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Brief Reports

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Self-organized criticality and fractal growth

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Growth processes are argued to develop self-organized critical states. This new characterization of growth phenomena yields insight into the origin of fractal pattern formation, and the associated exponents give information on scaling properties beyond that provided by the usual multifractal description. As a major example, the dielectric-breakdown η model is considered. The fractal dimension is estimated to be $D=\ln 3/\ln 2\simeq 1.585$ for $\eta=1$. This value is compared with results obtained for different geometries and with values found when lattice effects are present. Also, the limiting cases $\eta\rightarrow 0$ and $\eta\rightarrow\infty$ are discussed.

Spatial scaling structures originating from growth processes have been found to be extremely widespread in nature.[1] Careful experiments on viscous fingering,[2] dielectric breakdown,[3] and diffusion-limited growth[4] have been carried out, and various aggregation models[3,5–7] have been studied intensively in order to describe their fractal *outcome.* Still, there is a little understanding of the *evolution.* Dynamically, the interface is unstable and therefore triggered by fluctuations. The result is that the system eventually reaches a statistically stationary state where a rich ramified pattern is created. Since the state is an attractor of an intrinsic dynamics, it is called self-organized. A major observation is that this state can be described by power laws—the pattern eventually becomes scale invariant. Such a behavior is otherwise only well known from critical phenomena which occur at a *specific* value of a tuning parameter, e.g., temperature or magnetic field.

In this Brief Report, it is argued that growth processes *naturally* develop "critical" states. The analysis is based on the theory of branching processes,[8] and provides the "missing link" between fractal growth and self-organized critical phenomena, recently perceived by Bak *et al.*[9] in another class of coupled systems, also dynamically triggered by fluctuations. One example is a "sandpile," which is stable if the slope everywhere is less than or equal to a certain slope. Adding sand causes, on the one hand, the pile to grow; on the other hand, avalanches. The dynamically stationary state is obtained at the "critical" point where these two effects *exactly* balance. Since this critical state also is an attractor of an intrinsic dynamics, it is self-organized. The state is characterized by power-law distributions for the avalanches, both in lifetime and size. I shall argue that similar distributions characterize the self-organized critical growth phenomena.

One of the most studied growth models is the dielectric-breakdown model[3] (DBM) in two dimensions (2D) where (i) the Laplace equation $\nabla^2\phi=0$ is solved in the medium surrounding the cluster with boundary conditions $\phi=0$ on the cluster and $\phi=1$ on the surrounding boundary, (ii) the growth stochastically takes place at point i with probability

$$p_i\equiv\frac{|(\nabla\phi)_i|^\eta}{\sum_j |(\nabla\phi)_j|^\eta}\ , \tag{1}$$

where $(\nabla\phi)_i$ is the gradient of ϕ at i normal to the boundary. From the model presented here, the fractal dimension is found to be $D=\ln 3/\ln 2\simeq 1.585$ for $\eta=1$.

Consider first the formation of viscous fingers when one fluid displaces another fluid with higher viscosity. To understand why viscous fingers become scale invariant, one must follow the dynamical process that created them. Basically, (i) the flow can stop, (ii) the flow can continue, or (iii) the flow can branch, creating a new finger. However, eventually every finger *cannot* branch, since this would imply a *persistent* decrease of the average flow rate, and the system would never reach stationarity. Thus some of the fingers *must* stop growing. The system reaches stationarity exactly when successive branching has been broken down to the level where the flow barely survives. At this point extinction balances branching, and the growth process is stable with respect to fluctuations. It is in this sense that the dynamical stationary states for growth phenomena become critical. Common for the sandpile model and fractal growth, criticality expresses that information *only just* reaches infinity.

To be more specific, the flow above is dynamically modeled by the branching process where in each generation an "individual" is replaced by zero, one, or two descendants with probabilities C_0, C_1, and C_2, respectively [Fig. 1(a)]. On the average, the number of descendants increases by a factor of $C_1+2C_2=1+C_2-C_0$ from one generation to the next. At criticality where the family barely survives, $C_0=C_2$. At this point the structure of branches becomes scale invariant.[8,10] Removing the surviving paths (see below), the aggregate breaks into "subclusters" of extinct branches, corresponding to the avalanches for the sandpile. It is the distributions of the extinct branches, in size and lifetime, that describe the self-organized critical states for growth phenomena. The criticality ensures that the number of paths which eventually survives, also denoted the arms of the aggregate, is of order 1.

Experimentally, the surviving paths can be retrieved, observing where the interface moves in a small time period. If the growth probabilities p_i are known, the surviving paths can be found from multifractal analysis using the natural measure $m_i \equiv p_i$: The scaling properties of the measure with the size L of the aggregate are given by the distribution $N(\alpha)$ of the exponents $\alpha_i \equiv -\ln m_i/\ln L$, and the $f(\alpha)$ spectrum is defined by $f(\alpha)\equiv \ln N(\alpha)/\ln L$ in the limit of large L. Therefore the probability that the growth is governed by the exponent α is

$$P(\alpha)=N(\alpha)L^{-\alpha} \propto L^{f(\alpha)-\alpha} . \tag{2}$$

By integration,

$$1=\int P(\alpha)d\alpha \propto \int L^{f(\alpha)-\alpha}d\alpha \propto L^{f(\alpha^*)-\alpha^*} , \tag{3}$$

where α^* is defined by steepest descents, $f'(\alpha^*)=1$. From (3), $D_1 \equiv f(\alpha^*)=\alpha^*$ while $f(\alpha)$ must be less than α for all other values of α. Hence, $P(\alpha^*)=\delta(\alpha-\alpha^*)$ in the large-L limit, which is why D_1 is called the information dimension (for the natural measure). It is generally accepted[11] that $D_1=1$ for $\eta=1$. Analogous to (3) one has $\alpha^*=\int \alpha P(\alpha)d\alpha$, or for a finite system,

$$\ln p^* = \sum_i p_i \ln p_i , \tag{4}$$

where p^* is the probability associated with α^*. The surviving paths are those ending in points with $p_i \geq p^*$.

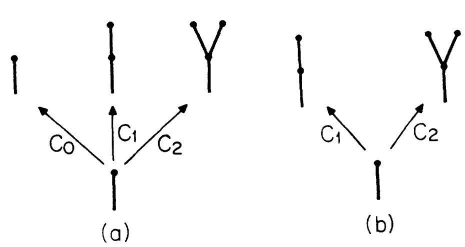

FIG. 1. Formation of spatial scaling structure described by branching processes. (a) Criticality appears for $C_0=C_2$. (b) From fine graining (Ref. 12).

From the description of the DBM as a critical branching process, information on the fractal dimension can be derived, using the fine graining along a one-dimensional cut through the cluster, recently introduced by Pietronero *et al.*[12] A cell is called full if it belongs to the cluster. Dividing all cells into two, a full cell is replaced by one or two full cells. The associated branching process is shown in Fig. 1(b), where C_1' and C_2' denote the probabilities that a full cell by fine graining is replaced by one or two, respectively. For our description to hold true, this fine graining must resemble a critical process; since by this procedure one cannot distinguish extinction from surviving without branching, one has the correspondence $C_1'=C_0+C_1$ and $C_2'=C_2$ to a critical branching process [Fig. 1(a)] where $C_0=C_2$. By fine graining the number of full cells in average increases by a factor $C_1'+2C_2$. The fractal dimension D of the cluster is

$$D=D_c+1 , \tag{5a}$$

where D_c is the fractal dimension of the one-dimensional intersection set, given by[12]

$$C_1'+2C_2=2^{D_c} . \tag{5b}$$

However, also the information dimension D_1 can be related to C_1' and C_2, remembering that in the large L limit almost all growth takes place on points with $\alpha=\alpha^*=D_1$, i.e., on sites with growth probability p^*. By fine graining, or equivalently by the branching process, *this must not change.* Thus

$$\ln(p^*|_{2L})=C_1'\ln(C_1'p^*|_L)+C_2\ln(C_2p^*|_L) , \tag{6}$$

i.e.,

$$D_1\ln 2=C_1'|\ln C_1'|+C_2|\ln C_2| . \tag{7}$$

Based on the observation that the tips of the cluster are more likely to grow when η is increased, the amount of branching, i.e., C_2, will decrease. In particular, when $\eta\rightarrow\infty$, $C_2\rightarrow 0$. Consequently, $C_1\rightarrow C_1'\rightarrow 1$, $D\rightarrow 1$, and $D_1\rightarrow 0$. Conversely, when η is *lowered* from infinity, C_2 and [by (7)] D_1 will *increase* from zero, reaching their *maximum* values $C_2=\frac{1}{2}$ and $D_1=1$ at $\eta=1$. Thus, for $\eta=1$, $C_1=0$, $C_0=C_1'=C_2=\frac{1}{2}$, and by (5),

$$D=\ln 3/\ln 2 \simeq 1.585 . \tag{8}$$

We notice that since $C_2=C_0$ must be obeyed for our simple branching model [Fig. 1(a)], C_2 cannot increase further as η becomes smaller than 1. To increase the value of C_2 loops must be created. If Ω denotes the fraction of loops per particle ($0\leq\Omega\leq 1$), the critical condition becomes $C_2=C_0+\Omega$.[13] Hence the presence of loops allows D to increase beyond the value given in (8). It is not clear how the relation (7) will change when loops are taken into account.

For $\eta=1$ and $\eta=0$, D_1 is known to be $D_1=1$. For $0<\eta<1$, one has $D_1\geq 1$. This follows from the relation between the $f(\alpha)$ spectrum for the *natural* measure and the spectrum for the *harmonic* measure $m_H \equiv |(\nabla\phi)_i| / \sum_j |(\nabla\phi)_j|$. The former spectrum is obtained from the latter by a contraction in the α direction by a factor of η, and a translation.[14,15] Thus the information dimension D_1 for the natural measure can be determined from the spectrum, $f_H(\alpha)$, for the harmonic measure as the value of $f_H(\alpha)$, where $f'_H(\alpha)=\eta$. The information dimension D_H for the harmonic measure equals the value of $f_H(\alpha)$ where $f'_H(\alpha)=1$. Since $D_H=1$ for the DBM independent of η,[11] the convexity of the spectrum ensures that $D_1 \geq 1$ for $\eta < 1$ (and $D_1 \leq 1$ for $\eta > 1$).

Extensive numerical simulations of the DBM in a cylindrical geometry have been carried out by Evertsz.[15] For $0 \leq \eta \leq 2$ the dimension D_c for a one-dimensional intersection set was determined at various heights from the basic growth line. After an initial growth region the dimension stabilizes at a value $D_c = 0.59 \pm 0.01$. The corresponding value $D=1.59$ is in perfect agreement with (8). For $\eta < 1$ an increasing value of D_c and thereby D is found. However, in accordance with the discussion above the presence of loops is clear for $\eta < 1$.

In a circular geometry, the behavior of D_c along a circular cut has not been studied. For off-lattice diffusion-limited aggregation (DLA), which is typically identified with the DBM for $\eta=1$,[12] the fractal dimension is found to be $D \simeq 1.7$,[16] which clearly is above the value in (8). In contrast, on a square lattice a crossover to a star-shaped object with four arms is observed at very large cluster sizes.[17] The crossover can be observed at smaller cluster sizes if noise reduction is introduced. The fractal dimension changes from a value[5] $D \simeq 1.67$, which is close to the off-lattice value, to a value[18] $D \simeq 1.57$ that is in good agreement with (8).

In conclusion, branching processes have been used to probe the underlying mechanism for fractal growth. The general arguments suggest that growth processes may evolve toward self-organized critical states, where the extinction *precisely* balances the branching. The critical state is characterized by exponents, which gives information on scaling properties beyond that provided by the $f(\alpha)$ spectrum. In particular, based on a model where branching processes develop independently, the fractal dimension is estimated to be $D=\ln 3/\ln 2$ for the DBM with $\eta=1$. This value is in excellent agreement with the dimension found in a *cylindrical* geometry, but does not agree with the value found for off-lattice DLA in a *circular* geometry. It would be very interesting to find the behavior of D_c in the latter case, and I urge studies in that direction.

The branching process in Fig. 1(a) has been used to describe the mean-field dynamics for self-organized critical phenomena.[10] However, the loops might change the critical exponents. To this end, we have for various growth models undertaken studies on the distributions of these branches as regard their sizes and associated lifetimes.[19] It will be interesting to see the extent to which the new critical exponents characterize growth phenomena.

Finally, it is noticed that the critical exponents are *experimentally accessible.* One can even estimate the value of p^*, since the small growth probabilities do not contribute to p^*, and the large probabilities p_i can be obtained by measuring the growth g_i along the interface in a small time period. Then $p_i \simeq g_i / \sum_j g_j$, p^* can be calculated from (4), and the surviving branches can be identified.

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