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Multistabilities and anomalous switching in the Lorenz-Haken model

Cun-zheng Ning and Hermann Haken
Institut für Theoretische Physik und Synergetik, Universität Stuttgart, Pfaffenwaldring 57/4, D-7000 Stuttgart 80, Federal Republic of Germany
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We show that pulses in the Lorenz-Haken model can be observed well below the well-known second threshold by using the anomalous switching technique. More specifically, if the laser system initially operates stably above the first threshold, the pulse motion can set in when we abruptly switch the pump to a new value, which is much smaller than the second threshold. This phenomenon is due to the existence of multistable attractors and anomalous switching between these attractors. By using the anomalous switching, physically distinguishable three, namely, two time-dependent and one stationary, attractors are numerically shown to coexist for some parameters.

Since the theoretical prediction of laser chaos by Haken[1] through the identification of the simpler laser equations with the well-known Lorenz equations,[2] the experimental observations of the chaotic behavior in lasers have become a very active chapter of laser physics. However, since the conflicting requirements for the occurrence of chaotic pulsations, namely, on the one hand the bad cavity and single-mode requirements and, on the other hand, the extremely high pumping condition, an experimental realization is difficult to achieve, and the only successful experiments were reported in Ref. 3. Indeed for a long time the very high pumping condition was a main obstacle to observe Lorenz-type chaos in lasers.[4,5] On the other hand, the Hopf bifurcation in the Lorenz equations is known to be subcritical.[2,6] This fact suggests that it might be possible to observe some stable time-dependent attractors below the Hopf bifurcation point, which means the existence of bistability between a time-dependent solution and the stationary-laser operation (constant laser intensity, or stationary convection in the case of fluids). Indeed, York and York[7] have numerically observed chaotic motions before the second threshold, the so-called preturbulence. Later, in the context of lasers, similar phenomena were discussed by Casperson[8] and by Narducci, Sadiky, Lugiato, and Abraham.[9] Casperson considered the question of where the pulsing state will end up if one starts initially from a pulsing state and decreases the pumping, i.e., he looked at the lower boundary of the pulsing solution. All these results indicate the coexistence of certain pulses with the steady lasing state. In this Rapid Communication we will show, by making use of this bistability and the phenomenon of anomalous switching,[10] that the pulsing threshold can be considerably reduced.

The reason behind this reduction is that the results of linear stability analysis become invalid in the case of abrupt change of parameters. In fact, the so-called second laser threshold and laser instabilities usually studied in most occasions refer to a kind of instability which occurs when certain control parameter changes adiabatically over the threshold predicted by linear analysis. For systems whose attractor is single valued, i.e., only one attractor for each set of parameters, this is the only kind of behavior. But many nonlinear systems possess more than one attractor for the same set of parameters. For such systems there exists still another kind of instability which occurs when one abruptly changes the parameters of the system with a finite amount but still below the instability threshold of linear stability analysis. In this case the system may be switched from one of its attractors to the other. A typical example of such phenomena is the so-called anomalous switching in optical bistable systems as discussed in Ref. 10. The authors of Ref. 10 considered a switching between the two branches of the solutions of optical bistability before the instability of the linear analysis is reached, due to an abrupt, finite increase in the external field so that the expression anomalous switching (AS) was coined.

Our starting point is the Lorenz-Haken model[1,2,5] describing the resonant single-mode ring laser with atoms of homogeneously broadened lines. The equations are given by

$$\dot{X}=k(Y-X)\,, \tag{1}$$

$$\dot{Y}=rX-Y-XZ\,, \tag{2}$$

$$\dot{Z}=-bZ+XY\,, \tag{3}$$

where X, Y, and Z refer to the electrical-field amplitude, the macroscopic polarization, and the inversion of the media, respectively. The relaxation constants are defined as $k=\kappa/\gamma_\perp$ and $b=\gamma_\parallel/\gamma_\perp$ with $\gamma_\parallel$ and $\gamma_\perp$ denoting the longitudinal and transversal relaxation constants of the atoms. The parameter r is related to the pumping power. And finally the time has been rescaled by $\gamma_\perp$. The well-known second threshold is given by[1,4]

$$k>b+1\,, \tag{4}$$

$$r_{\text{th}}=\frac{k(k+b+3)}{k-b-1}\,. \tag{5}$$

One can easily convince oneself of the big size of r_{th} and further that its minimal value is 9, nine times the first threshold, which is unity.

Our problem is now formulated as follows. Given the system originally operates near but above its lasing threshold, is it possible for the system to jump up to the pulsing state, which exists down to relatively low pumping values, by abruptly increasing the pumping with a finite amount smaller than the second threshold? How large is the minimal amount which suffices to make the system jump up to pulse? This problem is solved on the computer in the following way.

Suppose r_0 (a little bigger than 1, taken as 1.01 throughout this paper) is the pumping value where the system is initially operating. The corresponding stationary lasing state is $(r=r_0)$

$$X_{0,\pm}=\pm\sqrt{b(r-1)}\,, \tag{6}$$

$$Y_{0,\pm}=\pm\sqrt{b(r-1)}\,, \tag{7}$$

$$Z_{0,\pm}=r-1\,. \tag{8}$$

With $X_{0,\pm}$, $Y_{0,\pm}$, and $Z_{0,\pm}$ $(r=r_0)$ as initial conditions, Eqs. (1)-(3) are numerically solved for some new pumping value $r<r_{\rm th}$ and the evolution is traced to the stable stage to see whether the system ends up at the stationary state or at the pulsing state. The minimal value of r which makes the system switch to the pulsing state will be identified as the threshold for the anomalous switching and denoted as $r_{\rm AS}$, for a general discussion, see Ref. 11.

Figure 1 shows some results on $r_{\rm AS}$ and the corresponding second threshold for $k=2$ and $b\le 0.6$. We see that $r_{\rm AS}$ can be as small as nearly half of $r_{\rm th}$ for some small b values. For example, for $b=0.01$ the second threshold is 10.121 21 and the $r_{\rm AS}$ is 5.469.

Figure 2 shows the multistable behavior for $k=2$, $b=0.01$. The dashed lines represent the unstable attractors. The straight line $X=0$ is the state without lasing. The curve marked by SS stands for the stationary lasing state. A_1, A_2, and A_3 are three time-dependent attractors, which are discovered by using the anomalous switching technique. The time-dependent attractors in Fig. 2 are drawn in the following way. Given an r value, e.g., r_1, we can catch one of the pulsing solutions by using AS and we follow the evolution to the stable state and then sample the maximal absolute value of X within the final time interval which is larger than some characteristic time, e.g., the period of the pulse when the pulse is periodic. We then decrease or increase the r value around r_1 with a small amount and solve the equations with the final values of X, Y, and Z of the last time as our new initial values and sample another maximum of X. Repeating the steps again and again we can extend the curve in both directions and finally form the whole curve. For some r values the curve jumps to the lower one or rises to the upper one. The corresponding attractors will end up at that r value. No pulsing solution has been found for an r value smaller than the lower end of A_1. Therefore this lower end also represents the lower bound of the pulsing solution. As can be seen in Fig. 2, this end coincides with the AS threshold (represented by a dash-dotted line). This means that, starting from the lasing state near the first threshold, we can reach the pulsing state by AS technique as soon as the latter starts to exist. This coincidence[12] is, however, not necessarily true for all AS phenomena. Generally the AS threshold lies somewhere between the two ends of the bistable range as it occurs in optical bistable systems.[10] For our present system, we have checked this point for other parameters, e.g., those displayed in Fig. 1. No remarkable difference has been found between the lower end of A_1 and $r_{\rm AS}$.[12]

Now we give a brief phase-space explanation of this AS phenomenon based on our numerical simulation and leave the complete and more detailed discussion to a regular paper.[11] For initial pump value we choose the stationary lasing state as an initial condition. When we suddenly switch the pump, the system cannot respond to this pump switch rapidly enough and therefore cannot follow up the stationary state to the switched pump value. Instead the system stays at the same phase-space point. But for the new pump value this phase-space point now no longer corresponds to the stationary state, it can well fall into the basin of the pulsing state, which coexists with the stationary state. When this is the case, we have an AS phenomenon. The system will be switched to the pulsing state. Therefore our numerical simulations naturally leads to a definition of this AS threshold. *The AS thresh-*

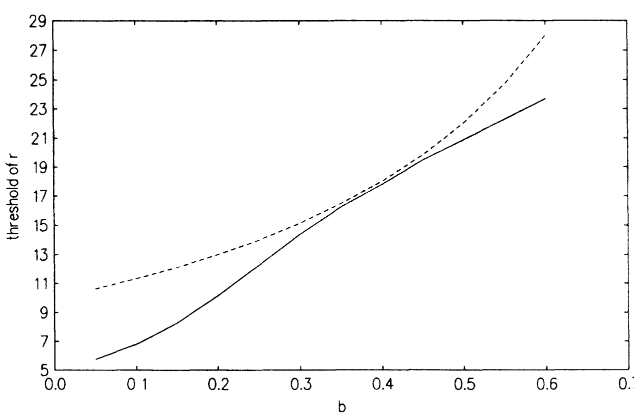

FIG. 1. The AS threshold (solid line) and the second threshold (dashed line) of the linear analysis with respect to b for $k=2$.

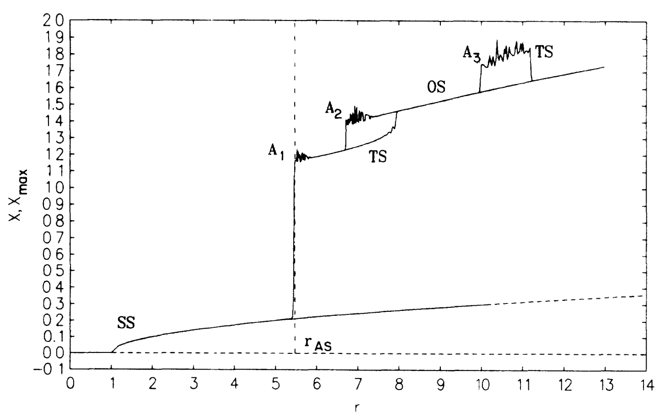

FIG. 2. The stationary states and the maximal amplitude of X with respect to r. TS refers to two-side- and OS to one-side-twisting pulsing, as explained in the text.

old is the pump value, at which the steady state with the initial pump value just falls on the boundary of the attracting basin of the pulsing state corresponding to the new switched pump. However, we shall not go into detail about this matter, further discussion will be presented elsewhere.[11]

It is also interesting to look at the multistabilities revealed by the AS technique. Figure 2 shows several parameter ranges of bistable and tristable behaviors[13] between the stationary-lasing state and various pulsing states. The tristability seems to be a new finding of this paper. Attractors marked by TS are two-side-twisting solutions around both stationary points $S_+:(X_{0,+}, Y_{0,+}, Z_{0,+})$ and $S_-:(X_{0,-}, Y_{0,-}, Z_{0,-})$ and that marked by OS is one-side-twisting around one of the two stationary states S_+ or S_-. The behavior that occurs on A_2 needs more explanation. The large portion of this attractor corresponds to the OS solution, as marked in Fig. 2. As we decrease r, such a solution undergoes period-doubling bifurcations until very high periods which seem to be difficult to follow. After r is decreased to a value smaller than $r=7.03$, the solution becomes TS-type and still with high periods, extending down to the lower end of this attractor. An example of the OS solution near this OS-TS transition is given in Fig. 3. As pointed out by Sparrow,[14] the TS strange attractors originate from homoclinic explosions. Here we see that the OS attractor, which can be attributed to the subcritical Hopf bifurcation of the stationary-lasing state, can change into a TS-type attractor or it can itself evolve into an OS chaotic attractor. We therefore generally have two kinds of origins of the pulsing states in this model, and a transition between them is also possible. The fact that both pulsing states can be chaotic or at least multiperiodic is indicated by the zigzag parts of the curves in Fig. 2 (see also the trajectory shown in Fig. 3).

Finally, we mention that the two-threshold phenomenon discussed by Zeghlache and Mandel[15] can be also explained as a consequence of the AS after taking into account our previous results[16] on the nature of the Hopf bifurcation (see also the discussion in Ref. 16 about this point). Another system we should mention is a dispersive optical bistable system. In a recent paper[17] we have uncovered many large amplitude time-dependent solutions, some of which are completely isolated from the stationary state. There we asked the question how to experimentally observe such attractors. Here we see that the AS technique could be used to serve this purpose.

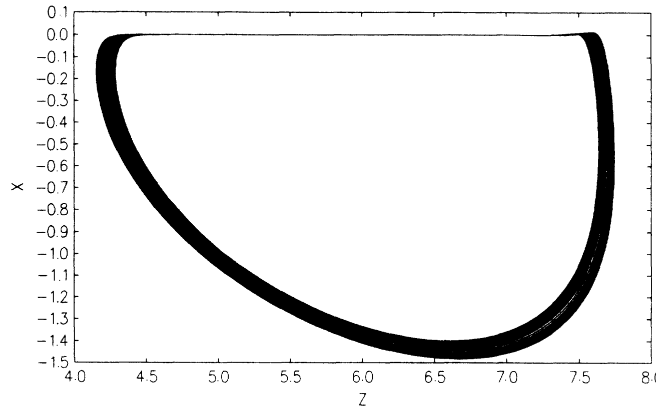

FIG. 3. An example of OS attractor caught by the AS technique, where $r=7.06$, $k=2.0$, and $b=0.01$. To ensure that the attractor is really OS type, we have traced the evolution to dimensionless time $\gamma_\perp t=18000$. After the transient stage, the solution is always OS type. The trajectory drawn in the figure is the evolution from $\gamma_\perp t=12000$ to $\gamma_\perp t=18000$.

To conclude, we would like to emphasize the significance of the much lower threshold due to the AS phenomenon. Previously the time-dependent attractors before the second threshold have been observed by many authors by choosing different initial conditions.[7-9] But how to experimentally observe such attractors is still a question to be answered, since these numerically arbitrary initial conditions are not always physically realizable. Of course, one can first go above r_{th} to reach the chaotic attractor and then adiabatically decrease the pumping to reach the lower end of the time-dependent attractor. This means that we have to attain the very high threshold, which, as we mentioned at the beginning, is a main difficulty for most of the normal lasers. The AS technique suggests a possible way to observe such lower pumping pulsings by starting from experimentally easily attainable initial conditions. This will certainly broaden the range of laser types which produce chaotic laser light. The findings of multistabilities formed by time-dependent attractors of different origins will also help further the understanding of this model.

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[12]For a more suitable definition of AS threshold and an explanation of this coincidence, see Ref. 11.

[13]Due to the symmetry property of this system under the transformation $X, Y, Z \rightarrow -X, -Y, Z$, a stable OS attractor twisting around one of the states, say, S_+ should also exist if the attractor around S_- is stable. Similarly, if an asymmetrical TS attractor which mostly twists around S_+ exists there should also exist another TS attractor which twists mostly around S_- to conserve this symmetry. Mathematically we therefore should say quartistability. But these two attractors reveal one physical existence. We ignore this mathematical difference.

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