterprise, it would be useful to be able to focus only on those λ_n's which are most important. In this regard, an elementary thermodynamic argument (minimization of free energy) will achieve a great simplification. I am assuming that the squeezing apparatus leaves the ρ_n's unaffected (i.e., it does not alter the ambient temperature T). The entropy S (not to be confused with the squeezing operator $\hat{S}$), however, will be affected as can be seen by the following argument. An order parameter for mixed states is[16] $\mathrm{Tr}(\hat{\rho}^2)$. When $\mathrm{Tr}(\hat{\rho}^2) = 1$, the system is in a pure state, perfectly ordered, with zero entropy. When the entropy increases, $\mathrm{Tr}(\hat{\rho}^2)$ decreases, and always has a value less than unity for mixed states. We can use Eq. (20b) and the unitarity of $\hat{S}$ to compare the squeezed and unsqueezed distributions:

$$\mathrm{Tr}\left[\hat{\rho}_s^2\right] = \sum_n \rho_n^2 + \sum_{n \neq n'} \rho_n \rho_{n'} |\langle n'|\hat{S}(\lambda_n - \lambda_{n'})|n\rangle|^2 \tag{28a}$$

$$\geq \sum_n \rho_2^2 = \mathrm{Tr}\left[\hat{\rho}^2\right] . \tag{28b}$$

If we assume that of all possible states given by Eq. (20), the most probable state is the one with minimum free energy ($E - TS$), then we need to pick the distribution of λ_n's which minimizes both Eqs. (24a) and (28a) subject to the constraint that, say, ΔX_1 has a certain squeezed value. This occurs when all the λ_n's have a uniform value λ, giving

$$\hat{\rho}_s = \hat{S}(\lambda)\hat{\rho}\hat{S}^{\dagger}(\lambda) \quad \text{(minimum free energy)} \tag{29}$$

which is the desired result. The corresponding squeezed values are

$$\Delta X_{1,2}^2 = e^{\mp 2\lambda}(\bar{n}+1/2)/\alpha^2 \tag{30a}$$

and

$$E = \left[\bar{n}+1/2\right]\hbar\omega\cosh(2\lambda) \tag{30b}$$

with a "minimum value" uncertainty product

$$\Delta X_1 \Delta X_2 = (\bar{n}+1/2)/\alpha^2 \tag{30c}$$

which for a system in thermal equilibrium has $\bar{n}$ given by the Bose-Einstein distribution:

$$\bar{n} = (e^{\hbar\omega/kT} - 1)^{-1} . \tag{30d}$$

In many cases a system will be both squeezed and displaced. Again, the state of minimum free energy is given by a unitary transformation $\hat{\rho} \rightarrow \hat{D}\hat{S}\hat{\rho}\hat{S}^{\dagger}\hat{D}^{\dagger}$, where $\hat{D}$ is the displacement operator. Since $\hat{S}$ and $\hat{D}$ do not commute, one must be careful to either preserve the proper order of products, or apply a unitary transformation to $\hat{S}$ (of the form $\hat{D}\hat{S}\hat{D}^{\dagger}$) to squeeze about a displaced position.

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