Universality in the lattice-covering time problem

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The lattice-covering time t is the expected time a random walk (RW) takes to visit all N lattice sites. Regular D-dimensional lattices with periodic and reflecting boundary conditions are considered. When $D=1$ these covering problems are equivalent to those of the first-visit type and they can be exactly solved. In contrast, when $D \geq 2$ the lattice-covering time problems are not reducible to any known lattice RW problem. The asymptotic ($N \to \infty$) behavior of t is studied using Monte Carlo methods and interesting questions regarding universality in the covering time problem are discussed.

I. INTRODUCTION

The theory of random walks has attracted much theoretical attention over the last fifty years due to its numerous applications in the physical, biological, and even social sciences.[1] Among them one can mention the theory of classical diffusion, trapping models in solid-state physics, transport phenomena in disordered structures, polymer physics, models of motion of microrganisms, and theory of income distribution in the economy.

In a recent report[2] we introduced a simple lattice random-walk (RW) problem, namely the lattice-covering time problem. The lattice-covering time t is the mean time a RW takes to visit *all* sites of a finite D-dimensional lattice. In particular, we are interested in the asymptotic limit of large N, with N being the total number of lattice sites. Although the lattice-covering time problem resembles some of the classical RW problems such as first-passage time,[3] site occupancy,[4,5] trapping,[6] and the number of sites visited by an m-step walk,[4,7] in $D \geq 2$ *it cannot be reduced* to any known[3–8] lattice RW problem. In contrast, lattice-covering time problems in $D=1$ are equivalent to some first-visit problems.

One could be puzzled as to how a random walk of fractal dimension 2 can fill spaces of arbitrarily high dimensions. This dilemma is easily resolved since, by the definition of the problem, one begins with a finite lattice of N sites that is covered by a RW in finite time, then the mean time for covering the lattice is calculated, and finally the "thermodynamic" limit of $N \to \infty$ is taken.

A related problem has been discussed in the mathematics literature where a continuous RW of finite width ϵ was considered.[9] There, one is interested in the asymptotic ($\epsilon \to 0$) limit of the expected time for the walk to cover a sphere of unit radius in D ($D \geq 3$) dimensions. One of the motivations of this problem is a data-analysis technique known as the Grand Tour.[10] It consists of producing one-dimensional (1D) projections of a multidimensional data set obtained as the sequence of data points visited by a RW in the data space. The time needed by the walk to be within a small distance ϵ of every data point is relevant because it gives a measure of the quality of the data inspection. The lattice-covering time problem is a discrete version of the Grand Tour.

The Grand Tour reminds one of the ergodic (or quasiergodic) problem[11] in some simple systems[12] as the data space is replaced by, say, a constant energy hypersurface in the phase space and one asks about the mean time for a system with some particular dynamics [from a given site (state of the system), only nearest-neighbor points in the phase space are accesible, all with equal probability] to visit all possible configurations. This, in turn, suggests an application of the covering time problem to Monte Carlo analysis[13] if one is interested in the speed at which configurations are sampled once equilibrium has been reached, say, in a microcanonical ensemble. In this case, not all configurations (points in the phase space) are to be visited, but only some fraction of them. This corresponds to a dilute covering time problem which we are presently studying.

One could attempt to reduce the lattice-covering time problem to some of the known lattice RW problems discussed in the literature.[3–8] For example, it is well known that the number of distinct sites S_m visited by an m-step RW behaves as $S_m \sim m^{1/2}(D=1)$, $S_m \sim m/\ln m(D=2)$, and $S_m \sim m(D \geq 3)$. Then, to "derive" the asymptotic behavior of t one could argue that the number of distinct sites to be visited by the walk should be N, identifying the number of steps m with the mean covering time t. Thus, one would conclude that $t \sim N^2(D=1)$, $t \sim N\ln N(D=2)$, and $t \sim N(D \geq 3)$ which is only right at $D=1$.

In fact, there are important differences between the covering time and other lattice RW problems discussed in the literature. The former is defined on finite lattices with the walk forced to visit *all* N sites and the "thermodynamic limit" of $N \to \infty$ taken afterwards, thus requiring statistics of walks with the number of steps larger than or equal to N. In contrast, the latter problems consider walks of a fixed number of steps in unbounded spaces, and usually the sites to be visited are not previously selected. These differences manifest in the asymptotic behavior of t which, except for $D=1$, differs from predictions based upon analogies with the usual lattice RW problems.

In this paper we present an extended version of our recent letter. There, the lattice-covering time problem was introduced and some of the material of the present work was briefly reported. This work also includes additional results obtained after the completion of the letter such as a more complete study of the covering problem with rigid boundaries in $1 \leq D \leq 4$. Finally, an universality hypothesis for the lattice-covering problem is made. Then, the $2 \leq D \leq 4$ periodic and rigid data are analyzed in the light of this hypothesis, helping to distinguish among two proposed forms for the asymptotic behavior of t.

II. THE LATTICE-COVERING TIME PROBLEM ON A D-DIMENSIONAL TORUS

In $D=1$ the lattice-covering time t_p [the subscript p denotes periodic boundary conditions (BC's)] is the mean time spent by a lattice walker in visiting all sites of a ring as illustrated in Fig. 1(a). For small lattices the problem is resolved by direct enumeration. For example, for $N=3$ one has

$$t_p=(1/2)\times 2+(1/2^2)\times 3+(1/2^3)\times 4+\cdots=3\ ,$$

where the integer of each term in the series is the time (number of steps) to complete the walk, and the fraction in parenthesis gives the normalized probability of finishing the walk after this time. Similarly, for $N=4$ one finds

$$t_p=(1/2^2)\times 3+(1/2^3)\times 4+(3/2^4)\times 5+(3/2^5)\times 6 +(7/2^6)\times 7+(7/2^7)\times 8+(15/2^8)\times 9 +(15/2^9)\times 10+\cdots=6\ .$$

In this way one obtains $t=1,3,6,10,15,\ldots$, for $N=2,3,4,5,6,\ldots$, respectively. This sequence of *integer* numbers for t together with the Monte Carlo (MC) data for larger lattices (a sample of which is shown in Table I) indicates that

$$t_p=N(N-1)/2\ , \tag{1}$$

which can be proven[14] by reducing the covering problem on rings to a first-visit problem. It is also equivalent to the problem of visiting N distinct sites on a 1D infinite lattice. This is illustrated in Fig. 1(b).

As discussed in Sec. III and Ref. 14 the 1D rigid lattice-covering time problems are also equivalent to some first-passage-time and trapping problems. In contrast, when $D \geq 2$ the lattice-covering time problems cannot be reduced to any known lattice RW. In fact, there is an essential difference between $D=1$ and $D \geq 2$. In the one-dimensional case there is only one path joining any two points (two paths on rings) while in higher dimensions there are many ways to connect two sites (in fact, infinite ways as $N \rightarrow \infty$). Hence, in $D=1$ a walk is completed after the end sites are visited (for example, considering the rigid BC case, a RW which starts at $s=1$ completes when site $s=N$ has been reached). In contrast, there is no analog to this in $D \geq 2$. Thus, the constraint to visit all sites is much stronger in $D \geq 2$ than in $D=1$ so changing the nature of the problem.

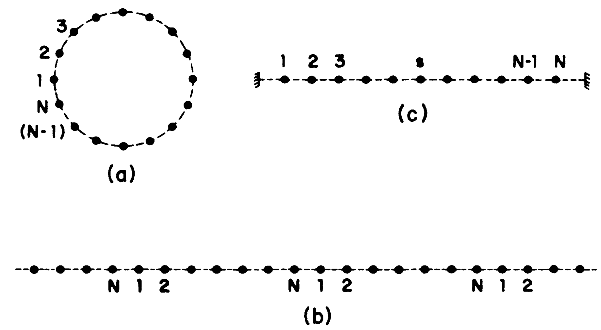

FIG. 1. (a) 1D periodic lattice (ring) of N sites. The expected time $\bar{t}(s)$ for a RW starting at site 1 to reach a site s is (Ref. 4) $\bar{t}(s)=(s-1)(N-s+1)$. (b) The lattice-covering time on rings is equivalent to the problem of visiting N distinct sites on a 1D infinite lattice. This is no longer true in $D \geq 2$. (c) A 1D lattice of N sites with rigid boundary conditions. If the walker is at site 1 (N) and attempts to move to the left (right) remains at the same site waiting one unit of time (reflecting BC).

We use MC methods to obtain the expected covering time for RW's on $(2 \leq D \leq 4)$ cubic lattices. Our data consist of several points (N,t) with N ranging from 10^2 to 10^5, each one obtained after averaging t over at least 500 simulations. Although we find that the power-law form

$$t_p \sim B_p N^{\alpha_p} \tag{2}$$

with $\alpha_p(D=2)\simeq 1.16$, $\alpha_p(D=3)\simeq\alpha_p(D=4)\simeq 1.09$ provides reasonable fits to the $2 \leq D \leq 4$ data as illustrated in Figs. 2 and 3, we think that this is not the right asymptotic behavior of t, which instead, we believe, is given by the forms

$$t_p \sim A_p N \ln^2 N,\quad D=2 \tag{3a}$$

$$t_p \sim A_p N \ln N,\quad D \geq 3\ . \tag{3b}$$

These forms, with the values of $A_p(D=2)=0.33$, $A_p(D=3)=1.55$, and $A_p(D=4)=1.30$, provide a somewhat better fit to our MC data as shown in Fig. 2 (3) for $D=2$ ($D=3$ and 4). In fact, the related work[9] of the

TABLE I. The error in t_p is estimated using the second moment of the t distribution.

N	t_p	$N(N-1)/2$	N	t_p	$N(N-1)/2$
5	10.08 ± 0.09	10	60	$(17.57\pm0.61)\times10^2$	1770
6	14.94 ± 0.13	15	120	$(71.29\pm1.85)\times10^2$	7140
8	27.80 ± 0.24	28	250	$(31.34\pm0.85)\times10^3$	31 125
10	45.02 ± 0.39	45	500	$(12.80\pm0.35)\times10^4$	124 750
12	66.31 ± 0.59	66	1000	$(49.25\pm1.68)\times10^4$	499 500

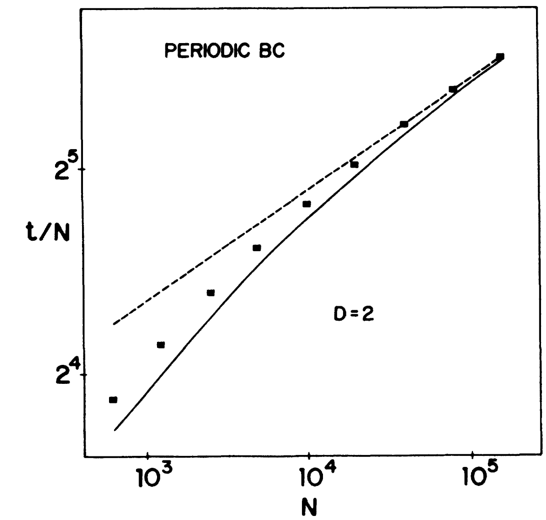

FIG. 2. ln-ln plot of (t/N) vs the total number of sites N for square lattices with periodic boundary conditions. The dashed line is a power-law fit $t \sim N^{\alpha_p}$ with $\alpha_p \simeq 1.16$ while the fit $t \sim A_p N \ln^2 N$ with $A_p = 0.33$ is shown by a solid line. For smaller values of N ($N \lesssim 10^4$) each point is the result of averaging 1000 simulations while for bigger lattices we average 500 simulations. Error bars representing the uncertainties in t due to the finite size of the MC ensembles (estimated from the second moment of the t distribution) would be about the size of the little squares.

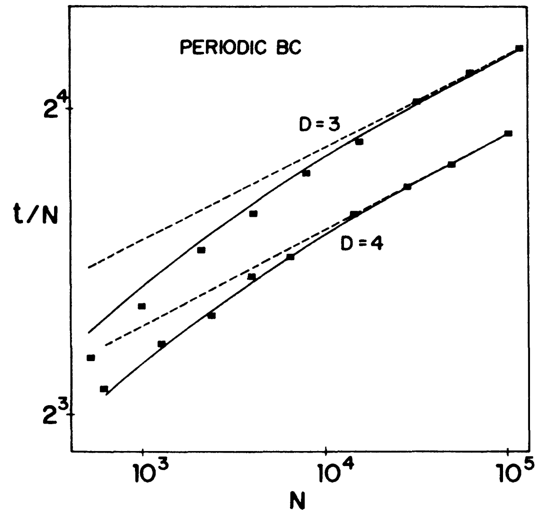

FIG. 3. Same as Fig. 2 for cubic and 4D-hypercubic lattices with periodic boundary conditions. In $D=3$ and 4, the power-law fits with $\alpha_p(3D) \simeq \alpha_p(4D) = 1.09$ are indicated by dashed lines, and the form $A_p N \ln N$, with $A_p(3D) = 1.55$ and $A_p(4D) = 1.30$, by solid lines.

mathematician strongly suggests logarithmic divergences. Moreover, problems involving random walks of finite variance on regular lattices have leading divergences in the form of integer or half-integer powers of N multiplied, in some cases, by logarithms (or integer powers of logarithms). In addition, in Sec. III we investigate the lattice-covering time with rigid boundaries and, as a by-product of our analysis, a further argument in favor of the logarithmic forms is obtained by making an universality hypothesis.

III. THE LATTICE-COVERING TIME PROBLEM WITH RIGID BOUNDARIES

The one-dimensional problem illustrated in Fig. 1(c) is the following. Given a lattice of N sites, the walk starts at either edge, sites $s=1$ or N, and the mean time t_R (where R stands for rigid BC's) to cover the lattice is calculated with reflecting boundary conditions, that is, every time the walker is at site $1(N)$ and attempts to step to the left (right), it remains at the same site waiting one unit of time. As in Sec. II, for small lattices one can obtain exact results for this problem by direct enumeration. Then, we find that $t_R = 0, 2, 6, 12, \ldots$, for $N = 1, 2, 3, 4, \ldots$. For instance, when $N=1$ the trivial answer is $t=0$; for $N=2$ we have

$$t_R = (1/2)\times 1 + (1/2^2)\times 2 + (1/2^3)\times 3 + \cdots = 2 \; .$$

Similarly, for $N=3$ one obtains

$$t_R = (1/2^2)\times 2 + (1/2^3)\times 3 + (2/2^4)\times 4 + (3/2^5)\times 5 + (5/2^6)\times 6 + (8/2^7)\times 7 + (13/2^8)\times 8 + \cdots = 6 .$$

For larger values of N, MC techniques were used to mea-

TABLE II. The quoted mean covering times are obtained by averaging over 5000 simulations for small lattices and averaging over 500 simulations for larger values of N ($N \geq 30$). The second moment of the t distribution is used as an estimate of the error in t. Predicted values of t for walks starting at an edge ($s=1$) and in the middle ($s=N/2$) are obtained from (4) and (6), respectively.

N	$t_R(s=1)$	$N(N-1)$	$t_R(s=N/2)$	$N(5N-6)/4$
4	11.9 ± 0.2	12	14.1 ± 0.2	14
6	30.1 ± 0.3	30	35.9 ± 0.3	36
8	56.4 ± 0.6	56	67.3 ± 0.7	68
10	89.8 ± 1.0	90	109 ± 10	110
30	864 ± 16	870	1089 ± 18	1080
60	3503 ± 40	3540	4443 ± 43	4410
120	$(1.43 \pm 0.04)\times 10^4$	1.428×10^4	$(1.74 \pm 0.05)\times 10^4$	1.782×10^4
250	$(6.08 \pm 0.25)\times 10^4$	6.225×10^4	$(7.82 \pm 0.28)\times 10^4$	7.775×10^4
500	$(2.51 \pm 0.10)\times 10^5$	2.495×10^5	$(3.11 \pm 0.14)\times 10^5$	3.1175×10^5
1000	$(1.02 \pm 0.05)\times 10^6$	9.990×10^5	$(1.21 \pm 0.06)\times 10^6$	1.2485×10^6

sure the covering time. A representative sample of our data is shown in Table II. The obtained sequence of *integer* numbers for t_R on small lattices together with the MC data for larger lattices strongly suggests that

$$t_R = N(N-1) , \tag{4}$$

which is just twice the value for rings of an equal number of sites. In fact, this result could have been obtained[14] from a known RW problem: $t_R(s=1)$ is the first-passage time through a site N of a nearest RW on a 1D lattice of N sites with reflecting BC's, starting the walk at site $s=1$.

We now consider a more general case by starting the RW at some arbitrary site s. It is clear that $t_R(s) \geq t_R(s=1)$, since to visit all sites a walk has first to visit either edge, and from there proceed to the other one. Then $t_R(s) = t_R(s=1) + t'$, where t' is the mean time a RW takes to reach either edge starting at s. For small lattices t' can be exactly calculated by direct enumeration. The first nontrivial case is $N=3$ and $s=2$, which gives $t'=1$. When $N=4$ and $s=2$ one finds

$$t' = (1/2)\times 1 + (1/2^2)\times 2 + (1/2^3)\times 3 + \cdots = 2 ,$$

and so on. For bigger lattices we measured t' (and t_R) for several lattice sizes starting the RW at various different sites (see Table II), concluding that

$$t' = (s-1)(N-s) , \tag{5}$$

$$t_R = N(N-1) + (s-1)(N-s) . \tag{6}$$

Alternatively, Eq. (5) could have been derived as follows. One constructs a ring of $N' = N-1$ sites joining (and identifying) sites 1 and N. Then t' is the expected time to reach site 1 starting the RW at site s on a ring of N' sites given by[4] $t' = (s-1)(N'-s+1)$ [see caption of Fig. 1(a)].

We have studied the lattice-covering time problem in $D=1$ using MC methods. We considered square, cubic, and 4D-hypercubic lattices of various sizes with N ranging from 10^2 to 10^5. Reflecting boundary conditions, as discussed for the $D=1$ case, are taken. The MC data clearly shows that, for large lattices, the covering time is now *independent of the starting site* in contrast to the one-dimensional case. Table III illustrates this fact by showing the ratio of the covering time starting from a corner of the lattice to one beginning in the middle, for a few lattice sizes in $D=2$, 3, and 4. This ratio is always very close to unity. Similar results were also obtained starting at an arbitrary site of the lattice.

Figures 4 and 5 display ln-ln plots of (t/N) versus the total number of sites in $D=2$, and in $D=3$ and 4, respectively. As for periodic boundary conditions we find that both the power-law behavior of Eq. (2) and the logarithmic forms of Eqs. (3) provide reasonable fits to the MC data. The power-law (dashed-line) fits of Figs. 4 and 5 have the values of $\alpha_R(D=2)=1.12$, $\alpha_R(D=3)=1.06$, and $\alpha_R(D=4)=1.07$, respectively, somewhat smaller than those for periodic boundary conditions. The fits with the logarithmic forms (3) (solid lines) give $A_R(D=2)=0.44$, $A_R(D=3)=1.96$, and $A_R(D=4)=1.71$, all of them about 30% bigger than the corresponding ones for the periodic case.

In Sec. IV we make a hypothesis of universality in the lattice-covering time problem. In the light of this hy-

TABLE III. t_R^{co} (t_R^{mi}) is the mean covering time of a walk that starts at a corner (in the middle) of the square, cube, or 4D hypercube, obtained by averaging over 500 simulations per point.

N	t_R^{co}/t_R^{mi}	N	t_R^{co}/t_R^{mi}	N	t_R^{co}/t_R^{mi}
25^2	1.034	8^3	0.978	5^4	1.020
35^2	1.009	10^3	0.999	6^4	0.998
50^2	0.961	13^3	1.006	7^4	0.980
100^2	1.001	16^3	0.960	8^4	1.009

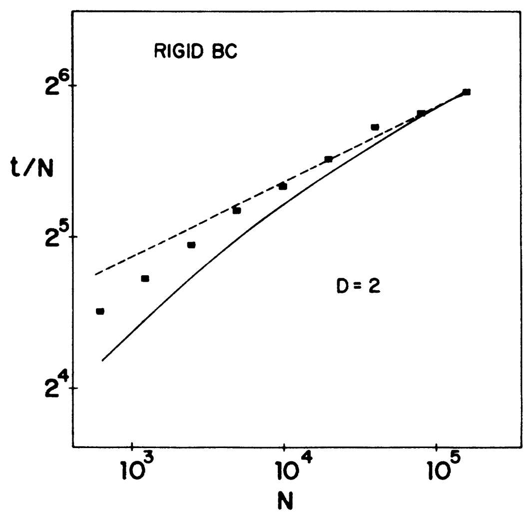

FIG. 4. Same as Fig. 2 for rigid (reflecting) BC with $\alpha_R = 1.12$ and $A_R = 0.44$.

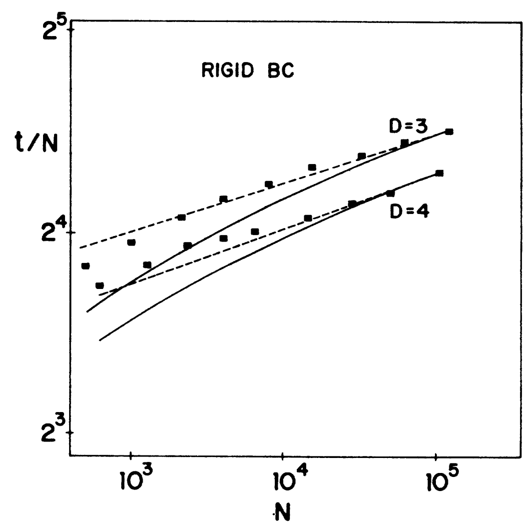

FIG. 5. Same as Fig. 3 for rigid BC with $\alpha_R(D=3)=1.06$, $\alpha_R(D=4)=1.07$, $A_R(D=3)=1.96$, and $A_R(D=4)=1.71$.

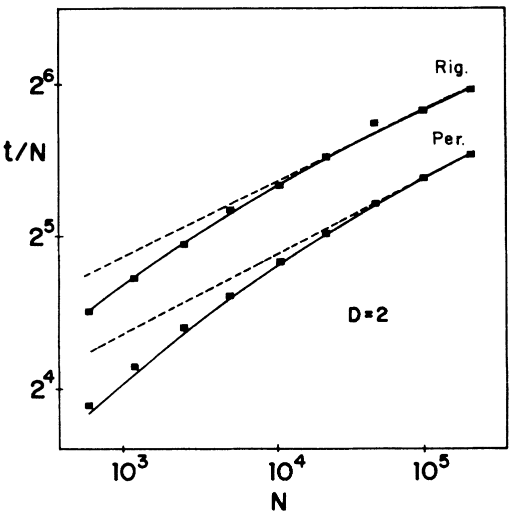

FIG. 6. ln-ln plot of (t/N) vs N for square lattice with periodic and rigid boundary conditions. The fits $t \simeq AN\ln^2 N(1+C/\ln N)$ with $A=0.30$, $C_R=5.39$, and $C_p=0.95$ are shown by solid lines. Dashed lines are used to indicate the power-law form $t \simeq BN^{\alpha}$ with $\alpha=1.14$, $B_p=8.58$, and $B_R=11.60$.

pothesis we analyze both the periodic and the rigid data in $2 \le D \le 4$.

IV. UNIVERSALITY IN THE COVERING TIME PROBLEM

We now study the question of universality in the lattice-covering time problem. By that we mean the following. If, say, the asymptotic behavior of t is the power-law form (2) one would expect that the exponent α

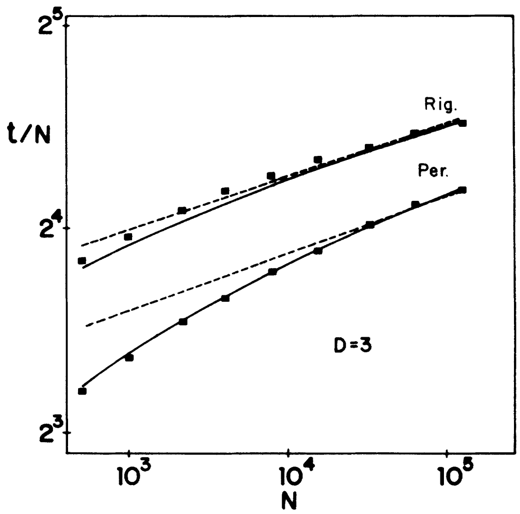

FIG. 7. Same as Fig. 6 for simple cubic lattices. Solid lines correspond to the logarithmic form $t \sim AN\ln N(1+C/\ln N)$ with $A=1.63$, $C_R=2.33$, and $C_p=-0.53$ while the power-law form (dashed line) $t \sim BN^{\alpha}$ has $\alpha=1.08$, $B_p=7.11$, and $B_R=9.04$.

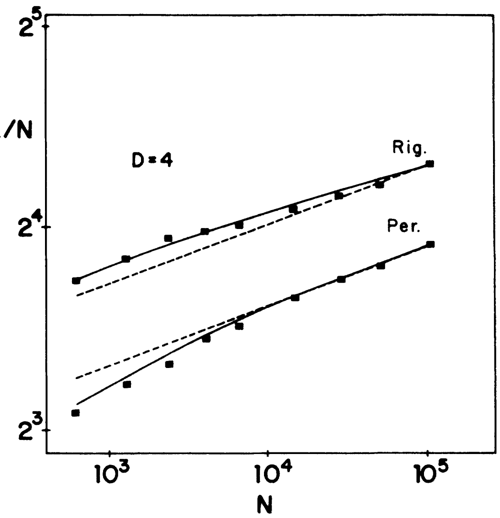

FIG. 8. Same as Fig. 7 for a 4D-hypercubic lattice. The parameters are $A=1.23$, $C_R=4.12$, $C_p=0.69$, $\alpha=1.08$, $B_p=5.91$, and $B_R=7.82$.

at a given D would be independent of the boundary conditions among other "irrelevant" features of the model such as the chosen (regular) lattice or the type of selected (finite variance) random walk. To the contrary, the nonuniversal amplitude B would depend on these details. To check this hypothesis, in Figs. 6–8 we plot both the periodic and rigid data in $D=2$, 3, and 4, respectively. Indeed we find that the values of $\{\alpha\}$ and $\{\beta\}$ in the captions of these figures provide reasonable fit to both the periodic and rigid data in $2 \le D \le 4$, as expected by the universality hypothesis. These results also predict $(t_R/t_p)=(B_R/B_p)=1.35$, 1.27, and 1.32 in $D=2$, 3, and 4, respectively. However, Table IV shows that the ratio (t_R/t_p) decreases about a constant value when the number of sites doubles. This trend can be accounted[15] for if instead $(t_R/t_p)=1+F/\ln N$ (see Table IV). In fact, the forms

$$t \sim AN\ln^2 N(1+C/\ln N), \quad D=2 \tag{7a}$$

$$t \sim AN\ln N(1+C/\ln N), \quad D \ge 3 \tag{7b}$$

where A is universal (i.e., independent of the BC) while

TABLE IV. The measured t_R and t_p are results of averaging 500 simulations per point. (t_R/t_p) can be fitted by $(t_R/t_p)=1+F/\ln N$, with $F(D=2)\simeq 4.0$, $F(D=3)\simeq 3.0$, and $F(D=4)\simeq 3.5$.

N	t_R/t_p	N	t_R/t_p	N	t_R/t_p
25^2	1.540	8^3	1.564	5^4	1.570
35^2	1.499	10^3	1.513	6^4	1.530
50^2	1.464	13^3	1.462	7^4	1.534
70^2	1.470	16^3	1.436	8^4	1.436
100^2	1.415	20^3	1.381	9^4	1.412
140^2	1.411	25^3	1.363	11^4	1.349
200^2	1.426	32^3	1.300	13^4	1.330
283^2	1.350	40^3	1.274	15^4	1.325
400^2	1.338	50^3	1.254	18^4	1.312

the constant C depends on the chosen BC, with the values of $\{A\}$ and $\{C\}$ in the captions of Figs. 6–8, not only explain our (t_R/t_p) data but also provide very good fits to our MC data for both boundary conditions in two, three, and four dimensions.

In this case, by universality we mean that the coefficient A of the leading divergent term (when $N \to \infty$) is independent of some irrelevant details such as the type of boundary condition, though it could, for example, depend on the lattice type. Of course the degree of universality (if it exists) needs to be further investigated.

V. CONCLUSIONS

We studied the lattice-covering time problem on regular cubic lattices in $1 \le D \le 4$ using MC methods and theoretical analysis. The expected times for covering one-dimensional lattices of N sites with periodic and reflecting BC's are obtained. Notice that the results for rigid BC's depend on the initial site since the problem does not possess translational invariance as in the periodic case. In contrast, we numerically verified that in dimensions higher than 1 the covering time with rigid boundaries is independent of the starting site in the large-N limit.

The one-dimensional rigid and periodic results can be derived by reducing these 1D covering problems to first-visit and trapping problems. On the contrary, in $D \ge 2$ the lattice-covering time problems are not reducible to any known lattice RW problem. The very different nature of the covering problem in $D=1$ and in $D \ge 2$ stems from the fact that in the former there is a unique path joining two given sites but there are infinitely many ways (as $N \to \infty$) of connecting two sites in higher dimensions.

It is found that both forms, the power law (2) and the logarithmic ones (3), adjust the rigid and the periodic MC data, although theoretical arguments presented here favor the latter form. Moreover, we studied the question of universality in the covering problem, in particular the dependence of the results on the chosen boundary condition. Making an universality hypothesis by assuming that certain quantities are independent on BC's, we found that the logarithmic forms (7) with A universal account nicely for all features of the MC data for both periodic and rigid BC in $D=2$, 3, and 4. Nevertheless, we believe that further numerical and analytical work is required to fully elucidate the asymptotic behavior of t.

There are several interesting lattice-covering time problems which deserve to be investigated. For instance, fractal lattices and other type of walks (Levy flights, infinite memory walks, . . .) can be considered. We are presently studying the dilute and the many-walker lattice-covering time problems as both posses very interesting features. For example, preliminary results show that the leading behavior of t can be drastically altered according to the site dilution or the walker density.

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[1]For a comprehensive review on random walks (RW) see G. H. Weiss and R. J. Rubin, Adv. Chem. Phys. **52**, 363 (1983). For a more recent review on RW on regular and disordered lattices see, J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987).

[2]A. M. Nemirovsky, H. O. Martin, and M. D. Coutinho-Filho (unpublished).

[3]S. Chandrasekhar, Rev. Mod. Phys. **35**, 1 (1943), reprinted in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).

[4]E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).

[5]P. Erdös and S. J. Taylor, Acta Math. Acad. Sci. Hung. **11**, 137 (1960).

[6]E. W. Montroll, J. Phys. Soc. Jpn. Suppl. **26**, 6 (1969); J. Math. Phys. **10**, 753 (1969); H. B. Rosenstock, Phys. Rev. **187**, 1166 (1969).

[7]A. Dvoretzky and P. Erdös, in *Proceedings of the Second Berkeley Symposium* (University of California Press, Berkeley, 1951), Vol. 33, p. 353; G. H. Vineyard, J. Math. Phys. **4**, 1191 (1963); E. W. Montroll, Proc. Symp. Appl. Math. **16**, 193 (1964).

[8]See Ref. 1 and references therein.

[9]P. Matthews, Ann. **16**, 189 (1988).

[10]D. Asimov, SIAM (Soc. Ind. Appl. Math.) J. Sci. Stat. Comput. **6**, 128 (1985).

[11]See, e.g., A. Münster, *Statistical Thermodynamics*, (Springer-Verlag, Berlin, Heilderberg, 1969), Vol. I.

[12]The stadium problem is an example of a simple system with an uniform measure on a bounded phase space. One should mention that while dynamical systems are deterministic (the dynamics is given by Hamilton's equations of motion), the lattice-covering time is, by definition, a stochastic problem. More general dynamical systems would have unbounded phase spaces with a complicated measure. A more general lattice-covering time problem, say, in a nonflat space could then be more appropriate to mimic some features of these sys-

tems.

[13]See, e.g., K. Binder and D. Stauffer, in *Applications of Monte Carlo Method in Statistical Physics,* edited by K. Binder (Springer-Verlag, Berlin, Heilderberg, 1987).

[14]It can be shown that the one-dimensional (1D) lattice-covering problems with periodic and with rigid boundary conditions are reducible to 1D first-visit and trapping problems. In turn, these problems are exactly solvable hence the conjectured forms (1) and (4)–(6) can be shown to be exact. See, C. S. O. Yokoi, A. Hernández-Machado, and L. Ramirez-Piscina (unpublished).

[15]Obviously, one could say that the trend of the decreasing value of (t_R/t_p) as N increases could be accounted by finite-size corrections to the power-law behavior. Since there is no theoretical guidance on the form of these corrections and this would introduce extra parameters, we have chosen not to pursue this line further. We should add that the fit $(t_R/t_p)=1+F/\ln N$ is, by no means, a unique one. For example, the form $A+BN^{-\delta}$, with $\delta \ll 1$ also provides a reasonably good fit to the data.