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Cellular-automaton model of earthquakes with deterministic dynamics

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A cellular-automaton model of threshold elements with deterministic dynamics is introduced. The model is a cellular-automaton version of the mechanical earthquake model invented originally by Burridge and Knopoff [Bull. Seismol. Soc. Am. **57**, 341 (1967)] and studied recently by Carlson and Langer [Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989)]. Randomness exists only in initial configurations. Numerical results show that the distribution function of the event magnitudes has a scaling region consistent with the Gutenberg-Richter law.

Self-organized criticality proposed by Bak, Tang, and Wiesenfeld[1] is the idea that a system can develop by itself to the critical state where it shows a scaling behavior. In their original paper they proposed, as an example, the "sand-pile model," which is a cellular-automaton model with stochastic dynamics, and they demonstrated that it has a critical state as an attracter of the dynamics for two or more spatial dimensions.

Recently Carlson and Langer[2] studied a simple model of an earthquake, which consisted of a one-dimensional chain of blocks and springs with each block being pulled through a pulling spring by a constant velocity. The system obeys the deterministic Newtonian equation but it shows very complicated behavior because they introduced velocity weakening friction acting on each block which prohibited its smooth motion. This type of model was originally introduced by Burridge and Knopoff[3] and is known among seismologists with its variants, but Carlson and Langer have established that the model shows a scaling behavior without any randomness being embedded except in the initial configurations. This spatially uniform version of the mechanical earthquake model, which we call the BKCL model after these people to distinguish it from the other versions of the model in this paper, seems to provide another example of self-organized criticality.[4]

One of the interesting features of the BKCL model is that the scaling behavior is found in the system with deterministic dynamics and no randomness built in, and that it is consistent with the Gutenberg-Richter law[5] for earthquake distribution. They also argued that this scaling law could be derived from a simple scaling argument.

If the Gutenberg-Richter law can be obtained by the scaling argument, the essential ingredients of the model for that become clearer and we should be able to construct a simpler model to attain deeper understanding. The purpose of this work is to construct a cellular-automaton model of an earthquake with deterministic dynamics which shows a behavior similar to the BKCL model.

The underlying physical idea is the same as with the BKCL model and the model consists of blocks and springs (Fig. 1). Suppose f_i is a total force which acts on the ith block through the springs attached to it. The forces f_i are related to the displacements of the blocks from their natural positions x_i by

$$f_i = -k_p(x_i - v_p t) + k_c(x_{i-1} + x_{i+1} - 2x_i)\,, \qquad (1)$$

where k_c and k_p are the spring constants for the connecting and the pulling springs, respectively, v_p is the pulling velocity, and t is time.

The dynamics of the model is defined as follows: As long as all the f_i's are smaller than a threshold value $f_{\rm th}$, all the blocks are stuck and x_i's are constant in time. In this time region the f_i increase continuously by a uniform rate $k_p v_p$ per unit time. As soon as one of the forces reaches the threshold $f_{\rm th}$, that block is assumed to slip by a certain distance to relax a certain amount of force δf. During this elementary process, all the other blocks are assumed to be stuck. Then part of the relaxed force δf will be distributed equally to the neighboring blocks. Namely, if the jth block is slipping, this process is given by the change of forces from f_i's to f_i''s as

$$\begin{aligned} &f_j = f_{\rm th} \rightarrow f_j' = f_{\rm th} - \delta f\,, \\ &f_{j\pm1} \rightarrow f_{j\pm1}' = f_{j\pm1} + \tfrac{1}{2}\Delta\delta f\,, \end{aligned} \qquad (2)$$

and all the other $f_i (i \neq j,\ j\pm1)$ are unchanged. In Eq. (2), Δ is a ratio of the distributed force to the relaxed force and is given by

$$\Delta = \frac{2k_c}{k_p + 2k_c}\,, \qquad (3)$$

according to Eq. (1).

If the neighboring forces $f_{j\pm1}$ before this process are small enough to make $f_{j\pm1}'$ smaller than $f_{\rm th}$, no more slipping ensues and all the forces start increasing uniformly again until the next event occurs. On the other hand, if the neighboring forces are close enough to $f_{\rm th}$ and $f_{j\pm1}' > f_{\rm th}$, then these blocks also start slipping and the forces will be relaxed according to the amount of excess forces over $f_{\rm th}$. The part of the relaxed force will be redis-

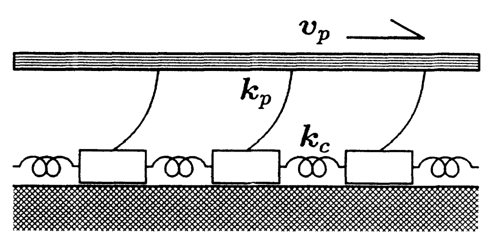

FIG. 1. Blocks and springs system.

tributed to the neighboring blocks again.

These elementary processes are defined by the following: Suppose f_j exceeds $f_{\rm th}$, then

$$f_j \rightarrow f_j' = \phi(f_j - f_{\rm th}) \,, \qquad f_{j\pm1} \rightarrow f_{j\pm1}' = f_{j\pm1} + \tfrac{1}{2}\Delta(f_j - f_j') \,. \tag{4}$$

This process will be repeated until all the forces become smaller than $f_{\rm th}$. The function ϕ defines how much force will be relaxed when the f_i's exceed the $f_{\rm th}$ and should satisfy the conditions

$$\phi(+0) = f_{\rm th} - \delta f \,, \qquad |\phi(x)| < f_{\rm th} \text{ for } x \geq 0 \,. \tag{5}$$

The parameter δf defines the smallest event where only a single block is involved. We will take ϕ as a decreasing function at least for small x in order that a small event can be amplified and lead to a large one, which is supposed to imitate an event caused by the velocity-weakening friction. The parameter α defined as

$$\left. \frac{d}{dx}\phi(x) \right|_{x=+0} = -\alpha \tag{6}$$

characterizes the way the forces are relaxed in the small events. The uniform increase of forces starts only after these processes settle, which means the duration time of the processes is assumed to be zero.

Each event consists of this kind of sequence of processes and is embedded in the uniform increase of forces. We define the moment of event[2] in the following way. Suppose the values of the forces and displacements just before and after the event are $\{f_i, x_i\}$ and $\{f_i', x_i'\}$, respectively. Then the moment of the event M is defined as

$$M \equiv \sum_i (x_i' - x_i) = k_p^{-1} \sum_i (f_i - f_i') \tag{7}$$

and the magnitude of the event as

$$\mu \equiv \log_{10} M \,. \tag{8}$$

The distribution function $\mathcal{R}(\mu)$ of the magnitude of events μ per block per unit time is introduced and it satisfies the sum rule

$$\int_0^\infty 10^\mu \mathcal{R}(\mu) d\mu = v_p \,, \tag{9}$$

which simply reflects the fact that each block is moving at v_p on the average.

There are three basic parameters for this system, namely Δ, δf, and α. The parameter Δ defined by (3) is related to k_c/k_p and characterizes the system "stiffness." If Δ is small, a large part of the forces is relaxed through pulling springs during events, consequently events tend to localize. On the other hand, if Δ is close to unity, most of the relaxed force will be taken over to neighboring blocks and events tend to extend over a large spatial region.

The parameter δf that defines the smallest event may be related to the pulling velocity v_p in Carlson and Langer's paper because the moment of the smallest event is proportional to v_p there.[2] It is important, however, to note that v_p itself plays no role in the present model.

The parameter α is "an amplification parameter." For large α, a small triggering slip can easily induce a large one. This parameter is analogous to the parameter α in Carlson and Langer's paper.[2]

We have done some numerical simulations. In the following we take $v_p = k_p = f_{\rm th} = 1$ and use

$$\phi(x) = \frac{(2-\delta f)^2/\alpha}{x + (2-\delta f)/\alpha} - 1 \,, \tag{10}$$

which is the simplest form satisfying Eqs. (5) and (6). The free boundary condition where f_i's are set to be zero outside the system is employed. Note that this does not correspond to the free boundary condition for the original blocks and springs system.

Figure 2 shows time sequence of events. Displacements $\Delta x_i = x_i' - x_i$ for each block during the events are plotted as a function of event time and position. The event sequence which is prepared with a random initial configuration of f_i's at $t = -2$ is shown. You can find some similarities to the corresponding plots for the BKCL model,[2] e.g., an almost periodic recurrence of small events (creeping events), irregular sequences of large events, precursor events before the large events, and quiet periods after them.

All these features are easily understood if you look at the plots of x_i for every sticking time region (Fig. 3). The same sequence as that in Fig. 2 is used in Fig. 3. The original configuration at $t = 0$ shown by the lowest curve moves upwards with an average velocity $v_p = 1$ to form many curves at every time the system sticks. Larger events ensue after localized regions that have fallen behind try to catch up by sequence of small events to make the overall configuration smoother. After the large events, the region has to wait some time for the next event because it has gone a little ahead compared with other parts of the system.

Distribution functions of events $\mathcal{R}(\mu)$ are plotted for some parameters in Fig. 4. Isolated peaks at the smallest

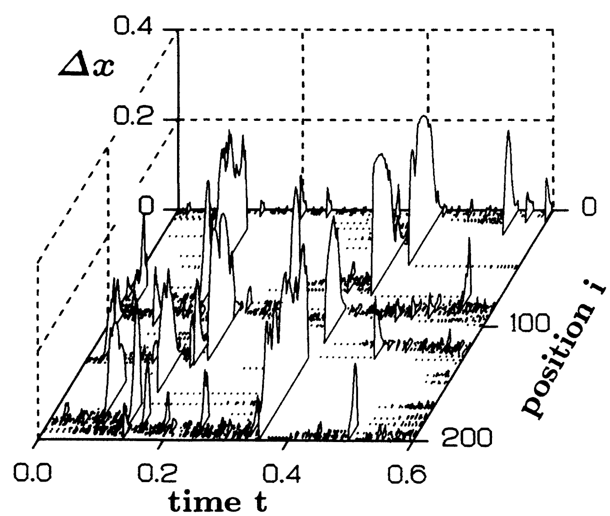

FIG. 2. Displacements during events Δx as a function of event time t and position i for $\Delta = 0.85$, $\alpha = 3.0$, $\delta f = 0.01$, and $L = 200$.

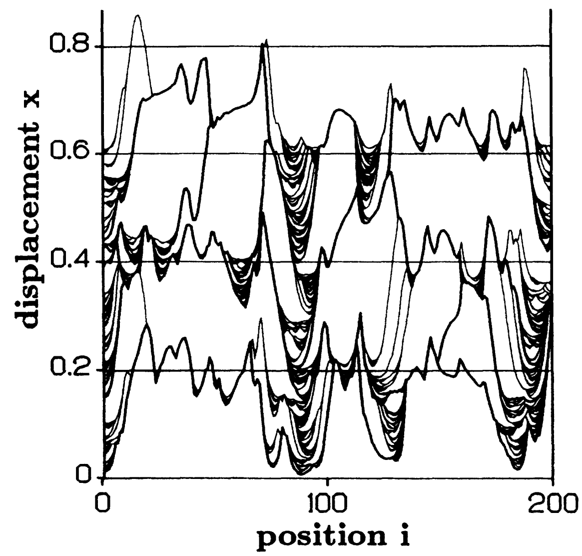

FIG. 3. Configuration sequence of the system x_i after each event. The lowest curve shows a configuration at $t=0$ and the system moves up with the average velocity $v_p=1$ afterward. The sequence of this figure is the same as that used in Fig. 2.

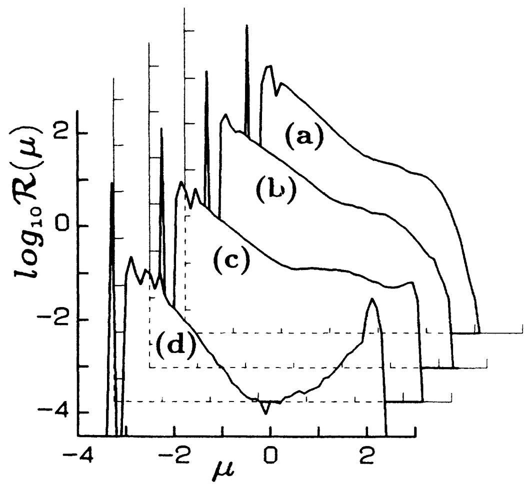

FIG. 4. $\log_{10}\mathcal{R}(\mu)$ vs $\mu=\log_{10}M$ for $\alpha=3.0$, $\delta f=0.01$, and (a) $\Delta=0.8$, (b) 0.85, (c) 0.9, and (d) 0.95.

μ correspond to the single block events of magnitude $\log_{10}[(1-\Delta)\delta f]$. There exists a scaling region where $\log_{10}[\mathcal{R}(\mu)]\sim\mu^{-b}$. The exponent b is not equal to unity and depends on parameters slightly although it is not far away from one. There are no great events which extend the whole system for smaller Δ, but for $\Delta\simeq 1$ there exists weight for the great events. This is simply because the system becomes stiff for $\Delta\simeq 1$ and the whole system tends to slip together.

We also have done the simulations with a different choice of ϕ which satisfies Eqs. (5) and (6), but the basic features described above are the same. In the present calculation, we employ the free boundary condition where no reflection occurs at the boundaries. Even if we use the periodic boundary condition, we do not expect any differences for small Δ because all the events are localized within the length smaller than the system size. For Δ close to one, however, the largest events can be much larger than the system size and travel around the whole system many times in the periodic boundary condition. This kind of situation does not seem to be physically interesting.

To summarize, being motivated by Carlson and Langer's work[2] on the mechanical model of earthquakes, we have constructed the cellular-automaton version of the model which shows similar behaviors to the BKCL model. Although the dynamics is deterministic and any randomness is not built in the model, it shows a rather irregular behavior and the magnitude distribution of events $\mathcal{R}(\mu)$ follows the power law μ^{-b} for small μ as is expected by the scaling argument.[2] The exponent b is close to one, which is consistent with the Gutenberg-Richter law, but this is not universal and depends slightly on the parameters.

Because the present cellular-automaton model is much simpler than the mechanical model, numerical simulation can be done easily. Note that the present model can be regarded as the $v_p\rightarrow 0$ limit in the BKCL model in the sense that the duration time of events is assumed to be zero. On the other hand, the smallest event is controlled through δf independently and this may have effects on the largest length scale as v_p does in the BKCL model.[2]

As Hwa and Kardar[6] pointed out, the conservation law is important for obtaining a scaling behavior. The present model does not conserve total force if $\Delta<1$, and this entails the existence of a largest length scale, but we can still have a scaling behavior in a small scale region.

Before concluding the report, let us discuss the relationship with other models. The model we studied here is similar to the one by Takayasu and Matsuzaki.[7] In their model, the f_i's are always set to be zero after slipping, namely, they took $\phi(x)=0$. This model shows only a trivial behavior without site dependent random increase rates of forces, with which they obtained a power law of magnitude distribution for two-dimensional systems. As for one dimension, however, it shows an exponential distribution and they concluded that the one-dimensional system with nearest-neighbor interaction never gives a power-law distribution. The present model has no intrinsic randomness, and randomness in configuration is self-generated by the function ϕ which leads to a power-law distribution even in one dimension

The sand-pile model by Bak and co-workers[1] is also very close to the present one. It can be derived from the present model by taking $\delta f=2$ and $\Delta=1$ and replacing the uniform increase of the f_i by stochastic discrete growing. In this model, the randomness is being supplied from outside. Although the original version of the sand-pile model in one dimension does not show a critical dynamics,[1] some variants have been demonstrated to develop a critical state even in one dimension.[8]

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[1]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988); J. Stat. Phys. **51**, 797 (1988).

[2]J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989).

[3]R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[4]The relevance of self-organized criticality to the earthquake problem has been disussed by several authors; A. Sornette and D. Sornette, Europhys. Lett. **9**, 197 (1989); K. Ito and M. Matsuzaki, J. Geophys. Res. (to be published); P. Bak and C. Tang, J. Geophys. Res. **94**, 15635 (1989).

[5]B. Gutenberg and C. F. Richter, Ann. Geofis. **9**, 1 (1956).

[6]T. Hwa and M. Kardar, Phys. Rev. Lett. **62**, 1813 (1989).

[7]H. Takayasu and M. Matsuzaki, Phys. Lett. A **131**, 244 (1988).

[8]L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).