Ruelle's rotation frequency for a symplectic chain of dissipative oscillators

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We present the calculation of Ruelle's rotation number (frequency) Ω for a chain of damped and driven coupled Duffing oscillators using the recently discovered symplectic structure of the system. A relation between the eigenvalues of periodic orbits and the winding number $|\Omega|/\omega$ is conjectured by making use of their Krein signatures. The occurrence and nonoccurrence of plateaus with respect to the driving frequency ω can be explained via this relation.

I. INTRODUCTION

The investigation of nonlinear dynamical systems has become a research area of increasing interest among physicists. The availability of high-speed computing devices including graphics has helped considerably in understanding basic features of nonlinear dynamical systems. Facts and data can be compiled for various systems and hints gathered for general laws to be handed over to the mathematician for proof. Therefore the numerical study of model systems has become a common practice and serves as starting point also in this article.

For dissipative dynamical systems interest is centered around the characterization of limit sets, mainly their attractors. In this context the calculation of ergodic quantities, especially the Lyapunov exponents, entropy, and information dimension, shall only be mentioned.[1]

For Hamiltonian dynamical systems—which do not have attractors—the interest is centered around ergodic (and similar) properties of their trajectories. For these systems, but extending to a wider subclass of dynamical systems which display a symplectic structure, Ruelle has defined a rotation number.[2] It has not yet been widely used, except for two-dimensional mappings,[3] and, through an independent approach, for driven dissipative single oscillators.[4–7] In the latter case a quantity called torsion frequency has been introduced. In the Appendix we show that this torsion frequency coincides with Ruelle's rotation number. The investigations of driven single oscillators have shown that the torsion frequency (i.e., Ruelle's rotation number) helps in understanding some bifurcation scenarios of the oscillators.[4,5]

Having these successes for single oscillators in mind, we approach the determination of Ruelle's rotation number for high-dimensional systems in the hope of rewarding results. In this article we present the numerical investigation of Ruelle's rotation number[2] for a chain of damped and driven Duffing oscillators. In particular, we study the connection between the eigenvalues of a periodic point $\mathbf{x}$ and the associated rotation number $\Omega(\mathbf{x})$. This study will lead to a deeper understanding of the meaning and relevance of Ruelle's rotation number for coupled oscillators, especially near bifurcation points.

Ruelle's rotation number is essentially based on the symplectic structure of Hamiltonian systems [see definitions (19) and (20)]. At first sight, Hamiltonian structure and dissipation seem to exclude each other. The reason the rotation number is defined for the chain of oscillators that we will introduce is that this chain belongs to a class of dissipative systems which preserve essential features of Hamiltonian systems.[8] This was discovered independently by Valkering *et al.*, who called these systems "dissipative Hamiltonian systems."[9]

The equations of motion for the system of N oscillators investigated can be written

$$\dot{q}_i=\frac{\partial H(t,\mathbf{q},\mathbf{p})}{\partial p_i},\quad \dot{p}_i=-\frac{\partial H(t,\mathbf{q},\mathbf{p})}{\partial q_i}-dp_i\ ,\qquad i=1,\ldots,N \tag{1}$$

the Hamiltonian H being periodic in time, i.e., $H(t,\mathbf{q},\mathbf{p})=H(t+T,\mathbf{q},\mathbf{p})$ for some period $T\neq 0$. [The Hamiltonian of the coupled oscillators considered is given explicitly in Sec. IV, Eq. (23).] Thus the equations of motion are closely related to the Hamiltonian equations. The only difference is the linear damping term with the damping constant d. If d goes to zero, the Hamiltonian equations are recovered. It can be shown[8,10] that this simple viscous damping does not destroy the symplectic structure of the system.

We now recall the notion of symplecticity and its modification to dissipative Hamiltonian systems. Introducing the matrix

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{2}$$

where each entry in J is an $N\times N$ block and 1 is the $N\times N$ identity matrix, Eq. (1) can be written more compactly as

$$\dot{\mathbf{x}}=J\nabla_{\mathbf{x}}H(t,\mathbf{x})-d(\mathbf{0},\mathbf{p})^{\mathrm{tr}}\equiv\mathbf{v}(t,\mathbf{x})\ ,\qquad \mathbf{x}=(\mathbf{q},\mathbf{p})^{\mathrm{tr}}\in\mathbb{R}^{2N} \tag{3}$$

$\nabla_{\mathbf{x}}H$ denoting the gradient of H with respect to $\mathbf{x}$. The presence of J is the hallmark of the "symplectic property."

Definition 1. A $2n \times 2n$ matrix A is said to be symplectic if and only if

$$AJA^{\mathrm{tr}} = J \tag{4}$$

holds. J is called the symplectic tensor.

For Hamiltonian systems it is well known[11] that the linearization of the flow is a symplectic matrix. For dissipative dynamical systems of the general form (3), we can find a generalization of the symplecticity relation (4) [see Eq. (7)].[8]

To state this result, we have to put forward some notation. Due to the periodic driving, the nonautonomous system (3) is equivalent to an autonomous system with phase space $M = S_T^1 \times \mathbb{R}^{2N}$. This latter is obtained in the usual way by adding the equation $\theta = 1$, i.e., by introducing time as new explicit state variable,

$$\begin{aligned} &\dot{z} = (\dot{\theta}, \dot{\mathbf{x}}) = \bar{\mathbf{v}}(z) = (1, \mathbf{v}(z)) , \\ &z = (\theta, \mathbf{x}) \in S_T^1 \times \mathbb{R}^{2N} \equiv M , \\ &\bar{\mathbf{v}}(z) \in T_z M . \end{aligned} \tag{5}$$

The circular component $S_T^1 = \mathbb{R}(\mathrm{mod}T)$ reflects the periodicity of the driving. Let ϕ^t be the flow map of this autonomous system. The periodic driving gives rise to a global Poincaré map P with respect to a Poincaré hyperplane $\Sigma(t_0)$ (Poincaré plane for short),

$$\Sigma(t_0) \equiv \{(t_0, \mathbf{x}) \in M \mid \mathbf{x} \in \mathbb{R}^{2N}\} . \tag{6}$$

P is then the stroboscopic flow map restricted to the Poincaré plane, i.e., $P = \phi^T|_{\Sigma(t_0)}$ (for details, see Refs. 8 and 12).

The linearization of $\phi|_{\Sigma(t_0)}$ with respect to a point $\mathbf{x}$ of the Poincaré plane $\Sigma(t_0)$ is denoted $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$. For the system introduced, $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ is represented by a $2N \times 2N$ matrix. Therefore, let $\mathbf{y} \in \mathbb{R}^{2N}$ be an infinitesimal perturbation of a point $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ of the nonautonomous system at some time t_0. Then, after time t, this perturbation will be given by $\mathbf{y}(t) = D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}\mathbf{y}$ in the linear approximation. Taking as time t the period T of the driving, one gets the linearization of the Poincaré map, i.e., $D_{\mathbf{x}}\phi^T|_{\Sigma(t_0)} = D_{\mathbf{x}}P$.

The dissipative system introduced preserves the symplectic structure because the following generalized symplecticity relation holds:[8]

$$D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)} J (D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})^{\mathrm{tr}} = e^{-dt} J \quad \forall t \in \mathbb{R} . \tag{7}$$

Thus the linearization with respect to a Poincaré plane of the flow map after some time t is symplectic times the scalar e^{-dt}. If d goes to zero, this scalar goes to 1 and we recover the symplecticity of Hamiltonian systems. Note that

$$S(t) \equiv e^{dt/2} D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)} \tag{8}$$

is symplectic in the usual sense. As a matter of fact, $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ is, times a scalar, a symplectic matrix. Therefore the geometric properties of symplectic matrices, i.e., the properties of eigenspaces and eigenvalues, which are investigated by the theory of symplectic geometry,[13] are essentially preserved. Perhaps the most striking result is the following.

Lemma 1. Let A be a $2N \times 2N$ matrix fulfilling

$$AJA^{\mathrm{tr}} = \lambda J, \quad \lambda \neq 0 .$$

Then, μ being an eigenvalue of A, λ/μ is an eigenvalue of A as well.

As a result of (7), many concepts developed for Hamiltonian systems can be applied to coupled oscillators. In fact, our analysis will show that understanding essential features of the behavior of the oscillators is only possible by referring to their symplectic structure. For example, we could show that the Lyapunov spectrum with respect to the Poincaré plane has the symmetry point $-d/2$ or, equivalently, the Lyapunov spectrum of the Poincaré mapping has the symmetry point $-dT/2$.[8] For $d = 0$ the center of symmetry of Hamiltonian systems is contained in this relation. Another consequence and the topic of this article is that the concept of Ruelle's rotation number can be transferred to damped and driven chains of oscillators.

In this article we investigate the meaning of Ruelle's rotation number. For symplectic high-dimensional systems, this number has been found to be a generalization of the torsion frequency well investigated for single oscillators.[4,5] At first sight, the surprising fact about this generalization is that also for high-dimensional systems, e.g., N oscillators, $N > 1$, it is only one rotation number which is associated with the system. Our analysis will elucidate this fact. Furthermore, we will conjecture a relation between the eigenvalues of $D_{\mathbf{x}}P^m$ and Ruelle's rotation number for a periodic point $P^m(\mathbf{x}) = \mathbf{x}$. This relation will make use of the Krein signatures, which give further information about the stability of symplectic systems.[14,13]

The paper is organized as follows. Before the rather abstract definition of Ruelle's rotation number for the high-dimensional case is given in Sec. III, its meaning for a single periodically driven oscillator is discussed. The geometrical meaning of the rotation number can easily be seen in this case of a two-dimensional Poincaré plane. In Sec. III Ruelle's rotation number is introduced in the context of a chain of oscillators. In Sec. IV we give the detailed equations of motion and present numerically calculated rotation numbers. With dependence on the external driving frequency ω, we calculate the associated winding number $|\Omega(\mathbf{x})|/\omega$ for periodic points $\mathbf{x}$. The calculation will demonstrate that the winding number shows plateaus for some bifurcations, while for others it does not. In Sec. V we analyze the special properties of the eigenspaces and eigenvalues of periodic orbits and give a relation between the eigenvalues and the winding number using their Krein signatures. In Sec. VI we test the given relation numerically. The most important verification of the conjectured relation will be that it will enable us to explain the occurrences and nonoccurrences of plateaus of the winding number.

II. TORSION FREQUENCY FOR SINGLE OSCILLATORS

For a single oscillator, Ruelle's rotation number turns out to be equal to the torsion frequency investigated in

Refs. 4 and 5. The proof for its equivalence is given in the Appendix. The torsion frequency was first introduced for periodic orbits to count the twists of the invariant manifolds of periodic attractors.[6,7] A related approach is the following. For a given periodic orbit, nearby starting trajectories are more or less twisted around it. Figure 1 shows a typical situation. The rotation of a neighbor trajectory starting at $(t_0, \mathbf{x}+\mathbf{y})$ in the same Poincaré plane $\Sigma(t_0)$ as $(t_0,\mathbf{x})$ is contained, in a linear approximation, in the linearization $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ of the flow with respect to $\Sigma(t_0)$. The torsion frequency gives the averaged radiant frequency of the rotation of the tangent vector $\mathbf{y}(t)=D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}\mathbf{y}$ around the reference orbit $(t_0+t, \mathbf{x}(t_0+t))$ [with $\mathbf{x}(t_0)=\mathbf{x}$] as time t goes to infinity.

Formally, the torsion frequency can be defined using the QR decomposition of the linearization $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$,[4]

$$D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}=Q(t)R(t) , \tag{9}$$

where $Q(t)$ is an orthogonal matrix and $R(t)$ an upper triangular matrix with positive diagonal elements. The QR decomposition is also used for the calculation of the Lyapunov spectrum. The Lyapunov exponents are determined by the positive diagonal elements of R as time t goes to infinity.[1] The orthogonal matrix Q contains the information about the torsion. For orientation-preserving flows, $\det D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ is always greater than 0 and therefore $Q(t)$ is a pure rotation, i.e., $\det Q(t)=1$ holds. Thus $Q(t)$ can be written as

$$Q(t)=\begin{bmatrix}\cos\Theta(t) & -\sin\Theta(t)\\ \sin\Theta(t) & \cos\Theta(t)\end{bmatrix} . \tag{10}$$

The angle Θ is uniquely determined up to a multiple of 2π. Setting $\Theta(0)=0$ (because $D_{\mathbf{x}}\phi^0|_{\Sigma(t_0)}=1$) and following $\Theta(t)$ continuously, the *torsion frequency* $\Omega(\mathbf{x})$ is defined by

$$\Omega(\mathbf{x})=\lim_{t\to\infty}\frac{\Theta(t)}{t} . \tag{11}$$

For single oscillators one can derive a differential equation for Θ and thus calculate $\Omega(\mathbf{x})$ just by solving a differential equation. This approach can be found in Ref. 5.

The above-defined torsion frequency was interpreted by Parlitz and Lauterborn as the (orbit-dependent) eigenfrequency of the nonlinear oscillator.[4] Furthermore, they used the torsion frequency to give a new definition for a (generalized) winding number for driven dissipative nonlinear oscillators that does not require the existence of an invariant torus. It coincides with the usual definition as long as an invariant torus exists. The *generalized winding number* w is defined by

$$w(\mathbf{x})\equiv\frac{|\Omega(\mathbf{x})|}{\omega}=\frac{|\Omega(\mathbf{x})|T}{2\pi} , \tag{12}$$

where $\omega=2\pi/T$ is the driving frequency. The winding number $w(\mathbf{x})$ describes the averaged rotation angle of a nearby trajectory during one period T of the excitation in units of 2π. For periodic oscillations with period mT [i.e., $P^m(\mathbf{x})=\mathbf{x}$], the *torsion number* n is defined by

$$n(\mathbf{x})=mw(\mathbf{x})=\frac{|\Omega(\mathbf{x})|mT}{2\pi} , \tag{13}$$

i.e., it gives the averaged rotation angle during one period mT of the oscillation in units of 2π. Parlitz and Lauterborn showed that the torsion number is a suitable quantity to classify saddle-node and period-doubling bifurcation curves (surfaces) in the parameter space of one-dimensional driven dissipative oscillators. Furthermore, they related the bifurcation superstructure of single driven oscillators to their resonances.[4,5,15]

For coupled oscillators the phenomenon of resonance also has much impact on their bifurcation structure (see, e.g., Ref. 16). Therefore the study of the local twist of the flow is also of interest for the high-dimensional case. For N oscillators the Poincaré plane $\Sigma(t_0)$ is $2N$ dimensional. The description of rotation is therefore much more complicated. Thus the above-described concept of torsion frequency cannot straightforwardly be extended to the case of coupled oscillators. For periodic orbits there seems to exist a natural extension by looking at the eigenspaces of $D_{\mathbf{x}}P^m$ to define a set of N torsion frequencies.[17,18] This approach would generalize the ideas in Refs. 19 and 7 and others where the torsion frequency of a single oscillator was introduced by looking at the twists of the invariant manifolds of a periodic orbit.

In the following we will investigate Ruelle's rotation number for high-dimensional symplectic systems. It may be considered a certain generalization of the torsion frequency concept, although there are some differences in the approach. (The torsion-frequency approach does not start from symplecticity, but is an operational approach which coincides—in the limit $t\to\infty$—with Ruelle's approach.)

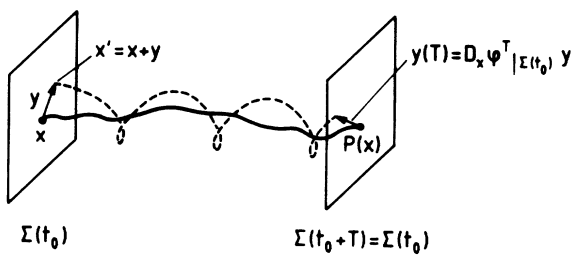

FIG. 1. A neighbor trajectory (dashed line) is twisted around a reference orbit (bold line) due to the torsion of the local flow. The rotation frequency gives the radiant frequency of this twisting for time t to infinity.

III. RUELLE'S ROTATION NUMBER (FREQUENCY)

In this section we present the definition of Ruelle's rotation number formulated for a chain of driven oscillators. By means of the polar decomposition of the linearization,

$$D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}=Q(t)P(t) , \tag{14}$$

one obtains a positive matrix

$$P(t)=(D_{\mathbf{x}}\phi^t|^{\mathrm{tr}}_{\Sigma(t_0)}D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})^{1/2}$$

and an orthogonal matrix $Q(t)$. The generalized symplecticity relation (7) implies that $Q(t)$ and $P(t)$ are, in addition, symplectic and symplectic times a scalar, respectively. The orthogonal matrix $Q(t)$ now enables one to associate an angle $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ with the linearization $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$. The angle comes in because the symplectic and orthogonal real $2n\times 2n$ matrices are strongly related to the unitary group $\mathrm{U}(n)$. This connection is given in the following. Because of symplecticity the orthogonal matrix $Q(t)$ has the form

$$Q(t)=\begin{bmatrix} V(t) & -W(t) \\ W(t) & V(t) \end{bmatrix}, \tag{15}$$

where $V(t)$ and $W(t)$ are real $n\times n$ matrices. If one defines, with these matrices, the complex matrix

$$U(Q(t))\equiv V(t)+iW(t)\ , \tag{16}$$

then $U(Q)$ is an $n\times n$ unitary matrix, $U(Q)\in\mathrm{U}(n)$.

Vice versa, it holds that for every unitary matrix $U\in\mathrm{U}(n)$ the related real $2n\times 2n$ matrix $Q(U)$, defined by

$$Q(U)\equiv\begin{bmatrix} \mathrm{Re}(U) & -\mathrm{Im}(U) \\ \mathrm{Im}(U) & \mathrm{Re}(U) \end{bmatrix}, \tag{17}$$

is orthogonal and symplectic, i.e., $Q(U)\in\mathrm{Sp}(2n,\mathbb{R})\cap\mathrm{O}(2n)$. Furthermore, it holds that

$$\begin{aligned} &Q(U_1U_2)=Q(U_1)Q(U_2)\quad \forall U_1,U_2\in\mathrm{U}(n)\ , \\ &U(Q_1Q_2)=U(Q_1)U(Q_2)\quad \forall Q_1,Q_2\in\mathrm{Sp}(2n,\mathbb{R})\cap\mathrm{O}(2n)\ . \end{aligned} \tag{18}$$

Equations (16)–(18) express that the symplectic and orthogonal matrices $\mathrm{Sp}(2n,\mathbb{R})\cap\mathrm{O}(2n)$ are a $2n$-dimensional real representation of the unitary group $\mathrm{U}(n)$.

By means of this connection, the angle Θ associated with the linearization $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ arises. Taking the determinant of the uniquely related matrix $U(Q(t))$, one gets

$$\det U(Q(t))\equiv\exp[i\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})]\ . \tag{19}$$

The angle $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ is uniquely defined up to a multiple of 2π. If one keeps track of these multiples (starting at time $t=0$ with 0), then the *rotation number* $\Omega(\mathbf{x})$ is defined by

$$\Omega(\mathbf{x})\equiv\lim_{t\to\infty}\frac{\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})}{t}\ . \tag{20}$$

Ruelle has proven that this quantity exists in the same sense as the Lyapunov spectrum exists, i.e., $\Omega(\mathbf{x})$ exists for μ-almost all $\mathbf{x}$ and, if μ is ergodic, $\Omega(\mathbf{x})$ is μ-almost everywhere equal to

$$\Omega(\mu)=\int\mu(d\mathbf{x})\Omega(\mathbf{x})\ . \tag{21}$$

Thus, for symplectic systems, it is another ergodic quantity which can be associated with an attractor. It has been called the rotation number by Ruelle. In order to distinguish this quantity, which has the dimension of a frequency, from other numbers (winding number, torsion number), we prefer to call it "*rotation frequency.*" In the Appendix we prove that the rotation frequency equals the torsion frequency for single driven oscillators.

The definition of the rotation frequency is strongly related to the topology of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. One can show that the group $\mathrm{Sp}(2n,\mathbb{R})$ is, topologically speaking, the product of S^1 and a simply connected space. This is done by use of the polar decomposition and an analysis of the Lie algebra of symplectic matrices. From an abstract point of view, the rotation frequency gives the averaged frequency of the curve defined by $\gamma(t)=D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ going around the hole. An excellent presentation of these arguments can be found in the Appendix of Ref. 20. Furthermore, the connection between the rotation number and Maslov indices which arise in the semiclassical quantization condition of Brillouin, Keller, and Einstein can be found in Ref. 20 and 21. This topological background is the reason that the mathematically correct definition of the rotation frequency has to be formulated by a function on the universal covering group of $\mathrm{Sp}(2n,\mathbb{R})$, as Ruelle does in Ref. 2. For the purposes of a concrete calculation of the rotation frequency, however, it is enough to remind the reader that one has to keep track of multiples of 2π when counting the angle Θ.

IV. NUMERICAL CALCULATION OF THE ROTATION FREQUENCY

In this section we first introduce the chain of oscillators for which all numerical calculations were performed. It is a ring of $N=4$ mass points with nearest-neighbor coupling where the interaction potential Φ (realized by a spring) is given by a single-well Duffing potential, in our case $\Phi(x)=x^2/2+x^4/4$. The chain is excited at one mass point by a periodic external force proportional to the space coordinate $y(t)$ of the limit cycle of the van der Pol oscillator,

$$\ddot{y}+(y^2-1)\dot{y}+\omega_0^2 y=0\ . \tag{22}$$

This driving is used with a view toward biological applications where an active oscillator may be connected to passive ones. This driving also shows that the investigations do not depend on simple sinusoidal excitation.

Thus the Hamiltonian H, which, with Eq. (3), gives the equations of motion, has the form

$$H(t,q_1,\ldots,q_N,p_1,\ldots,p_N)=H(t,\mathbf{q},\mathbf{p})=\sum_{i=1}^{N}\frac{p_i^2}{2}+\sum_{i=1}^{N}\Phi(q_i-q_{i-1})+fy(t)q_1\ . \tag{23}$$

$\mathbf{q}\in\mathbb{R}^N$ is the tuple of the displacements of the N oscillators and $\mathbf{p}\in\mathbb{R}^N$ is the momentum vector of the oscillators. The periodic boundary conditions can be expressed as $q_0=q_N$, $q_{N+1}=q_1$. The Hamiltonian considered is periodic in time with the period $T=T(\omega_0)$ of the van der Pol oscillator, i.e., $H(t+T,\mathbf{q},\mathbf{p})=H(t,\mathbf{q},\mathbf{p})$. The parameters we will vary are f, which measures the strength of the excitation, and ω_0, which is approximately equal to the frequency $\omega=2\pi/T$ of the driving for $\omega_0>0.5$.[22]

We now want to describe the numerical procedure we used to calculate the rotation frequency. In order to follow the angle $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$, the trajectory $\phi^t(t_0,\mathbf{x})$ has to be discretized, e.g., in n steps with discretization time τ, i.e., $t=n\tau$. Let t_k be the time and $\mathbf{x}_k$ the point of the nonautonomous systems after k steps, i.e., $t_k=t_0+k\tau$ and $\mathbf{x}_k=\pi\circ\phi^{k\tau}(t_0,\mathbf{x})$ [π being the projection on the second argument, i.e., $\pi(t,\mathbf{x})=\mathbf{x}$]. With this notation the positive matrices P_k and orthogonal matrices Q_k are recursively defined by

$$D_{\mathbf{x}_k}\phi^{\tau}|_{\Sigma(t_k)}P_k=P_{k+1}Q_{k+1}\ , \tag{24}$$

starting the polar decomposition with $P_0=1$. This recursion relation is resolved to

$$D_{\mathbf{x}}\phi^{n\tau}|_{\Sigma(t_0)}=P_nQ_n\cdots Q_2Q_1\ . \tag{25}$$

This follows from (24), but can most easily be seen in Fig. 2, from which the idea behind (24) becomes clear. To get the iteration scheme (25), we use a slightly different type of the polar decomposition [than that stated in (14)], the "PQ decomposition," i.e., $A=PQ$ and $P=(AA^{\mathrm{tr}})^{1/2}$, but this is justified because it is easy to see that the polar decomposition and the PQ decomposition lead to the same orthogonal matrix Q, and thus to the same angle Θ that we want to compute. In the following we refer to the PQ decomposition as polar decomposition too.

Because of the uniqueness of the polar decomposition of $D_{\mathbf{x}}\phi^{n\tau}|_{\Sigma(t_0)}=P(n\tau)Q(n\tau)$, it holds that

$$\begin{aligned}P(n\tau)&=P_n\ ,\\ Q(n\tau)&=Q_n\cdots Q_2Q_1\ .\end{aligned} \tag{26}$$

Therefore the total rotation is decomposed into successive small rotations. If n is chosen large enough and the discretization time τ small enough, then the absolute value of the rotation angle $\Theta(Q_i)$ will be less than 2π. Because of the representation relation (18) and the definition of the angle Θ, (19), the angle after time $n\tau$ will be given by

$$\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})=\Theta(Q(n\tau))=\sum_{i=1}^{n}\Theta(Q_i)\ . \tag{27}$$

If n is chosen too small, then relation (27) will only be fulfilled up to a multiple of 2π. The best criterion to know that the discretization time is small enough to really keep track of all the rotations is that the right-hand side (rhs) of (27) has to become independent of n for increasing n.

When performing our first calculations of $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$, we noticed that with decreasing discretization time τ the points $e^{i\Theta(Q_i)}$ on the complex unit circle belonging to the angles $\Theta(Q_i)$ were all lying below the real axis. Therefore, when counting $\Theta(Q_i)$ positive, i.e., $\Theta(Q_i)\in[0,2\pi)$, the $\Theta(Q_i)$ would increase with decreasing discretization time τ. In contrast, the absolute value of the angle $\Theta(Q_i)$ counted negative, i.e., $\Theta(Q_i)\in(-2\pi,0]$, would decrease with decreasing step size τ, as it must. Therefore the sum in (27) only became constant for increasing n when counting the angle $\Theta(Q_i)$ negative.

Let us remark that this is known for single oscillators. There the torsion frequency is necessarily always negative. This is an artifact of the chosen coordinate system in the Poincaré plane. Choosing the coordinate system $(\mathbf{p},\mathbf{q})$ instead of $(\mathbf{q},\mathbf{p})$, we would obtain a positive rotation frequency, as one can convince oneself of for the single oscillator if one takes the meaning of p i.e., $(p=\dot{q})$ into consideration.

This and the numerical tests suggest that also in the case of coupled oscillators the rotation angles will always be negative (in the coordinate frame chosen). Therefore we count the calculated angle negative, i.e., $\Theta(Q_i)<0$.

As a criterion for the discretization time τ being small enough, we increased for the first period T of excitation the number of steps n till

$$|\Theta(Q_i)|<\pi\quad\forall\ i=1,\ldots,n \tag{28}$$

holds with $T=n\tau$ and $Q(T)=Q_n\cdots Q_1$.

For the calculation of $\Theta(D_{\mathbf{x}}\phi^{mT}|_{\Sigma(t_0)})$ of the remaining m periods, the discretization time τ was then kept fixed.

Before we present examples of numerically calculated rotation frequencies, we discuss the principal numerical difficulties which go along with the procedure described. The calculation of $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ contains an explicit determination of $P(t)$ [see Eqs. (25) and (26)], but the eigenvalues of $P(t)$ increase (or decrease) asymptotically like $e^{\lambda_i(\mathbf{x})t}$, where $\lambda_i(\mathbf{x})$ are the Lyapunov exponents. Therefore the well-known difficulties of numerical instability are recovered that would also arise when calculating the Lyapunov exponents without using a Gram-Schmidt reorthonormalization or the equivalent iterated QR decomposition.[1,23]

One is tempted to circumvent these difficulties by using a (numerically stable) iteration scheme analogous to the iterated QR decomposition, i.e.,

$$D_{\mathbf{x}}\phi^{n\tau}|_{\Sigma(t_0)}=Q_nP_n\cdots P_1\ . \tag{29}$$

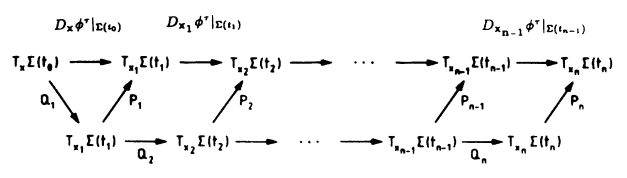

FIG. 2. Iteration scheme (24) used to compute Ruelle's rotation frequency.

The drawback of this (numerically stable) decomposition is that it does not lead to a polar decomposition because the product of two positive matrices is, in general, not a positive matrix again. So, in contrast to the upper triangular matrices R, the positive matrices do not form a group.

Using an iterated QR decomposition (which leads to a real QR decomposition),

$$D_{\mathbf{x}}\phi^{n\tau}|_{\Sigma(t_0)}=Q_nR_n\cdots R_1=Q(n\tau)R(n\tau) \tag{30}$$

does not help either because, for general dimensions, the matrices R_i are not symplectic. Thus the orthogonal matrix Q_n is not symplectic and therefore not connected in the described way to an angle $\Theta(Q_n)$. To conclude, at the moment there seems to exist a principal problem in calculating $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ for arbitrarily long times t using a discrete procedure. Therefore the rotation frequency $\Omega(\mathbf{x})$ cannot be computed with arbitrary accuracy. For nonperiodic orbits the results are not satisfying, but for periodic orbits one can estimate the limit $\Omega(\mathbf{x})$ from $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ well enough to investigate $|\Omega|$ with dependence on the driving frequency ω.

The numerical calculation of $D_{\mathbf{x}_k}\phi^{\tau}|_{\Sigma(t_k)}$ by integrating the variational equations is described in full detail in Ref. 8. In Fig. 3(a) we show for $N=4$ oscillators the absolute value of

$$\Omega(\mathbf{x},mT)\equiv\frac{\Theta(D_{\mathbf{x}}\phi^{mT}|_{\Sigma(t_0)})}{mT} \tag{31}$$

plotted against the periods m of excitation. For the chosen parameter values ($f=0.05$, $d=0.2$, $\omega_0=2.1$), the Poincaré mapping has a fixed point. At the beginning, $\Omega(\mathbf{x},mT)$ approaches the limit quite well, but at $m\approx 110$ the calculation obviously goes wrong. The numerical difficulties become evident. The eigenvalues of $P(mT)$ differ too much. Therefore the polar decomposition (which was carried out by a singular value decomposition routine of the package of computer codes LINPACK) cannot be done accurately. To stop the computation before this event and to estimate the limit $\Omega(\mathbf{x})$, we used our knowledge about the matrix

$$P(t)=(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}D_{\mathbf{x}}\phi^t|^{\mathrm{tr}}_{\Sigma(t_0)})^{1/2}\ .$$

Due to the symplecticity relation (7) of $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$, $P(t)$ fulfills the same relation with scalar e^{-dt}. Therefore the eigenvalues of $P(t)$ can be grouped into pairs $(\mu_i, e^{-dt}/\mu_i)$, $i=1,\ldots,N$. Thus, let $\mu_1(t)\geq\cdots\geq\mu_{2N}(t)$ be the eigenvalues of $P(t)$ ordered according to size. Then the following relation holds:

$$\mu_{2N-i+1}(t)=\frac{e^{-dt}}{\mu_i(t)},\quad i=1,\ldots,N\ . \tag{32}$$

The deviation from (32) was used to stop the calculation when the SVD routine starts to work unsatisfactorily. We define

$$\Delta_i(t)\equiv\left|\mu_{2N-i+1}(t)-\frac{e^{-dt}}{\mu_i(t)}\right|,\quad i=1,\ldots,N$$

and

$$\Delta(t)\equiv\sum_{i=1}^{N}\Delta_i(t)\ .$$

To control the precision of the computation, we required

$$\Delta(mT)<10^{-3} \tag{33}$$

to hold. The check of the inequality (33) does not require much more numerical effort because the eigenvalues of $P(t)$ appear as singular values during the calculation.

In Fig. 3(b) we show the same computation as shown in Fig. 3(a), but this time requiring (33) to hold to continue the computation. The violation of (33) happened here after $m=73$ periods of excitation. We estimated the limit $\Omega(\mathbf{x})$ by averaging over windows of 10 periods and choosing the average value with the smallest standard deviation. The plots shown in Fig. 3 suggest that the finite time rotation frequency $|\Omega(\mathbf{x},mT)|$ is still decreasing. Therefore the estimated value of $|\Omega(\mathbf{x})|$ tends to be too high.

For a small excitation amplitude ($f=0.05$), we present in Fig. 4 the rotation frequency and the winding number $|\Omega|/\omega$ with dependence on the bifurcation parameter ω_0. To obtain an overview of the dynamical behavior of the chain, an energy bifurcation diagram has simultaneously been calculated. The method of energy bifurcation diagrams was developed by Geist and Lauterborn.[16] Instead

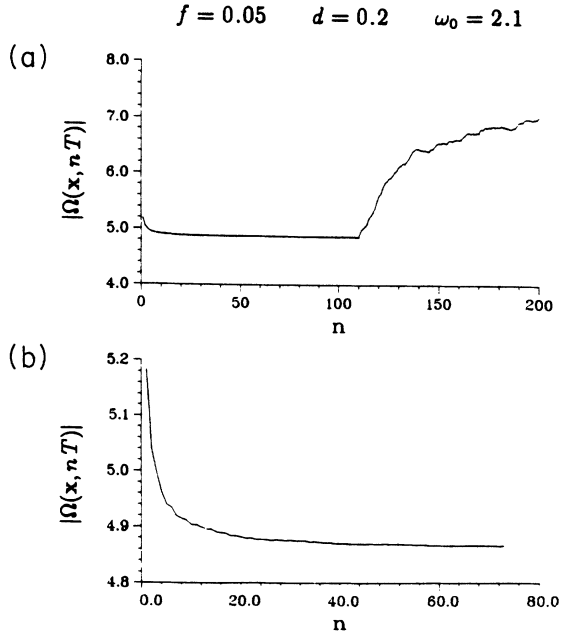

FIG. 3. (a) For $N=4$ Duffing oscillators and some fixed point $\mathbf{x}$ of the Poincaré mapping, the absolute value of the finite rotation frequency $|\Omega(\mathbf{x},nT)|$ is plotted against the period of excitation n. At the beginning $\Omega(\mathbf{x},nT)$ approaches a limit quite well, but at $n\approx 110$ the calculation obviously breaks down. (b) The same calculation as in (a), but this time requiring criterion (33) to hold in order to stop the computation before the SVD routine starts to work unsatisfactorily.

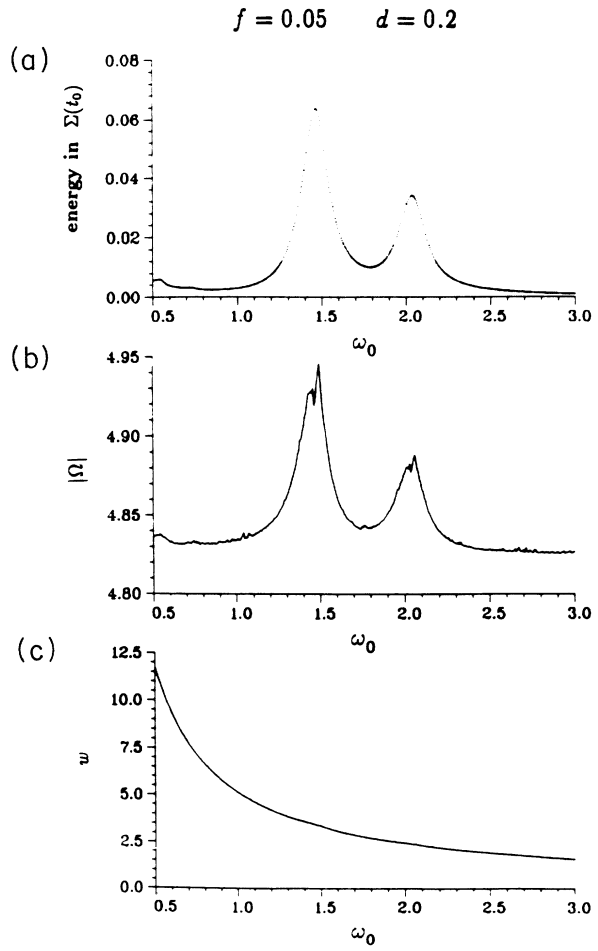

FIG. 4. Dependence on the bifurcation parameter ω_0 (which is approximately proportional to the oscillation frequency $\omega=2\pi/T$ of the van der Pol oscillator used to drive the chain) for $N=4$ Duffing oscillators and low excitation ($f=0.05$) of (a) the energy bifurcation diagram [the Poincaré plane $\Sigma(t_0)$ was defined at the time t_0 when the van der Pol oscillator went through zero with positive velocity], (b) Ruelle's rotation frequency $|\Omega(\mathbf{x})|$, (c) the associated winding number $w(\mathbf{x})=|\Omega|/\omega$.

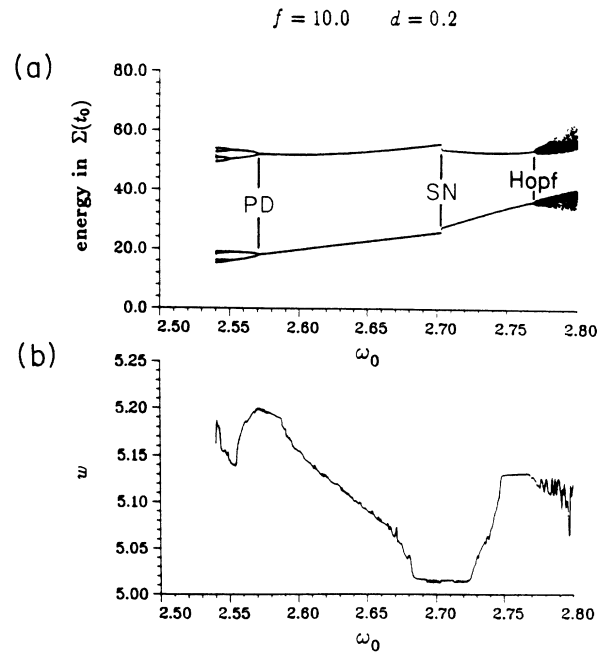

FIG. 5. For high excitation ($f=10$) the winding number $w(\mathbf{x})$ [panel (b)] with corresponding energy bifurcation diagram (a) is shown vs the bifurcation parameter ω_0. Around the PD bifurcations there seems to be no plateau in the winding number, while at the SN bifurcation and the Hopf bifurcation a plateau seems to occur.

of a coordinate of the successive points $P^m(t_0,\mathbf{x})$, $m=1,\ldots,n_p$, in the Poincaré plane, their energy $H(P^m(t_0,\mathbf{x}))$ is plotted against the bifurcation parameter ω_0. The main advantage of this procedure is that occurrences of resonances can easily be detected. The maxima occurring in the energy can be explained by the normal modes of the harmonic limit.[16] As can be seen in Fig. 4, the rotation frequency $|\Omega|$ shows maxima for the same ω_0 values. Consequently, the winding number w flattens around the corresponding ω_0 values.

At this point the question arises as to how to interpret the notion of the rotation frequency for the multidimensional case ($N>1$). For single oscillators the strong relation between resonances, bifurcations, and the winding number is well investigated. In particular, it is known that the winding number shows plateaus around saddle-node and period-doubling bifurcations. Therefore it is of interest to investigate the behavior of the winding number of the high-dimensional case around bifurcations. *A priori* one cannot decide whether plateaus in the winding number should occur or not, because the deeper reason is not yet understood.

In Fig. 5 we show the energy bifurcation diagram and the winding number w for $N=4$ oscillators for an excitation ($f=10$) where already many bifurcations have taken place. Besides period-doubling (PD) bifurcations at $\omega_0\approx 2.550, 2.554, \ldots$ and a saddle-node (SN) bifurcation at $\omega_0\approx 2.70$, a Hopf bifurcation can be seen at $\omega_0\approx 2.762$. At the Hopf and SN bifurcations there seems to be a plateau in the winding number, while for the PD bifurcations this is not the case. Because of the difficulties of the computation of the rotation frequency, we cannot be sure at this stage of the investigation whether plateaus occur and, if they do, why.

In the next section we will give a relation between the eigenvalues and the winding number of a periodic point $\mathbf{x}$ that will answer the questions raised above.

V. EIGENVALUES AND THE WINDING NUMBERS OF PERIODIC POINTS

First, we repeat the relation between the winding number and the eigenvalues of a periodic point $\mathbf{x}$ for the single oscillator ($N=1$). If the two eigenvalues of $D_x P^m$ are not real, $\mu_{1,2}=re^{\pm i\varphi(\mathbf{x})}$, then $\varphi(\mathbf{x})$ is related to the torsion frequency by

$$\varphi(\mathbf{x})=[mT|\Omega(\mathbf{x})|]\mathrm{mod}2\pi$$
$$=[mw(\mathbf{x})2\pi]\mathrm{mod}2\pi\ . \qquad (34)$$

If $\mu_{1,2}$ are real, then the eigenvalues are both positive or both negative and φ has to be replaced by 0 or π in (34), respectively. Equation (34) explains the plateaus in the winding number w if for a saddle-node or period-doubling bifurcation one eigenvalue approaches the unit circle on the positive or negative axis, respectively (because φ, being 0 or π, stays constant for an interval of the bifurcation parameter).

In order to get a similar relation for the case of N oscillators, we analyze the definition of the rotation frequency. The angle $\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})$ used in the definition of the rotation frequency was defined by

$$\det U(Q(t))=\exp[i\Theta(D_x\phi^t|_{\Sigma(t_0)})] . \tag{35}$$

Let $e^{i\theta_i(t)}$, $i=1,\ldots,N$, be the eigenvalues of $U(Q(t))$. Then, according to (35), we obtain

$$\left[\sum_{i=1}^{N}\theta_i(t)\right]\mathrm{mod}2\pi=\Theta(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})\mathrm{mod}2\pi . \tag{36}$$

Therefore, a relation (if any) analogous to (34) between the rotation frequency $|\Omega|$ and the eigenvalues $\mu_i=r_ie^{i\theta_i}$, $i=1,\ldots,2N$, of $D_{\mathbf{x}}P^m$ of an m-periodic point $\mathbf{x}$ must have the following form:

$$\left[\sum_{i=1}^{N}\varphi_{l_i}\right]\mathrm{mod}2\pi=(mT|\Omega|)\mathrm{mod}2\pi . \tag{37}$$

The problem which has to be solved to check (37) is to pick N angles φ_{l_i}, out of the $2N$ phases of the $2N$ eigenvalues of $D_{\mathbf{x}}P^m$. Because the definition of the rotation frequency is essentially based on the symplecticity of the system, the selection of the N angles φ_i must be guided by the symplectic properties of $D_{\mathbf{x}}P^m$. Again, we use the formation of pairs of the eigenvalues. Due to

$$D_{\mathbf{x}}P^mJ(D_{\mathbf{x}}P^m)^{\mathrm{tr}}=\lambda J \quad \text{with } \lambda=e^{-dmT} , \tag{38}$$

the eigenvalues of $D_{\mathbf{x}}P^m$ can be ordered in N pairs $(\mu_i, \lambda/\mu_i)$, $i=1,\ldots,N$. If one looks at the phases of one pair, $\mu=re^{i\varphi}$ and $\lambda/\mu=(\lambda/r)e^{-i\varphi}$, one notices that each pair determines one angle φ. So there is a natural reduction from $2N$ angles to N angles. But there is still a problem remaining.

Instead of φ, one can also associate $2\pi-\varphi$ with the pair $(\mu, \lambda/\mu)$, as can be seen from

$$\begin{aligned}&\mu=re^{i\varphi}=re^{-i(2\pi-\varphi)} ,\\ &\lambda/\mu=\lambda/re^{-i\varphi}=\lambda/re^{i(2\pi-\varphi)} .\end{aligned} \tag{39}$$

For $N=1$ this ambiguity in the choice of φ is easily fixed, but it needs more trial-and-error checking for general N because N angles are involved in the sum occurring in (37). Therefore we must have a rule for the selection of φ or $2\pi-\varphi$. Again, we will use the symplectic properties of $D_{\mathbf{x}}P^m$.

First, we have to discuss the configuration of the eigenvalues of symplectic (times a scalar) real matrices in more detail. Since $A=D_{\mathbf{x}}P^m$ is real, it follows that complex eigenvalues occur in quadruplets,

$$\mu,\ \lambda/\mu,\ \bar{\mu},\ \lambda/\bar{\mu} , \tag{40}$$

unless $\mu\bar{\mu}=\lambda$, i.e., $|\mu|=\sqrt{\lambda}$, when they occur in complex-conjugate pairs $(\mu,\bar{\mu}=\lambda/\mu)$ on the circle of radius $\sqrt{\lambda}$, while real eigenvalues come in pairs $\mu,\lambda/\mu\in\mathbb{R}$. Therefore the eigenvalues lie symmetric to the real axis and the circle with radius $\sqrt{\lambda}$. We will call the latter circle the symplectic circle, in order to refer to it more easily. For ordinary symplectic matrices ($\lambda=1$), this circle coincides with the unit circle which describes, in addition, the linear stability of the system.[24] For the systems considered, $\lambda<1$ holds and the double role of the unit circle is removed.

For the eigenvalues lying on the symplectic circle, one can apply the concept of the Krein signatures. Krein developed his theory investigating the stability of symplectic maps. The main point is that eigenvalues lying on the symplectic circle can only leave the circle after a collision on the symplectic circle.[14,13] However, not any collision causes the eigenvalues to leave the circle. Krein showed that one can assign signs (the so-called Krein signatures) to the eigenvalues on the symplectic circle. One

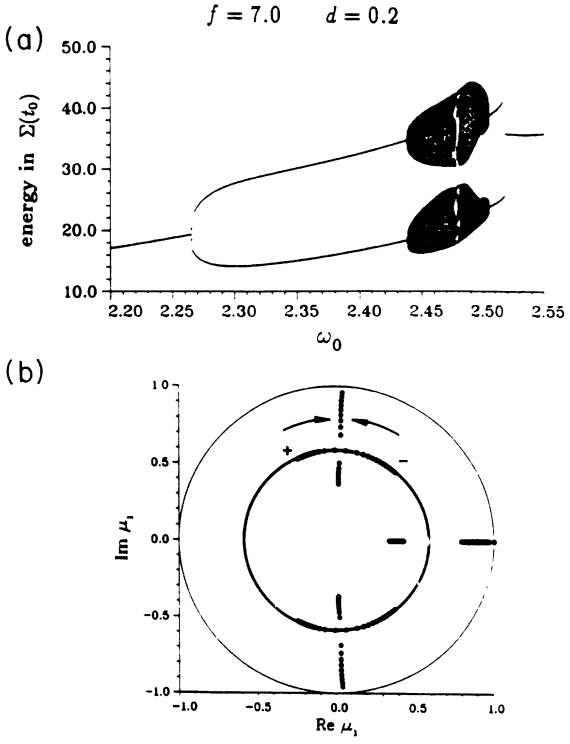

FIG. 6. (a) Energy bifurcation diagram for $N=4$ Duffing oscillators at $f=7$. (b) The eigenvalues of $D_{\mathbf{x}}P^2$ for a two-periodic point $\mathbf{x}$ is given for 21 parameter values ω_0 in the range $2.35\le\omega_0\le2.43$. (Due to the translational invariance of the equations of motion, there is always one eigenvalue of $D_{\mathbf{x}}P^m$ equal to 1, and therefore another one equals $e^{-dmT(\omega_0)}$.) The symplectic circles with radius $e^{-d2T(\omega_0)/2}$ for the ω_0 values 2.35 and 2.43 are drawn. They are hardly distinguishable. At $\omega_0\approx2.3945$ a Krein collision can be seen. The Krein signatures before the collision are marked. The Krein collision here is a precursor of the Hopf bifurcation that occurs at $\omega_0\approx2.436$ [see (a)].

main result is then that the collision of eigenvalues of the same sign does not do any harm. The eigenvalues can only leave the symplectic circle if two eigenvalues (and, at the same time, the complex-conjugate pair) of different Krein signatures collide.

In Fig. 6 the eigenvalues of $D_{\mathbf{x}}P^2$ for a two-periodic point $\mathbf{x}$ is given in the range $2.35 \leq \omega_0 \leq 2.43$. At $\omega_0 \approx 2.3945$ a Krein collision can be seen. Figure 6 strikingly shows how the symplectic properties of $D_{\mathbf{x}}P^2$ determine the behavior of the eigenvalues.

It turns out now that the Krein signature will give us a criterion for selecting the angle φ or $2\pi-\varphi$ of a pair $(\mu=\sqrt{\lambda}e^{i\varphi}, \lambda/\mu=\bar{\mu}=\sqrt{\lambda}e^{i(2\pi-\varphi)})$ on the symplectic circle. Note that here the problem of associating an angle with the pair $(\mu, \lambda/\mu)$ coincides with the ambiguity of notation concerning which eigenvalue to call μ and which to call $\bar{\mu}$.

In the following we recall the definition of the Krein signature and state the connection between the Krein signature and the chosen angle φ. First, we restrict ourselves to the case of a simple (multiplicity 1) eigenvalue $\mu \in \mathbb{C}\setminus\mathbb{R}$ $(|\mu|=\sqrt{\lambda})$. Let $\xi=\mathbf{u}+i\mathbf{v}\in\mathbb{C}^{2N}$ be the complex eigenvector belonging to $\mu=re^{i\varphi}$ and $\mathcal{U}_\mu \subset \mathbb{R}^{2N}$ the invariant two-dimensional subspace of $\mathbb{R}^{2N}$ (spanned by $\{\mathbf{u},\mathbf{v}\}$) associated with μ. Then, one can formulate the definition of the Krein signature as follows.

Definition 2. The pair of eigenvalues $(\mu,\bar{\mu}=\lambda/\mu) \in \mathbb{C}\setminus\mathbb{R}$ (with multiplicity 1) is said to have positive (negative) Krein signature, if

$$[A\mathbf{w}, \mathbf{w}] > 0 \quad (<0) \quad \text{for } \mathbf{w}\in\mathcal{U}_\mu \subset \mathbb{R}^{2N} . \tag{41}$$

The symbol $[,]$ denotes the nondegenerate antisymmetric bilinear form, called the skew-scalar product which takes the standard form in the chosen coordinate system

$$[\mathbf{u},\mathbf{v}] \equiv (\mathbf{u}, J\mathbf{v}), \quad \mathbf{u},\mathbf{v}\in\mathbb{R}^{2N} , \tag{42}$$

where $(,)$ is the Euclidean scalar product in $\mathbb{R}^{2N}$.

That the definition of Krein signature in (41) makes sense and does not depend on the chosen vector $\mathbf{w}\in\mathcal{U}_\mu$ can be justified abstractly, but for our purposes the following calculation is better suited.

Lemma 2. Let $\mu=re^{i\varphi}$, $\bar{\mu}=\lambda/\mu\in\mathbb{C}\setminus\mathbb{R}$, $\mathbf{u},\mathbf{v}\in\mathcal{U}_\mu$ be as above. Let $\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}$ be an arbitrary vector in $\mathcal{U}_\mu$; then it holds that

$$[A\mathbf{w}, \mathbf{w}] = r(c_1^2+c_2^2)(\sin\varphi)[\mathbf{u}, \mathbf{v}] . \tag{43}$$

This is straightforwardly calculated, but from (43) one sees readily that the Krein signature $\sigma_K(\mu)$ of μ is given by

$$\sigma_K(\mu) = \operatorname{sgn}\{(\sin\varphi)[\mathbf{u}, \mathbf{v}]\} . \tag{44}$$

If we had made the calculation based on $\bar{\mu}$, then φ would have to be replaced by $2\pi-\varphi$ and $[\mathbf{u}, \mathbf{v}]$ by $[\mathbf{u}, -\mathbf{v}]$. The factors would change sign, but the product would keep its sign as it must be. Therefore we can always use the following convention.

Convention. That eigenvalue of the pair $(\mu, \lambda/\mu)\in\mathbb{C}\setminus\mathbb{R}$ on the symplectic circle is called μ, for which the corresponding eigenvector $\xi\in\mathbb{C}^{2N}$ fulfills

$$[\operatorname{Re}(\xi), \operatorname{Im}(\xi)] = [\mathbf{u}, \mathbf{v}] > 0 . \tag{45}$$

The phase φ of $\mu=re^{i\varphi}$ is the angle associated with the pair $(\mu, \lambda/\mu)$.

Using the convention made above, it is $\sin\varphi$ which determines the Krein signature, i.e., $(\mu, \lambda/\mu)$ is positive if $\sin\varphi>0$ and negative if $\sin\varphi<0$ holds, respectively. Therefore a pair of eigenvalues changes sign when crossing the real axis.

We now discuss the case of a pair of eigenvalues $(\mu, \lambda/\mu)\in\mathbb{C}\setminus\mathbb{R}$ on the symplectic circle with multiplicity k. Again, we denote with $\mathcal{U}_\mu$ the corresponding $2k$-dimensional invariant subspace.

Definition 3. The pair $(\mu,\bar{\mu}=\lambda/\mu)\in\mathbb{C}\setminus\mathbb{R}$ has definite positive Krein signature if

$$[A\mathbf{w}, \mathbf{w}] > 0 \quad \forall \mathbf{w}\in\mathcal{U}_\mu, \quad \mathbf{w}\neq 0 \tag{46a}$$

holds, and definite negative Krein signature if

$$[A\mathbf{w}, \mathbf{w}] < 0 \quad \forall \mathbf{w}\in\mathcal{U}_\mu, \quad \mathbf{w}\neq 0 \tag{46b}$$

holds.

Not all eigenvalues on the symplectic circle (of multiplicity >1) have definite Krein signature. For example, if two eigenvalues of different signature collide and merge, the resulting pair of eigenvalues will have mixed signature (for definition, see Ref. 24).

We now recall Krein's theorem in a version from Ref. 24.

Krein's theorem. Let $(\mu,\bar{\mu}=\lambda/\mu)\in\mathbb{C}\setminus\mathbb{R}$ have definite Krein signature. Then A has diagonal Jordan form on the invariant space $\mathcal{U}_\mu$ and the eigenvalues cannot leave the symplectic circle under small perturbations.

As a result of Krein's theorem, for an eigenvalue μ (of multiplicity k) with definite Krein signature there are $2k$ complex eigenvectors $\xi_i, \bar{\xi}_i$, $i=1,\ldots,k$, with $A\xi_i=\mu\xi_i$, and the product

$$(\sin\varphi)[\operatorname{Re}(\xi_i), \operatorname{Im}(\xi_i)], \quad i=1,\ldots,k \tag{47}$$

has, for all i, the same sign, which is given by the (definite) Krein signature. Again, we can make the convention that the notation of μ is chosen such that

$$[\operatorname{Re}(\xi_i), \operatorname{Im}(\xi_i)] > 0, \quad i=1,\ldots,k . \tag{48}$$

The angle $k\varphi$ is then associated with the pair $(\mu=re^{i\varphi}, \bar{\mu}=\lambda/\mu)$.

Next, we describe the consequences of our convention for the angle associated with a quadruplet, which is generated by a Krein collision. Because of Krein's theorem the eigenvalues μ_1 and μ_2 before the collision must have different signatures. So let μ_1 be the eigenvalue with positive Krein signature and μ_2 that with negative Krein signature. Then, $\varphi_1\in(0,\pi)$ and $\varphi_2\in(\pi,2\pi)$. Let φ be the angle of μ_1 at the collision point. φ_2 will then be equal to $2\pi-\varphi$ when the eigenvalues coalesce. Therefore the sum of φ_1 and φ_2 will be 2π. When the eigenvalues have left the symplectic circle to form a quadruplet, they are coupled together and can no longer change angles independently. Therefore the sum of the associated angles will remain 2π. For eigenvalues of multiplicity greater than 1, similar considerations hold; only 2π has to be replaced

by $k2\pi$. This, however, is not of importance in order to state a relation between the phases of the eigenvalues and the rotation frequency, because, in any case, we come to the conclusion that a quadruplet generated by a Krein collision does not give any contribution to the sum in (37) when counted mod2π.

Finally, we describe the angle associated with a real pair $(\mu, \lambda/\mu) \in \mathbb{R}$. We remark that

$$\varphi \bmod 2\pi = (2\pi - \varphi) \bmod 2\pi \quad \text{for } \varphi = 0 \text{ or } \varphi = \pi \tag{49}$$

holds. Therefore we do not have to find a criterion for real eigenvalues to select an angle. A real pair of eigenvalues with multiplicity k will be counted with 0 if $\mu > 0$ and with $k\pi$ if $\mu < 0$ in the sum occurring in our relation (37).

Let us summarize the determination of the phases φ_i of the map $D_x P^m$. The eigenvalues are first ordered to pairs $(\mu, \lambda/\mu)$. The related angles for the sum in (37) are then determined through use of the following rules.

(1) A pair $(\mu, \lambda/\mu)$ on the symplectic circle with definite Krein signature and multiplicity k counts with $k\varphi$ [where φ has to be determined by Krein signature (45)].

(2) A quadruplet generated by a Krein collision counts with $k2\pi$.

(3) A real pair $(\mu, \lambda/\mu)$ counts with 0 if $\mu > 0$ and with $k\pi$ if $\mu < 0$, respectively.

Making this selection of the angles $\varphi_i(\mathbf{x})$ of $D_x P^m$, we conjecture that

$$\left[\sum_{i=1}^{N} \varphi_i(\mathbf{x}) \right] \bmod 2\pi = [m w(\mathbf{x}) 2\pi] \bmod 2\pi$$
$$= [mT|\Omega(\mathbf{x})|] \bmod 2\pi \tag{50}$$

holds. We do not claim that rules (1)–(3) contain all possible configurations of eigenvalues. For example, the case of a pair of eigenvalues on the symplectic circle with mixed signature is not considered. If $D_x P^m$ has diagonal Jordan form on $\mathcal{U}_\mu$, then the extension of convention (48) is straightforward. In our numerical studies of the chain of oscillators, more general cases of mixed signature did not occur. Therefore we think that, at the moment, for practical purposes, the rules given will be sufficient to get a first insight in the relation between the rotation frequency and the eigenvalues of a periodic point. In the next section we will show numerical tests which support relation (50).

VI. NUMERICAL RESULTS

First, we want to check relation (50) for fixed parameter values. To avoid clumsy notation we call the lhs of (50) Θ_K (K denotes Krein) and the rhs Θ_Ω. In Figs. 7(a) and 7(c) the eigenvalues of $D_x \phi^T|_{\Sigma(t_0)} = D_x P$ and the corresponding symplectic circle of radius $e^{-dT/2} = e^{-d\pi/\omega}$ are shown for a small excitation ($f = 0.05$) and for two ω_0 values ($\omega_0 = 1.5$ and 2.1) (also see Fig. 1). One eigenvalue has multiplicity 2. Its definite Krein signature is indicated by the double sign. For these parameter values the calculated values of Θ_K and Θ_Ω are $\Theta_K = 2.36$ and $\Theta_\Omega = 2.39$ for $\omega_0 = 1.5$, and $\Theta_K = 2.18$ and $\Theta_\Omega = 2.19$ for $\omega_0 = 2.1$. Reminding the reader of the difficulties in the numerical determination of the rotation frequency, the numerical results for the parameter values shown are in good agreement with (50).

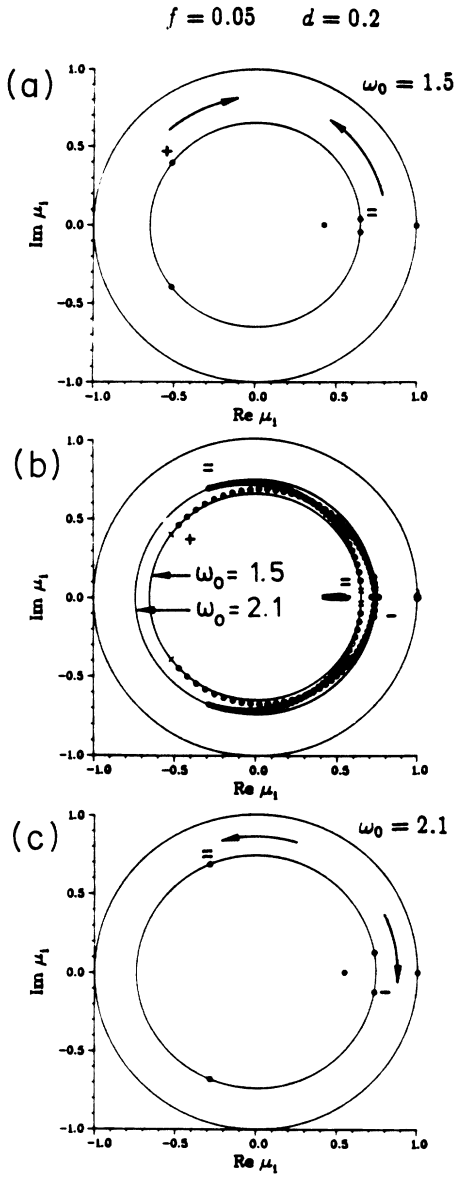

FIG. 7. In (a) and (c) the eigenvalues of $D_x P$ of a period-1 point $\mathbf{x}$ and the corresponding symplectic circles of radius $e^{-dT(\omega_0)/2}$ are shown for a small excitation ($f = 0.05$) and two ω_0 values (a) $\omega_0 = 1.5$ and (c) $\omega_0 = 2.1$. One eigenvalue has multiplicity 2. Its definite Krein signature is indicated by the double sign. (b) The eigenvalues of $D_x P$ for 40 ω_0 values in the range $1.5 \le \omega_0 \le 2.1$. The pair of eigenvalues with positive Krein signature for $\omega_0 = 1.5$ moves on the symplectic circle (with varying radius $e^{-dT(\omega_0)/2}$) and crosses the real axis at $\omega_0 \approx 2.05$, where it changes its Krein signature. Doing this it leaves the symplectic circle and becomes real for a short ω_0 interval.

Next, we investigate how these eigenvalues of $D_x P$ (and with them their Krein signatures) develop when raising the parameter ω_0 from 1.5 to 2.1. In Fig. 7(b) the eigenvalues of $D_x P$ are plotted in the range $1.5 \leq \omega_0 \leq 2.1$. As can be seen, the pair of eigenvalues with positive Krein signature corresponding to $\omega_0 = 1.5$ crosses the real axis for an ω_0 value slightly less than 2.1, where it changes its Krein signature. Doing this, it leaves the symplectic circle and becomes real for a short ω_0 interval ($\omega_0 \approx 2.04-2.06$). The same is true for $\omega_0 \approx 1.47$, where the other pair (with multiplicity 2 and negative Krein signature corresponding to $\omega_0 = 1.5$) leaves the symplectic circle and stays on the positive real axis for a short ω_0 interval ($\omega_0 \approx 1.47-1.49$). This explains the maximum in $|\Omega|$ which happens if one pair of eigenvalues leaves the symplectic circle for a short ω_0 interval. For, varying on the real axis, the pair does not give any contribution to the sum in (50). $T = 2\pi/\omega$, however, decreases for increasing ω_0, and therefore $|\Omega|$ must increase in order to satisfy (50). One can interpret this phenomenon as follows. Leaving the symplectic circle is a prerequisite for an instability or a bifurcation to occur. For the small excitation ($f = 0.05$) the system is still stable, but the saddle-node bifurcations for higher excitations are already influencing the system. It is also important that the eigenvalues leaving the symplectic circle at $\omega_0 \approx 1.47$ and 2.04 are different pairs. In Ref. 16 the connection between the normal modes of the chain and the resonances is stressed. That different pairs of eigenvalues are responsible for different resonances therefore supports the conjecture that the eigenvalues of $D_x P$ are related to the normal modes, as stated in Ref. 17.

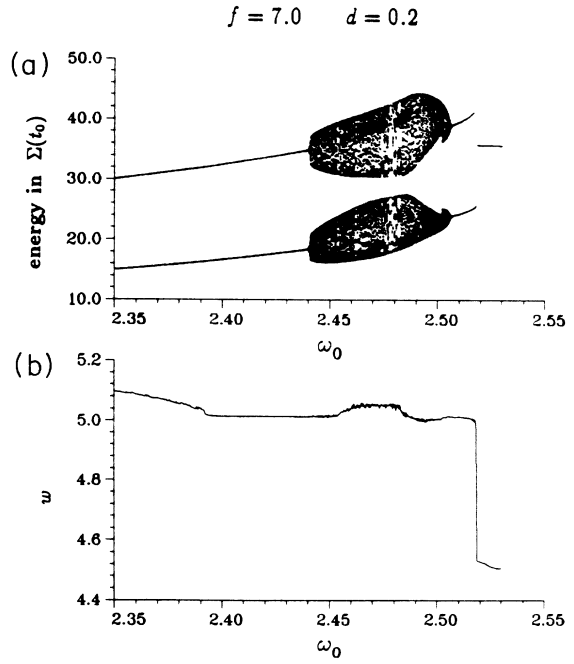

FIG. 8. Winding number (b) with dependence on ω_0 and corresponding energy bifurcation diagram (a). To understand the behavior of the winding number for ω_0 between 2.35 and 2.43, compare the corresponding eigenvalues of $D_x P^2$ in Fig. 6(b). The plateau of the winding number starts right after the Krein collision at $\omega_0 \approx 2.3945$ when all eigenvalues have left the symplectic circle. This is in agreement with the given relation (50), including the rules to select the angles φ_i.

To check (50) we compared Θ_K and Θ_Ω for a variety of different parameter values. All the numerical tests performed lead to an agreement of Θ_K and Θ_Ω within 0.3. For almost all values, Θ_Ω was slightly greater than Θ_K, but this should be expected, because $|\Omega|$ seems to be calculated too high (see, e.g., Fig. 3). However, the most striking hint for the validity of (50) is that it helps to explain the behavior of the winding number $w(\mathbf{x})$ at bifurcations by looking at the corresponding eigenvalues of $D_x P^m$.

First, we consider the Hopf bifurcation shown in Fig. 6. At $\omega_0 \approx 2.3945$, a Krein collision occurs and a pair of eigenvalues splits off the symplectic circle. From the rules given to count the angles, it follows that in the range $2.3945 < \omega_0 < 2.44$ the sum of phases remains a multiple of 2π. Therefore we expect, according to relation (50) (with $m = 2$), in this range a winding number w that is constant at an integer or half-integer value. The numerical calculations of the winding number (see Fig. 8) confirm this. As can be seen, it is exactly after the Krein collision that the winding number becomes constant. The deviation from an integer value (here $w = 5$) is due to the numerical error made because the calculation has to be stopped before the final value of Ω is reached (see Fig. 3).

Now we discuss the behavior of the winding number near the bifurcations in Fig. 5. In Fig. 9(a) the eigenvalues of $D_x P^2$ are plotted in the range $2.572 \leq \omega_0 \leq 2.590$ at the PD bifurcation. As can be seen, there is one pair of eigenvalues left on the symplectic circle. During the variation of ω_0 this pair changes its angle φ. This explains, together with Eq. (50), that the winding number does not show a plateau at the PD bifurcation ($\omega_0 \approx 2.572$). At the SN bifurcation [Fig. 9(b)] [see Fig. 5(a)], in contrast, all eigenvalues have left the symplectic circle and Θ_K is zero. Again, Eq. (50) then predicts a plateau for the winding number around this SN bifurcation. Comparing the calculated winding number $w \approx 5.01$ at this bifurcation and Eq. (50), it is only $w = 5$ that is compatible with (50). Again, the difference of 0.01 must be explained by the numerical error we made. This numerical error is much larger than the fluctuations in a plateau. Therefore it is somewhat surprising that one can even detect a plateau in the plots of the winding number. The explanation is that the calculated winding number has a systematic tendency to be too high.

The Hopf bifurcation [see Figs. 5(a) and 9(c)] is not caused by the Krein collision, which can be seen in Fig. 9(b). The quadruplet of Fig. 9(b) goes back to the symplectic circle for increasing ω_0. The Hopf bifurcation at $\omega_0 \approx 2.76$ comes from a Krein collision with the pair of eigenvalues which made the saddle-node bifurcation and one pair of the former quadruplet in Fig. 9(b). One pair of the quadruplet stays on the symplectic circle. Because its angle does not change very much during the variation of ω_0, it is almost a plateau which can be seen in Fig. 5 at the Hopf bifurcation, but the pair of eigenvalues causes

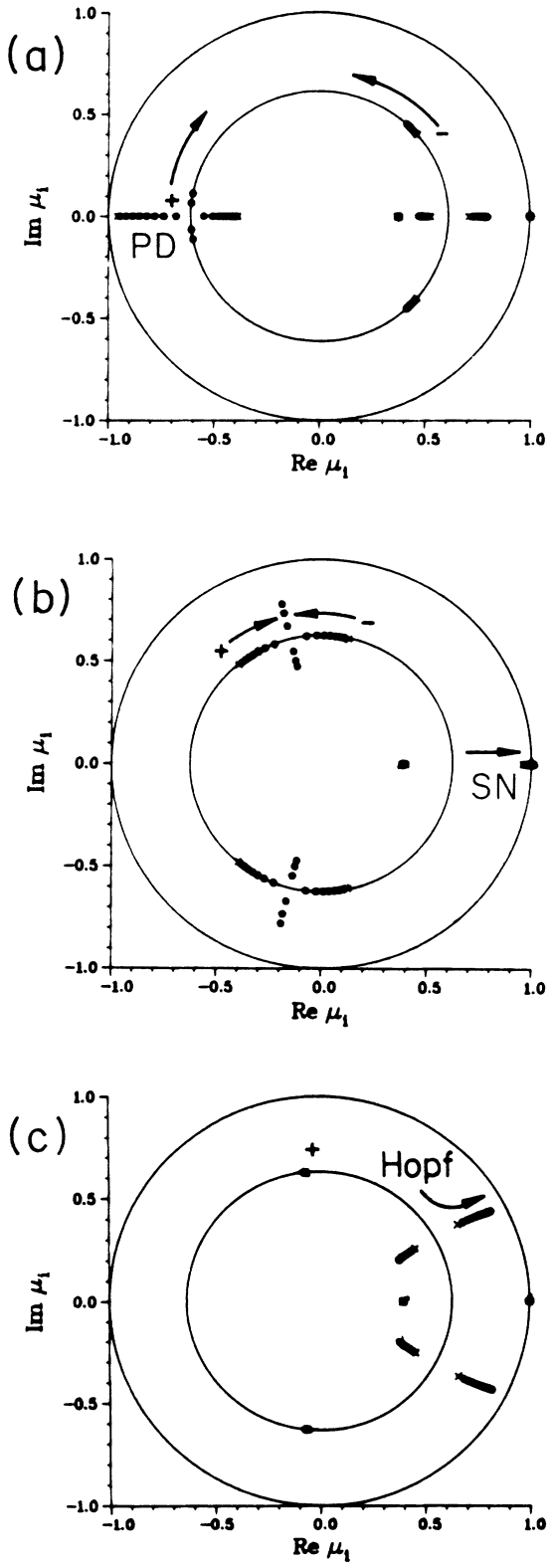

FIG. 9. Investigation of the behavior of the winding number in Fig. 5(c) ($f=10$) near the bifurcations by looking at the corresponding eigenvalues of D_xP^2. (a) The eigenvalues of D_xP^2 in the range $2.572 \leq \omega_0 \leq 2.59$ at the PD bifurcation ($\omega_0 \approx 2.572$). (b) Eigenvalues of D_xP^2 for $\omega_0 \approx 2.66-2.70$ at the SN bifurcation. (c) Eigenvalues of D_xP^2 for $\omega_0 = 2.75-2.76$ at the Hopf bifurcation ($\omega_0 \approx 2.76$).

the winding number to have the value $w \approx 5.13$. If all eigenvalues had left the symplectic circle, then w would have to be 5.0 or 5.25 in order not to contradict Eq. (50).

Summarizing, the numerical calculations confirm our conjecture (50), but also show the numerical problems that are still unsolved if one tries to calculate Ruelle's rotation frequency for high-dimensional systems. A true confirmation of (50) can, of course, only be given by an analytical proof.

VII. SUMMARY AND CONCLUSIONS

A ring of coupled Duffing oscillators with periodic driving is investigated. Due to a linear viscous damping term, this system belongs to a class of dissipative dynamical systems which preserve essential features of Hamiltonian dynamics. In particular, the generalized symplecticity relation (7) of the linearized flow holds and imposes constraints on the motion.

For Hamiltonian systems the unit circle in the complex plane plays a double role. On one hand, it determines the stability of periodic solutions; on the other, it is a center of symmetry for the eigenvalues because of the symplecticity of the linearization. For dissipative Hamiltonian systems this "degeneracy" is resolved. Besides the unit circle, which still describes the stability and local bifurcations, there is the symplectic circle of radius $e^{-mdT/2}$ which plays a crucial role. An analysis of the eigenvalues of D_xP^m without considering how these eigenvalues are lying with respect to the symplectic circle will be incomplete. The theory of the Krein signatures[14,13] can be applied to these systems.

As a further implication of (7), Ruelle's rotation frequency (number) is defined for the ring of oscillators considered. It could be shown that for single oscillators Ruelle's rotation frequency (number) Ω coincides with the torsion frequency of single oscillators. We calculated the winding number $w = |\Omega| / \omega$ as a function of the external driving frequency, focusing our attention on the behavior of the winding number around bifurcations. For periodic points we could relate the sum of N angles determined by the $2N$ eigenvalues of D_xP^m to the winding number (50). The selection of the angles in (50) is not haphazard, but is founded on the symplecticity of D_xP^m in two ways. First, it uses the formation of symplectic pairs $(\mu, \lambda/\mu)$ to reduce the number of relevant angles. Secondly, it takes the symplectic orientation of the complex eigenvectors into account, which is connected with the Krein signatures to derive a uniform convention for the angle of the pair of eigenvalues $(\mu, \lambda/\mu)$. A major consequence of this convention is that the sum of the phases of a quadruplet generated by a Krein collision is a multiple of 2π. On the whole, it follows that the winding number shows a plateau when all eigenvalues have left the symplectic circle. On the other hand, when a pair is still on the symplectic circle and varies while another pair of eigenvalues gives rise to a bifurcation (i.e., crosses the unit circle), the winding number will not show a plateau. This is independent of the type of bifurcation considered. In particular, for high-dimensional systems with intermediate excitation amplitude there will generally be pairs

left on the symplectic circle, and thus a plateau cannot be expected. If one wants to use a notion of winding numbers which necessarily become constant around bifurcations, one has to introduce a set of winding numbers, but much work has still to be done before a profound mathematical basis for these concepts can be provided.

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APPENDIX: EQUIVALENCE OF RUELLE'S ROTATION FREQUENCY (NUMBER) AND THE TORSION FREQUENCY FOR SINGLE OSCILLATORS

In this appendix we give the proof for the equivalence of the torsion frequency defined through the QR decomposition [(9)–(11)] and Ruelle's definition in two dimensions. The peculiarity of two dimensions is the fact that each 2×2 matrix A with positive determinant, i.e., $\det A>0$, is symplectic times a scalar. In particular, every orthogonal matrix Q with $\det Q=1$, i.e., $Q\in \mathrm{SO}(2)$, is symplectic, i.e.,

$$\mathrm{SO}(2)=\mathrm{SO}(2)\cap \mathrm{Sp}(2,\mathbb{R}) .$$

Therefore the definition of Ruelle's rotation frequency applies to all driven single oscillators even if they do not satisfy the equations of motion considered [Eq. (3)].

The relation between SO(2) and U(1) turns out to be the well-known representation of a matrix, $Q\in \mathrm{SO}(2)$,

$$Q=\begin{bmatrix}\cos\Theta & -\sin\Theta\\ \sin\Theta & \cos\Theta\end{bmatrix}\in \mathrm{SO}(2) . \tag{A1}$$

The definition of $U(Q)$ [Eq. (16)] according to symplecticity gives

$$U(Q)=\cos\Theta+i\sin\Theta=e^{i\Theta}\in \mathrm{U}(1) .$$

Thus the angle Θ associated with a special orthogonal matrix $Q\in \mathrm{SO}(2)$ according to (A1) is exactly the angle associated with $Q\in \mathrm{SO}(2)\cap \mathrm{Sp}(2,\mathbb{R})$ because of its relation to the unitary group U(1). At this point the differences between the two definitions of a torsion or rotation frequency reduce to the fact that Ruelle's definition is based on the polar decomposition,

$$D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}=Q_P(t)P(t) ,$$

and the former definition (11) on the QR decomposition,

$$D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}=Q_R(t)R(t) .$$

For closed curves $\gamma(t)=D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ one could now justify abstractly using topological arguments that the two definitions of a torsion or rotation frequency coincide, but because $\gamma(t)=D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$ is not a closed curve, we have to investigate the differences between the two definitions in more detail.

By means of the QR decomposition and the polar decomposition, two different orthogonal matrices $Q_R(t)$ and $Q_P(t)$ are related to the linearization $D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}$, as are, therefore, two different angles that we denote Θ_R and Θ_P, respectively, i.e.,

$$\begin{aligned}&\Theta_P(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})\equiv\Theta(Q_P(t)) ,\\ &\Theta_R(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})\equiv\Theta(Q_R(t)) .\end{aligned} \tag{A2}$$

For clarity we denote the corresponding rotation (torsion) frequencies that arise when keeping track of the multiples of 2π as $\Omega_P(\mathbf{x})$ and $\Omega_R(\mathbf{x})$, i.e.,

$$\Omega(\mathbf{x})\equiv\Omega_P(\mathbf{x})=\lim_{t\to\infty}\frac{\Theta_P(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})}{t} \tag{A3}$$

and

$$\Omega_R(\mathbf{x})=\lim_{t\to\infty}\frac{\Theta_R(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})}{t} . \tag{A4}$$

Now we show that in the infinite time limit the possible differences for finite time vanish, i.e., that $\Omega_R(\mathbf{x})=\Omega_P(\mathbf{x})$ holds. Using the polar decomposition of $R(t)$, we obtain

$$R(t)=Q_A(t)P_R(t) , \tag{A5}$$

and

$$\begin{aligned}D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)}&=Q_P(t)P(t)\\ &=Q_R(t)Q_A(t)P_R(t) .\end{aligned} \tag{A6}$$

Because of the uniqueness of the polar decomposition it holds that

$$\begin{aligned}&P_R(t)=P(t) ,\\ &Q_P(t)=Q_R(t)Q_A(t) .\end{aligned} \tag{A7}$$

Thus $Q_P(t)$ and $Q_R(t)$ differ only by the orthogonal matrix $Q_A(t)$, which is given by [combine (A6) and (A7)]

$$Q_A(t)=R(t)P(t)^{-1} . \tag{A8}$$

We now show that the angle $\Theta(Q_A(t))$ is bounded. Let $r_{ij}(t)$ and $\tilde{p}_{ij}(t)$ be the entries of $R(t)$ and $P(t)^{-1}$, respectively. Using that $R(t)$ is an upper triangular matrix (here $r_{21}=0$), we obtain from Eq. (A8)

$$Q_A(t)=\begin{bmatrix}r_{11}(t)\tilde{p}_{11}(t)+r_{12}(t)\tilde{p}_{21}(t) & r_{11}(t)\tilde{p}_{12}(t)+r_{12}(t)\tilde{p}_{22}(t)\\ r_{22}(t)\tilde{p}_{21}(t) & r_{22}(t)\tilde{p}_{22}(t)\end{bmatrix} . \tag{A9}$$

Therefore we obtain the expression for the angle $\Theta(Q_A(t))$,

$$\begin{aligned} e^{i\Theta(Q_A(t))} &= r_{22}(t)\tilde{p}_{22}(t)+ir_{22}(t)\tilde{p}_{21}(t) \\ &= r_{22}(t)\tilde{p}_{22}(t)\left[1+i\frac{\tilde{p}_{21}(t)}{\tilde{p}_{22}(t)}\right] \\ &\equiv c(t)[1+i\lambda(t)] , \end{aligned} \tag{A10}$$

with

$$\begin{aligned} &\lambda(t)\equiv\tilde{p}_{21}(t)/\tilde{p}_{22}(t) , \\ &c(t)\equiv r_{22}(t)\tilde{p}_{22}(t)\in\mathbb{R} \text{ and } c(t)\neq 0 . \end{aligned} \tag{A11}$$

$\lambda(t)$ is well defined because $\tilde{p}_{22}(t)$ is always different from zero. This can easily be seen. The assumption $\tilde{p}_{22}(t)=0$ leads to $\det P(t)^{-1}=-\tilde{p}_{12}(t)\tilde{p}_{21}(t)=-[\tilde{p}_{12}(t)]^2<0$ and this contradicts the positivity of $P(t)$. Furthermore, note that the curve $\gamma(t)\equiv\exp[i\Theta(Q_A(t))]$ never crosses the negative real axis. This is so because $\lambda(t)=0$ implies that $\tilde{p}_{21}(t)=0$, but then $\tilde{p}_{22}(t)>0$ [because of the positivity of $P(t)^{-1}$] holds and, with $r_{22}(t)>0$, $c(t)=r_{22}(t)\tilde{p}_{22}(t)>0$ also follows. Therefore we obtain for $\lambda(t)=0$ that $c(t)=\gamma(t)=\exp[i\Theta(Q_A(t))]>0$ holds and this means, of course, $\gamma(t)=1$. Because the angle $\Theta(Q_A(t))$ depends continuously on t, it follows that the curve $\gamma(t)$ can never go around the origin and, consequently, $\Theta(Q_A(t))$ is constrained to the interval $(-\pi,\pi)$.

This boundedness leads to the desired equality of $\Omega_R(\mathbf{x})$ and $\Omega_P(\mathbf{x})$. With (A7) we have

$$\Theta_P(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})=\Theta_R(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})+\Theta(Q_A(t)) .$$

However, the differences between the finite rotation (torsion) frequencies $\Omega_P(\mathbf{x},t)$ and $\Omega_R(\mathbf{x},t)$ vanishes for t to infinity because of the boundedness of $\Theta_A(Q(t))$, which leads to

$$\lim_{t\to\infty}\frac{\Theta(Q_A(t))}{t}=0 .$$

Therefore we finally get the desired result,

$$\begin{aligned} \Omega_P(\mathbf{x}) &= \lim_{t\to\infty}\frac{\Theta_P(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})}{t} \\ &= \lim_{t\to\infty}\frac{\Theta_R(D_{\mathbf{x}}\phi^t|_{\Sigma(t_0)})}{t} \\ &= \Omega_R(\mathbf{x}) . \end{aligned} \tag{A12}$$

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[1]J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[2]D. Ruelle, Ann. Inst. Henri Poincaré **42**, 109 (1985).
[3]A. Lambert, R. Lima, and R. Vilela Mendes, Physica D **34**, 366 (1989).
[4]U. Parlitz and W. Lauterborn, Z. Naturforsch. **41a**, 605 (1986).
[5]U. Parlitz and W. Lauterborn, Phys. Rev. A **36**, 1428 (1987).
[6]P. Beiersdorfer, J.-M. Wersinger, and Y. Treve, Phys. Lett. **96A**, 269 (1983).
[7]T. Uezu, Phys. Lett. **93A**, 161 (1983).
[8]U. Dressler, Phys. Rev. A **38**, 2103 (1988).
[9]T. P. Valkering, G. Derks, and E. van Groesen, in *Singular Behavior and Nonlinear Dynamics*, edited by St. Pnevmaticos, T. Bountis, and Sp. Pnevmaticos (World Scientific, Singapore, 1988), p. 457.
[10]H. J. van Opheusden and T. P. Valkering, Nonlinearity **2**, 357 (1989).
[11]H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980).
[12]J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).
[13]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).
[14]M. G. Krein, *Topics in Differential and Integral Equations and Operator Theory* (Birkhäuser, Basel, 1983).
[15]U. Parlitz and W. Lauterborn, Phys. Lett. **107A**, 351 (1985).
[16]K. Geist and W. Lauterborn, Physica D **31**, 103 (1988).
[17]K. Geist, Ph.D. dissertation, University of Göttingen, 1989.
[18]U. Parlitz (private communication).
[19]P. Beiersdorfer, Phys. Lett. **100A**, 379 (1984).
[20]R. G. Littlejohn, Phys. Rep. **138**, 193 (1986).
[21]R. G. Littlejohn and J. M. Robbins, Phys. Rev. A **36**, 2953 (1987).
[22]D. Guicking and K. Haars (unpublished).
[23]K. Geist, U. Parlitz, and W. Lauterborn, Progr. Theor. Phys. (to be published).
[24]J. E. Howard and R. S. MacKay, Math. Phys. **28**, 1036 (1987).