Free Brownian motion of a particle driven by a dichotomous random force

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Dynamic behavior of free Brownian motion of a particle driven by dichotomous (random telegraphic) colored noise has been treated based on the ordinary Langevin equation. The treatment makes direct use of the characteristic function without using the Fokker-Planck equation. The conditional probability density in the case where inertial effects are neglected has been obtained exactly as a function of the position of the particle, and it is shown that starting from the Dirac δ function at $t=0$, the probability density exhibits δ functions at both extremes at early times, and the amplitudes of the δ functions gradually vanish as time goes on. It is shown that this presence of δ functions at the early stage is associated with dichotomous noise. The characteristic function for the Langevin equation with complete consideration of inertial effects has also been obtained exactly.

I. INTRODUCTION

The importance of dichotomous or random telegraphic colored noise has long been realized, first by communication engineers,[1] and subsequently by physcists and chemists.[2] Dichotomous noise takes either the value of $+E$ or $-E$ with a frequency γ at time t and gives rise to an exponentially decaying correlation function. The Fokker-Planck equations for a fluctuating system with dichotomous noise have been obtained by Klyatskin and by Kitahara, Horsthemke, and Lefever[2] with a view to investigating the equilibrium or stationary properties. Ishii and Kitahara[3] treated dynamic problems based on a class of stochastic differential equations in the Laplace-transformed domain of time. However, we note that essential results for the dynamic processes have not been satisfactorily obtained for the conditional probability density as an explicit function of space and time.

In this paper we shall consider two fundamental problems whose properties in the case of white noise are fully known,[4] namely, free Brownian motion of a particle driven by dichotomous noise. To this end, we start with the ordinary Langevin equation whose formal solution enables us to obtain the characteristic function directly by using the method of averaging without reference to the Fokker-Planck equation. By Fourier inversion of the characteristic function, we shall calculate the conditional probability density of the position of the particle, x, at time t for case (i) where inertial effects in the Langevin equation are ignored, which corresponds to the high-viscosity limit treated by Einstein for the diffusion of a particle in a fluid with a white-noise random force. It is found that the conditional probability density starting with the Dirac δ function centered at the initial position at $t=0$ still has Dirac δ functions as time goes on at the two extremes which correspond to the case where the process consists entirely of $+E$ or $-E$ up to t without exception. These δ functions appear explicitly in the early stage and vanish as time goes on, because they are multiplied by the factor $\exp(-\gamma t/2)$. We show that this appearance of the δ functions at the early stage is limited not only to the present particular cases but also to the general one with dichotomous noise. It should be pointed out that, even though the density is not a continuous, finite function at the ends, average values obtained from the density behave normally. The non-Gaussian character of the present nonwhite Markov dichotomous noise thus will be demonstrated explicitly.

Secondly, we shall calculate the characteristic function exactly for case (ii) where inertial effects are fully taken into account based on the Langevin equation. The probability density as a function of the velocity of the particle v and t, this time is represented by a convolution integral. Numerical calculations to obtain the conditional probability density also lead to δ functions at the extremes at short time. This case also corresponds to an L,R circuit subject to a randomly fluctuating dichotomous e.m.f. and Brownian motion of a particle in a harmonically bound potential with neglect of inertial effects.

II. THEORY

A. Calculation of conditional probability density in case (i)

We write the equation of motion by neglecting inertial effects in the Langevin equation as follows:

$$\frac{dx(t)}{dt}=\lambda(t) \; . \tag{1}$$

This equation of motion corresponds to the case which Einstein investigated, Brownian motion where $\lambda(t)$ is white noise. But in the present work we assume that $\lambda(t)$ in Eq. (1) is centered dichotomous noise with the following properties:

$$\langle \lambda(t) \rangle = 0 \; , \tag{2}$$

$$\langle \lambda(t_1)\lambda(t_2) \rangle = E^2 \exp(-\gamma|t_1-t_2|) \; . \tag{3}$$

Note that by keeping E^2/γ constant and letting $\gamma \to \infty$, we obtain the δ function on the right-hand side of Eq. (3), which corresponds to the case for white noise. Equation (1) can be readily integrated formally and it is well known that the Laplace transform of the characteristic function $\Phi(u,s)$ with respect to t is given by

$$\Phi(u,s)=L(\langle \exp[-iux(t)]\rangle)$$
$$=L\left[\left\langle \exp\left[-iu\int_0^t \lambda(t')dt'\right]\right\rangle\right]$$
$$=\int_0^\infty \langle \exp(-iux)\rangle e^{-st}dt$$
$$=\frac{1}{s+[u^2E^2/(s+\gamma)]} \tag{4}$$

where the initial position $x(0)=x_0=0$ without the loss of the generality in the formulation. In obtaining Eq. (4), we have used the assumption that $\lambda(t)$ is a Poisson process so that the nonoverlapping time ranges are independent, namely,

$$\langle \lambda(t_1)\lambda(t_2)\cdots\lambda(t_{2m-1})\lambda(t_{2m})\rangle=\langle\lambda(t_1)\lambda(t_2)\rangle\langle\lambda(t_3)\lambda(t_4)\rangle\cdots\langle\lambda(t_{2m-1})\lambda(t_{2m})\rangle , \tag{5}$$

where

$$t_1\geq t_2\geq\cdots\geq t_{2m} .$$

The inverse Fourier transform of both sides in Eq. (4) with respect to u may be readily obtained. The result is

$$L\left[P(x,t)\right]=\frac{1}{2E}[(s+\gamma')+\gamma']F(x,s) , \tag{6}$$

where

$$\gamma'=\frac{\gamma}{2} ,$$

$$F(x,s)=\frac{1}{S}\exp\left[-\frac{x}{E}S\right] , \tag{7}$$

in which

$$S=[(s+\gamma')^2+\gamma'^2]^{1/2}$$

and $P(x,t)$ represents the conditional probability density. The inverse Laplace transform of $F(x,s)$ with respect to s can be obtained (see p. 249 of Ref. 5)

$$L^{-1}(F(x,s))=e^{-\gamma' t}I_0(\gamma^*)H\left[t-\frac{x}{E}\right] , \tag{8}$$

where $I_0(z)$ is the modified Bessel function, $H(z)$ is the Heaviside step function, and

$$\gamma^*=\gamma'\left[t^2-\frac{x^2}{E^2}\right]^{1/2} .$$

Similarly, it follows that

$$L^{-1}((s+\gamma')F(x,s))=e^{-\gamma' t}\left[\gamma'^2\frac{t}{\gamma^*}I_1(\gamma^*)H\left[t-\frac{x}{E}\right]+\delta\left[t-\frac{x}{E}\right]\right] . \tag{9}$$

Therefore $P(x,t)$ for $x\geq 0$ is given by

$$P(x,t)=\frac{\gamma}{4E}e^{-\gamma' t}H\left[t-\frac{x}{E}\right]\left[I_0(\gamma^*)+\gamma'\frac{t}{\gamma^*}I_1(\gamma^*)\right]+\frac{1}{2E}e^{-\gamma' t}\delta\left[t-\frac{x}{E}\right] \quad (x\geq 0) . \tag{10}$$

Note that $P(x,t)$ is an even function of x. It is immediately evident that $P(x,t)$ is not Gaussian and has δ functions at $t_+=(x/E)$ and $t_-=-(x/E)$, which are pronounced particularly at short times where the factor exp $(-\gamma' t)$ is not sufficiently small. It is seen that these positions correspond to the case in which the random force consists of nothing but $+E$ all the way up to t, or $-E$, respectively, for t_+ and t_-. These positions may be regarded as boundaries for $P(x,t)$ in the white-noise case where δ functions are absent. It should be noted that even though $P(x,t)$ at the early stage behaves rather anomalously, the characteristic function $\Phi(u,s)$ in Eq. (4) is not discontinuous. In fact, it is obvious from Eq. (4) that

$$\langle \exp[-iux(t)]\rangle=e^{-\gamma' t}\left[\cosh(\gamma_u t)+\frac{\gamma'}{\gamma_u}\sinh(\gamma_u t)\right] , \tag{11}$$

where

$$\gamma_u=\tfrac{1}{2}(\gamma^2-4u^2E^2)^{1/2} ,$$

which corresponds to the overdamped oscillator case.

B. Calculation of the characteristic function for case (ii)

Now, we shall move to another fundamental case of Brownian motion which is governed by the ordinary Langevin equation:

$$dv(t)/dt=-\beta v(t)+\lambda(t) \tag{12}$$

with the assumptions in Eqs. (2) and (3) for $\lambda(t)$. This equation also corresponds to the circuit equation for an (L,R) system subject to the fluctuating dichotomous emf, if $v(t)$ is regarded as the current $i(t)$. The particular case of $\beta=0$ in Eq. (12) corresponds to Eq. (1). In addition, Eq. (12) is the equation of motion for a particle undergoing Brownian motion under a harmonically bound potential with neglect of inertial effects in which case $v(t)$ is the position of the particle.

The formal solution for Eq. (12) is given by

$$v(t)=e^{-\beta t}\int_0^t e^{\beta t'}\lambda(t')dt' , \tag{13}$$

where $v(0)=v_0=0$ has been assumed. To calculate the characteristic function for this case, we should know the moments:

$$\langle v^{2m}(t)\rangle=(2m)!e^{-2m\beta t}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{2m}}\exp[\beta(t_1+t_2+\cdots+t_{2m})]\langle\lambda(t_1)\lambda(t_2)\cdots\lambda(t_{2m})\rangle dt_1dt_2\cdots dt_{2m} , \tag{14}$$

where m is an integer. Equation (2) enables us to find

$$\langle v^{2m+1}(t)\rangle=0 .$$

In view of Eqs. (3) and (5), we find after using the Laplace-transform technique that

$$\begin{aligned}\langle v^{2m}(t)\rangle=(2m)!\left[\frac{E^2}{2\beta^2}\right]^m\Bigg[&\frac{1}{0!m!(c+1)(c+3)\cdots[c+(2m-1)]}\\&+\frac{e^{-2\beta t}}{1!(m-1)!(-c+1)(c+1)(c+3)\cdots[c+(2m-3)]}\\&+\frac{e^{-4\beta t}}{2!(m-2)!(-c+1)(-c+3)(c+1)(c+3)\cdots[c+(2m-5)]}\\&+\cdots+\frac{e^{-2m\beta t}}{m!0!(-c+1)(-c+3)\cdots[-c+(2m-1)]}\\&-2\frac{e^{-(\gamma+\beta)t}}{0!(m-1)!(c+1)(-c+1)(-c+3)\cdots[-c+(2m-1)]}\\&-2\frac{e^{-(\gamma+3\beta)t}}{1!(m-2)!(c+1)(c+3)(-c+1)(-c+3)\cdots[-c+(2m-3)]}\\&-\cdots-2\frac{e^{-[\gamma+(2m-1)\beta]t}}{(m-1)!0!(-c+1)(c+1)(c+3)\cdots[c+(2m-1)]}\Bigg] ,\end{aligned} \tag{15}$$

where

$$c=\frac{\gamma}{\beta} .$$

In order to obtain the characteristic function from Eq. (15), we sum each coefficient for $\exp(-2m\beta t)$ and that for $\exp\{-[\gamma+(2m-1)\beta]t\}$, finding that

$$\begin{aligned}\Phi(u,t)=&{}_1F_1\left[\frac{(c+1)}{2};\frac{z}{2}\right]{}_1F_1\left[\frac{(-c+1)}{2};\frac{z}{2}e^{-2\beta t}\right]\\&+2z\frac{e^{-(\gamma+\beta)t}}{(c+1)(-c+1)}{}_1F_1\left[\frac{(-c+3)}{2};\frac{z}{2}\right]{}_1F_1\left[\frac{(c+3)}{2};\frac{z}{2}e^{-\beta t}\right] ,\end{aligned} \tag{16}$$

where

$$z=-u^2\frac{E^2}{2\beta^2},\quad {}_1F_1(a;z)=1+\frac{z}{1!a}+\frac{z^2}{2!a(a+1)}+\frac{z^3}{3!a(a+1)(a+2)}+\cdots ,$$

in which ${}_1F_1(a;z)$ is a confluent hypergeometric function. Equation (16) can be rewritten by using the identity

$${}_1F_1\left[\nu+1;-\frac{y^2}{4}\right]=\Gamma(\nu+1)\left[\frac{y}{2}\right]^{-\nu}J_\nu(y)$$

as

$$\begin{aligned}\Phi(u,t)=&\Gamma(\tfrac{1}{2}(c+1))\Gamma(\tfrac{1}{2}(-c+1))\left[\frac{uE}{2\beta}\right]^{-(1/2)(c-1)}J_{(1/2)(c-1)}\left[\frac{uE}{\beta}\right]\\ &\times\left[\left[\frac{uE}{2\beta}e^{-\beta t}\right]^{(1/2)(c+1)}J_{-(1/2)(c+1)}\left[\frac{uE}{\beta}e^{-\beta t}\right]\right]\\ &+\frac{e^{-(\gamma+\beta)t}}{(c+1)(-c+1)}\left[\frac{uE}{\beta}\right]^2\Gamma(\tfrac{1}{2}(-c+3))\Gamma(\tfrac{1}{2}(c+3))\left[\frac{uE}{2\beta}\right]^{(1/2)(-c+1)}\\ &\times J_{(1/2)(-c+1)}\left[\frac{uE}{\beta}\right]\left[\left[\frac{uE}{2\beta}e^{-\beta t}\right]^{(1/2)(c+1)}J_{(1/2)(c+1)}\left[\frac{uE}{\beta}e^{-\beta t}\right]\right]\ .\end{aligned}\tag{17}$$

In order to calculate the inverse Fourier transform of Eq. (17), we find it useful to note the relations (see p. 123 of Ref. 5):

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}{}_1F_1\left[\frac{(c+1)}{2};-\frac{1}{4}\left[\frac{uE}{\beta}\right]^2\right]e^{-iux}du=\frac{1}{\sqrt{\pi}}\frac{\Gamma((c=1)/2)}{\Gamma(c/2)}\frac{\beta}{E}\left[1-\left[x\frac{\beta}{e}\right]^2\right]^{(c/2)-1},\\ &\frac{1}{2\pi}\int_{-\infty}^{\infty}2z\,{}_1F_1\left[\frac{(-c+3)}{2};\frac{z}{2}\right]e^{-iux}du=\frac{1}{\sqrt{\pi}}\frac{\beta}{E}\frac{\Gamma((-c+3)/2)}{\Gamma((-c+2)/2)}\left[1-\left[\frac{x\beta}{E}\right]^2\right]^{-(c/2)-2}c\left[c(c+1)\left[\frac{x\beta}{E}\right]^2+1\right],\\ &\frac{1}{2\pi}\int_{-\infty}^{\infty}{}_1F_1\left[\frac{(c+3)}{2};-\frac{z}{2}e^{-2\beta t}\right]e^{-iux}du=\frac{1}{\sqrt{\pi}}\frac{\Gamma((c+3)/2)}{\Gamma((c+2)/2)}\frac{\beta}{E}e^{\beta t}\left[1-\left[x\beta\frac{e^{\beta t}}{E}\right]^2\right]^{c/2}\ .\end{aligned}\tag{18}$$

Then the inverse can be obtained by using the convolution theorem stating that if

$$\int_{-\infty}^{\infty}f(x)e^{-iux}dx=\int_{-\infty}^{\infty}f_1(x)e^{-iux}dx\int_{-\infty}^{\infty}f_2(x)e^{-iux}dx\ ,$$

then

$$f(x)=\int_{-\infty}^{\infty}f_1(x-x')f_2(x')dx'\ .$$

III. DISCUSSION

It has not been noted in previous studies as far as the author knows that $P(x,t)$ has δ functions at the boundaries as seen in Eq. (10). The δ functions arise from the fact that at short times there is still a relatively high probability that the Brownian particle takes all steps entirely to the right or left without a single failure. The location of the boundaries can be readily checked in Eq. (1) by putting $\lambda(t)=E$ or $-E$, which leads to

$$x_+=Et \quad\text{and}\quad x_-=-Et\ ,$$

where x_+ and x_- represent the positions of the particle which takes all steps to the right or left. Even for the general case where

$$\frac{dx(t)}{dt}=M(x)+N(x)\lambda(t)\tag{19}$$

the position of the δ functions can be found easily by putting $\lambda(t)=E$ or $-E$ in the starting stochastic differential equation. This indicates that a general system subject to random dichotomous noise will exhibit a conditional probability density with δ functions at boundaries particularly at short times with small γ. Although it has been shown in previous studies[2] that the equilibrium or stationary density is nonzero in a limited range, it does not exhibit δ functions at the extremes of the range. This can be understood in view of the fact that as t approaches infinity, we can totally rule out the possibility of all successes or failures. Hence in equilibrium, the density does not have spikes, whereas in the dynamic case for small t, it does. The white-noise case does not lead to spikes in the dynamics, because except at $t=0$, the random variable can take, in principle, any value from $-\infty$ to ∞ with the prescribed probability and this can also be seen from the fact that the white-noise limit is obtained by keeping E^2/γ while letting γ tend to infinity, which will remove the probability of all successes or failures. We can regard the appearance of the δ functions as the presence of moving boundaries at $x=x_+$ and x_- with the probability (not the density) of $\frac{1}{2}\exp(-\gamma' t)$ of finding the particle at each boundary, which follows by integrating the second term on the right-hand side of Eq. (10) with respect to x. Hence the total probability of finding the particle between the boundaries is given by $1-\exp(-\gamma' t)$. If we calculate the probability from ∞ to x_1 which is greater than zero, and less than x_+ by integrating the second term in Eq. (10), it suddenly goes up at $x=x_+$ from zero to $\frac{1}{2}\exp(-\gamma' t)$ and increase due to the contribution from the area between the boundary and x_1. Therefore, physically, the presence of the δ functions in $P(x,t)$ at the boundaries arising from dichotomous noise can be reduced to that of the moving boundaries with a time-varying ability of attracting and reflecting the particle. It is interesting to note that in the early stage dichotomous noise leads to a probability density which is a mixture of the δ functions at the boundaries with the ordinary smooth non-Gaussian distribution between them. It should be pointed out that the same conclusion is also true for the general case where Eq. (19) is valid.

Plots of $P(x,t)$ obtained from Eq. (10) are shown in Fig. 1 together with the corresponding Gaussian cases (dotted curves) and with the δ functions represented by vertical lines where the relative contribution from the functions is indicated by their heights. It is obvious in Eq. (10) that if we do not take into account the δ functions, the normalization condition that

$$\int_{-\infty}^{\infty} P(x,t)dx = 1$$

cannot be fulfilled, because we do not take into account the probability concentrated at the boundaries. This is another reason why the δ function must be present in $P(x,t)$. This fact also may be checked for $P(x,t)$ in Fig. 1 for short time by comparing the corresponding Gaussian curve where the area under the curve for $P(x,t)$ without the δ functions for the dichotomous noise is considerably smaller than that for the Gaussian curve. It should be mentioned that although $P(x,t)$ has discontinuities due to the presence of the δ functions, the average which is the integral with $P(x,t)$ takes ordinary smooth behavior.

In order to check the white-noise limit in Eq. (10), let us put $\gamma \to \infty$ keeping (E^2/γ) constant. It follows in view of the relation

$$\lim_{\gamma^* \to \infty} I_\nu(\gamma^*) = \frac{e^{\gamma^*}}{(2\pi\gamma^*)^{1/2}}$$

that

$$P(x,t) = \left[4\pi \frac{E^2}{\gamma} t \right]^{-1/2} \exp\left[-\frac{x^2}{4(E^2/\gamma)t} \right],$$

which is nothing but the Gaussian distribution for the motion of a random walker with the variance $\langle x^2(t) \rangle = 2E^2 t/\gamma$. Figure 2 shows plot of $P(x,t)$ for $\gamma = 1$ from which it is seen that the time evolution is highly different from that of Fig. 1 in that the area made under $P(x,t)$ is significantly smaller than 1, which is $1-\exp(-\gamma t/2)$, and the former is far from the Gaussian distribution (dotted curves).

Although it seems difficult to discuss even the white-noise limit in Eq. (16) by considering the dynamics, the equilibrium or stationary property of $\Phi(u,t)$ can be found, because

$$\lim_{t \to \infty} \Phi(u,t) = {}_1F_1\left[\frac{(c+1)}{2}; \frac{z}{2} \right].$$

In view of Eq. (18), it follows that

$$P_{\text{eq}}(v) = \frac{1}{\sqrt{\pi}} \frac{\Gamma((c+1)/2)}{\Gamma(c/2)} \frac{\beta}{E} \left[1 - \left[\frac{v\beta}{E} \right]^2 \right]^{(c/2)-1}, \tag{20}$$

where $P_{\text{eq}}(v)$ is the equilibrium density. This agrees fully with the equilibrium density deduced from the method

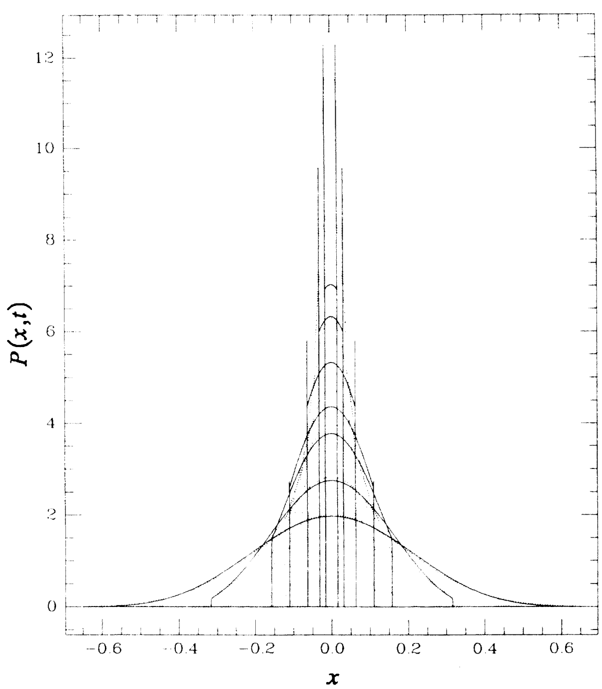

FIG. 1. Plots of $P(x,t)$ in Eq. (9) vs x as a function of t. The time t is 0.05, 0.1, 0.2, 0.35, 0.5, 1.0, and 2.5 from the top to the bottom curves viewed from $x=0$ with $\gamma=10$ and $E^2=0.1$. Dotted curves represent the corresponding Gaussian process when white noise is introduced. The spikes represent δ functions, and their decrease in height as t becomes large represents the relative contribution from the δ functions.

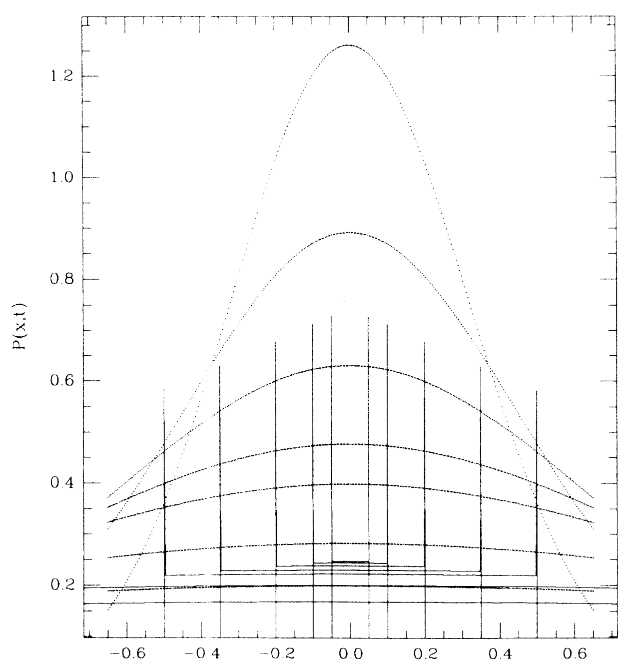

FIG. 2. Plots of $P(x,t)$ in Eq. (9) vs x as a function of t. The time t is 0.05, 0.1, 0.35, 1.0, and 2.0 from the top to the bottom curves viewed from $x=0$ with $\gamma=1$ and $E^2=1.0$. Dotted curves represent the corresponding Gaussian process when white noise is introduced. The spikes represent δ functions, and their decrease in height as t becomes large represents the relative contribution from the δ functions.

based on the Fokker-Planck equation,[2] which was not used in our derivation of Eq. (20), confirming the agreement of the two independent approaches. The white-noise limit can be obtained for Eq. (20) by using the relation

$$\lim_{c\to\infty}\frac{\Gamma((c+1)/2)}{\Gamma(c/2)}=\sqrt{(c/2)}$$

leading to

$$P_{\rm eq}(v)=\left[\frac{\beta\gamma}{2\pi E^2}\right]^{(1/2)}\exp\left[-\frac{v^2\beta\gamma}{2E^2}\right].$$

This relation gives the Boltzmann distribution after using the fluctuation-dissipation theorem. It is obvious in Eq. (20) that the dichotomous noise leads to a non-Boltzmann distribution as $t\to\infty$ and $P_{\rm eq}(v)$ has the transition point at $c=2$; for $c>2$ $P_{\rm eq}(v)$ has the maximum at $v=0$, whereas for $c<2$, it has minimum at $v=0$.

Although we could obtain the moments and the characteristic function exactly in case (ii) [cf. Eq. (16)], the probability density $P(v,t)$ was expressed by the convolution integral. Thus we have carried out numerical calculations of case (ii) whose results are shown in Fig. 3 again together with the white-noise case for $\gamma=10$. It becomes also clear that dichotomous noise leads to the δ functions at the boundaries just as in case (i).

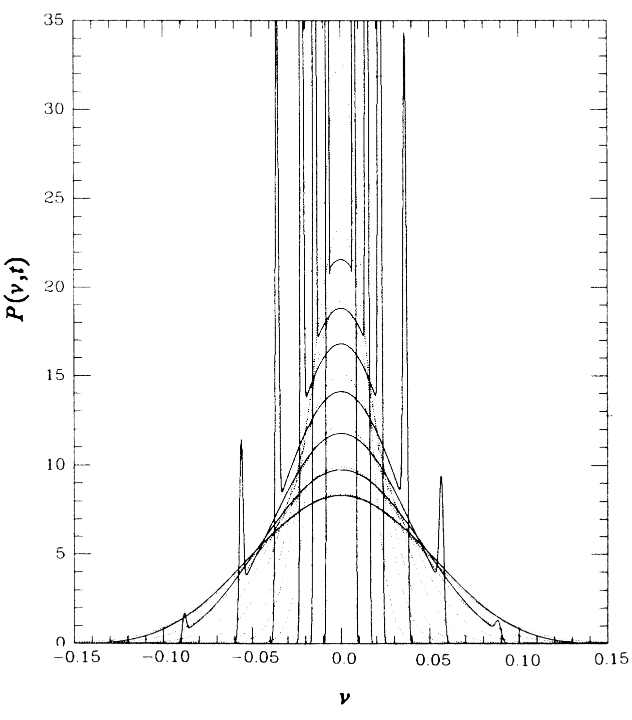

FIG. 3. Plots of $P(v,t)$ for case (ii) vs v as a function of t for $E=0.5$, $\beta=1.0$, and $\gamma=50$. The time t is 0.015, 0.03, 0.045, 0.075, 0.12, 0.195, and 0.3 from the top to the bottom curves viewed from $v=0$. Dotted curves represent the corresponding Gaussian process when white noise is introduced. The spikes this time have finite widths due to the fact that the solid curves are obtained from a numerical analysis, but these spikes essentially should be identical to those indicated by vertical lines in Fig. 1.

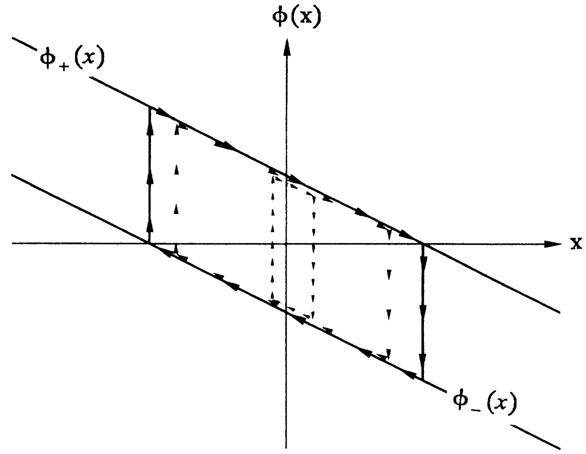

FIG. 4. Limit cycle plot for $[dx(t)/dt]=-\beta x(t)+\lambda(t)$.

Finally we show why we have the noise-induced transition at $c=2$. If we write

$$\phi_+(x)=-\beta x+E$$

and

$$\phi_-(x)=-\beta x-E$$

by putting $\lambda(t)=E$ and $-E$ in Eq. (12), we see that ϕ's represent the velocity as shown in Fig. 4, which may be regarded as a phase diagram. It is seen easily that the motion should be confined within the parallelogram shown by the bold lines, because once $dx(t)/dt=0$ is reached, the system no longer moves. Hence it must be on either the upper or lower line with transition whose frequency is determiend by γ. Thus if γ is large, whose case is indicated by the short-dashed line in Fig.4, since starting from $x=0$, the system on the upper line makes the movement of x increase, because of the positive velocity in the region, while that on the lower line makes it decrease so that the motion will be mostly confined near $x=0$ with frequent transition between the two lines, $P(x,t)$ has the maximum at the stable point of $x=0$. Whereas, if γ is small, whose case is indicated by the dotted line in Fig. 4, even though $x=0$ is still the stable point, the system can stay on either line longer, which eventually drives the system near the points where $dx(t)/dt=0$, leading to $P(x,t)$ with the minimum at $x=0$. We therefore see that there should be the noise-induced transition which is independent of E. This kind of plot, which is similar to a limit cycle, becomes quite useful when we consider the dynamic process intuitively. In our case of dichotomous noise, the trajectory the system takes is either $\phi_+(x)$ or ϕ_- and the transition between the two states is governed statistically so as to satisfy the Chapman-Kolmogorov equation which arises from the assumption that the process is Markovian.

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