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Path integrals and non-Markov processes. II. Escape rates and stationary distributions in the weak-noise limit

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The path-integral formalism developed in the preceding paper [McKane, Luckock, and Bray, Phys. Rev. A **41**, 644 (1990)] is used to calculate, in the weak-noise limit, the rate of escape Γ of a particle over a one-dimensional potential barrier, for exponentially correlated noise $\langle \xi(t)\xi(t')\rangle = (D/\tau)\exp\{-|t-t'|/\tau\}$. For small D, a steepest-descent evaluation of the appropriate path integral yields $\Gamma \sim \exp(-S/D)$, where S is the "action" associated with the dominant ("instanton") path. Analytical results for S are obtained for small and large τ, and (essentially exact) numerical results for intermediate τ. The stationary joint probability density for the position and velocity of the particle is also calculated for small D: it has the form $P_{st}(x,\dot{x}) \sim \exp[-S(x,\dot{x})/D]$. Results are presented for the marginal probability density $P_{st}(x)$ for the position of the particle.

I. INTRODUCTION

This paper is the second in a series devoted to the study of a class of non-Markov processes using path-integral methods. The first of these,[1] to be referred to as paper I, establishes the general formalism (and contains extensive references to earlier work). The present paper is concerned exclusively with obtaining concrete results in the limit of weak noise, where the path integrals discussed in paper I can be evaluated explicitly by the method of steepest descents. A preliminary account of this work was given in Refs. 2 and 3.

We will be primarily interested in two quantities: (i) the rate of escape Γ of a particle over a potential barrier, and (ii) the stationary joint probability distribution $P_{st}(x,\dot{x})$ for the position and velocity of the particle. Both of these have, for small noise strength D, an exponential dependence on D. For example, $\Gamma \sim \exp(-S/D)$, where S is the extremal action for a path $x(t)$ connecting the stable (or metastable) minimum of the potential and the unstable maximum (see Fig. 1). Such a path is an "instanton" of the theory,[4] and the leading exponential contribution to Γ can be obtained from the simplest one-instanton calculation. To calculate the prefactor requires allowing for multi-instantons, and including small fluctuations around the extremal path. Such refinements introduce considerable technical complexities, and will therefore be deferred to a planned future publication.[5]

The system we consider consists of an overdamped particle moving in one dimension in a potential $V(x)$ and subjected to a random noise $\xi(t)$. The Langevin equation is

$$\dot{x} = -V'(x) + \xi(t) , \tag{1}$$

where overdots and primes indicate derivatives with respect to x and t, respectively. An additional "inertial" term, proportional to $\ddot{x}$, on the left-hand side of (1) introduces no new points of principle. The effects of such a term will be discussed in detail in a separate paper.[6]

The noise $\xi(t)$ will be assumed to be Gaussian (other possibilities will be discussed briefly below), with zero mean, so that it is completely specified by its second moment. We will be concerned almost exclusively with exponentially correlated noise:

$$\langle \xi(t)\xi(t')\rangle = (D/\tau)\exp(-|t-t'|/\tau) , \tag{2}$$

which, as discussed in paper I, represents the simplest departure from white noise. This is connected with the fact (see paper I) that this one-dimensional non-Markov process is equivalent to a two-dimensional Markov process. In particular, the probability density functional for $\xi(t)$ [cf. Eq. (21) of paper I],

$$P[\xi] = \mathcal{N} \exp\left[-\frac{1}{4D}\int_{-\infty}^{\infty} dt(\xi^2 + \tau^2\dot{\xi}^2) \right] , \tag{3}$$

contains only zero- and first-order time derivatives of ξ. A change of variable from $\xi(t)$ to $x(t)$, using (1), yields the probability density functional for $x(t)$

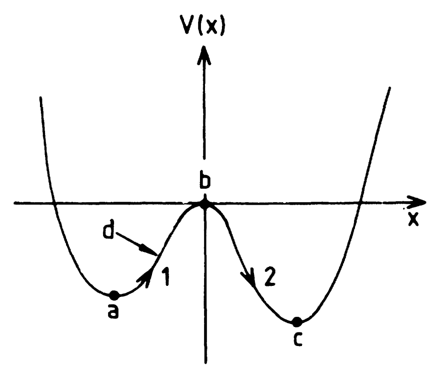

FIG. 1. Typical potential considered in this paper. The rate of escape Γ from the left-hand well is governed by the "action" S of the instanton associated with the "uphill" path 1: $\Gamma \sim \exp(-S/D)$. The instanton associated with the "downhill" path 2 has zero action. The point d is the inflection point of the potential on the uphill path.

$$P[x]=\mathcal{N}J[x]\exp(-S[x]/D)\ , \tag{4}$$

$$S[x]=\tfrac{1}{4}\int_{-\infty}^{\infty}dt[(\dot{x}+V')^2+\tau^2(\ddot{x}+\dot{x}V'')^2]\ , \tag{5}$$

where $J[x]$, the Jacobian of the transformation, is given in paper I. Since $J[x]$ is independent of D, it does not enter physical quantities at the leading exponential order considered in this paper.

As discussed in paper I, the forms (3) and (5) are appropriate for paths defined on the infinite time interval $(-\infty,\infty)$. If we wish to consider processes occurring on finite (or semi-infinite) time intervals, additional "surface" terms must be included, as in Eqs. (54) and (55) of paper I. We will return to this point in Sec. VI, which deals with the stationary probability density generated by Eq. (1).

The outline of this paper is as follows. The equations which determine the escape rate are derived in Sec. II, and solved analytically for small and large τ in Secs. III and IV, respectively. Numerical results for intermediate τ are presented in Sec. V. Section VI is concerned with the stationary probability distribution: a formal expression is derived for small D and numerical results presented. Analytic results, which exhibit an interesting change of behavior as τ passes through a critical value, are derived for the stationary distribution near the top of the barrier. Section VII describes various extensions of the instanton approach, in particular to multiplicative noise and to field theory. The paper concludes with a discussion and summary of the results.

II. ESCAPE RATE FOR EXPONENTIALLY CORRELATED NOISE

Our goal in this section is to calculate the rate Γ for a particle, initially located at the point a in the left-hand well in Fig. 1, to escape over the barrier into the right-hand well. At this point we can leave Γ somewhat loosely defined—we could, for example, take it to be the reciprocal of the "mean-first-passage time," being the mean time to make the first traversal of the barrier. All definitions will give the same result at the level of the leading exponential behavior discussed here. The definition will be made more precise in the next paper of this series,[5] when prefactors are calculated. In particular, it can be shown[5] that the probability that the particle does not cross the barrier in time T has the form $\exp(-\Gamma T)$.

Consider the conditional probability (density) $P_{1|1}(c,T/2|a,-T/2)$ to find the particle at point c at time $T/2$ given that it was at point a at time $-T/2$. In general, as was stressed in paper I, this quantity is not well defined until one specifies the distribution of velocities at time $-T/2$ (or at some earlier time). Such subtleties need not concern us here, however: for the instanton paths $x_c(t)$ which dominate the path integral for $P_{1|1}(c,T/2|a,-T/2)$, the velocity $\dot{x}_c$ (and indeed all higher derivatives) vanish at the turning points a,b,c of the potential for $T\to\infty$. Since the error in the action associated with taking T to infinity is exponentially small in T, we will set $T=\infty$ *ab initio*: T reappears in the prefactor for the conditional probability (see below) as a consequence of the time-translational invariance of the action.[4]

The instanton path $x_c(t)$ is obtained by minimizing the "action" $S[x]$ over paths satisfying $x(-\infty)=a$ and $x(\infty)=c$. In fact, it will turn out that the path can be split into two subpaths, connecting the points a,b and b,c, respectively. The former ("uphill") path has a finite, nonzero action while the action for the latter ("downhill") path is precisely zero. This is in accord with physical expectations: the descent from the unstable maximum b to the new minimum c is a "free" descent, and proceeds in the absence of external noise.

For exponentially correlated noise, Eq. (2), the action is given by Eq. (5). By inspection, the zero-noise Langevin equation, $\dot{x}=-V'$, gives zero action for all τ, and generates the desired "downhill" instanton path, with $x(-\infty)=b$ and $x(\infty)=c$. The "uphill" path, satisfying $x(-\infty)=a$ and $x(\infty)=b$, corresponds to a nontrivial extremum of the action. The extremal condition $\delta S[x]/\delta x(t)=0$ yields the fourth-order differential equation

$$-\ddot{x}+V'V''+\tau^2(\ddddot{x}+3\ddot{x}\dot{x}V'''+\dot{x}^3V''''-\dot{x}^2V''V'''-\ddot{x}V''^2)=0\ . \tag{6}$$

Multiplying by $\dot{x}$ and integrating with respect to t gives[7]

$$\dot{x}^2-V'^2=\tau^2(2\dot{x}\dddot{x}-\ddot{x}^2+2\dot{x}^3V'''-\dot{x}^2V''^2)\ . \tag{7}$$

The integration constant (the "energy") vanishes for the instanton paths, because $\dot{x}$ and higher derivatives vanish when $V'=0$, i.e., at the turning points a,b,c of the potential.

In the original work using this approach,[2] Eq. (7) was solved analytically for small and large τ, for general potentials, and numerically for general τ for the quartic bistable potential $V(x)=-x^2/2+x^4/4$ (in dimensionless variables). The numerical solution requires fixing three boundary conditions. Setting the logarithmic derivative of x for $t\to\mp\infty$, to the values obtained by linearizing (6) around a and b, respectively (and retaining only the dominant exponential in the solution), provides two of the boundary conditions. For the third boundary condition, we remove the invariance of the action under time translations, $x(t)\to x(t+t_0)$ [which follows from the observation that the integrand of the action functional (i.e., the "Lagrangian") contains no explicit time dependence], by fixing $x(0)$ to any value in the interval (a,b), e.g., $x(0)=(a+b)/2$. This approach is a little inelegant, however, as in practice the conditions on the logarithmic derivative have to be applied at large but finite positive and negative times, the contributions from the outer regions computed analytically, and insensitivity to the choice of boundary points verified.

Fortunately, however, a much more elegant approach is available: translational invariance in time suggests that we introduce the velocity $y=\dot{x}$ and solve for the function $y(x)$ instead of for $x(t)$. Then $\ddot{x}=yy'$, $\dddot{x}=y(yy''+y'^2)$, and (7) becomes

$$y^2-V'^2=\tau^2(2y^3y''+y^2y'^2+2y^3V'''-y^2V''^2)\ . \tag{8}$$

This is now only a second-order differential equation, defined on the *finite* interval (a,b), with the simple boundary conditions

$$y(a)=0=y(b) . \quad (9)$$

The action can also be written directly in terms of $y(x)$:

$$S[y]=\frac{1}{4}\int_a^b (dx/y)[(y+V')^2+\tau^2 y^2(y'+V'')^2] . \quad (10)$$

One can verify that the extremal condition $\delta S[y]/\delta y(x)=0$ reproduces (8).

Finally, the leading-order contribution to $P_{1|1}(c,T/2|a,-T/2)$ has the form $\text{const}\times T\exp(-S/D)$, where S is now the extremal action. The prefactor is obtained by including small fluctuations around the extremal path $x_c(t)$, and will be discussed in detail elsewhere.[5] In particular, the factor T arises from the above-mentioned invariance of $S[x]$ under time translations. (It might be thought that there should be a separate factor of T for the uphill and downhill paths, giving a factor T^2 overall. In fact, the uphill and downhill instantons interact, and the integration over the relative coordinate yields a T-independent contribution to the prefactor. A single factor of T arises from integrating over the "center of mass" of the instanton pair). Identifying the coefficient of T as the escape rate gives

$$\Gamma=\text{const}\times\exp(-S/D) . \quad (11)$$

Before presenting the results of solving (8) [with boundary conditions (9)] numerically, it is instructive to consider first the exactly soluble white-noise limit (i.e., $\tau=0$), and expansions around this limit in powers of τ^2. In Sec. IV, we will consider the opposite limit of large τ, and obtain for the first time the leading correction to this limit. These expansions for small and large τ are valid for general potentials $V(x)$. Finally, in Sec. V, we will present numerical results for general values of τ for the special case of the "quartic bistable" potential (16).

III. SMALL-τ EXPANSION

For white noise, $\tau=0$ and (8) gives

$$y=\pm V' . \quad (12)$$

Since $y\equiv\dot{x}>0$ for both "uphill" ($V'>0$) and "downhill" ($V'<0$) paths (i.e., paths 1 and 2, respectively, in Fig. 1) the upper and lower signs in (12) correspond to uphill and downhill instanton solutions respectively. The corresponding action is given by (10) with $\tau=0$, i.e., $S=0$, for the downhill solution and

$$S\equiv S_0=\int_a^b dx\, V'(x)=V(b)-V(a)\equiv\Delta V \quad (13)$$

for the uphill solution, the subscript zero indicating the white-noise limit. Inserting this result into (11) gives the standard Arrhenius formula $\Gamma\sim\exp(-\Delta V/D)$. The prefactor is known for white noise from calculations based on the Fokker-Planck equation.[8] It will be calculated explicitly within the path-integral approach in a future publication.[5]

The white-noise result is the starting point for a systematic expansion in powers of τ^2. Because of the stationary property of $S[y]$, knowing $y(x)$ to order τ^{2n} determines $S[y]$ to order τ^{2n+2}. Since the downhill path has zero action for all τ, we consider only the uphill path from now on. Setting

$$y(x)=\sum_{n=0}^{\infty}\tau^{2n}y_n(x) , \quad (14)$$

substituting into (8), and equating coefficients of τ^{2n}, one obtains

$$y_0=V' ,$$

$$y_1=2V'^2V''' ,$$

$$y_2=14V'^3V'''^2+8V'^2V''^2V'''+10V'^3V''V''''+2V'^4V''''' ,$$

and

$$S=\Delta V+\tau^2\int_a^b dx\, V'V''^2-\tau^4\int_a^b dx\, V'^3V'''^2$$
$$+\tau^6\int_a^b dx(V'^5V''''^2-6V'^4V'''^3-4V'^3V''^2V'''^2)$$
$$+O(\tau^8) . \quad (15)$$

This extends by one term the result quoted in Ref. 2. The $O(\tau^2)$ term was first derived in Ref. 9. Obtaining higher-order terms is straightforward but tedious.

Equation (15) is a special case of the small-τ expansion presented in Ref. 3 (see also paper I) for colored noise with a general correlator

$$\langle\xi(t)\xi(t')\rangle=(D/\tau)C(|t-t'|/\tau)$$

for which all the moments of C exist. Reference 3 also shows how (15) may be generalized to non-Gaussian noise.

While (15) is valid for general potentials, the numerical results obtained for general τ must preforce be restricted to specific potentials. The form for $V(x)$ most commonly studied in the literature is the "quartic bistable" potential. In dimensionless units this is

$$V(x)=-x^2/2+x^4/4 . \quad (16)$$

For this potential (15) becomes, after normalizing by $\Delta V=\frac{1}{4}$,

$$S/\Delta V=1+\tfrac{1}{2}\tau^2-\tfrac{6}{5}\tau^4+\tfrac{279}{35}\tau^6+O(\tau^8) . \quad (17)$$

Before leaving this section, a comment on the nature of the small-τ expansion is in order. A perturbative treatment of a differential equation like (8), which involves expanding in the coefficient of the highest-order derivative, yields a singular expansion. In particular, terms in $x_c(t)$ [or equivalently $y_c(x)$] which vanish faster than any power of τ for $\tau\to 0$ are not picked up by the expansion. In view of this, the small-τ expansion is likely to be at best asymptotic.

Finally we remark that since the instanton method is based on a steepest descent approach valid for small D, the limit $D\to 0$ is implicitly taken before the limit $\tau\to 0$ in the above results. To obtain the analogous results when the limits are taken in the opposite order requires a different technique.[10]

IV. LARGE-τ EXPANSION

The large-τ limit can be understood at a number of levels. The leading term, Eq. (20) below, was first derived in Ref. 9, using a path-integral approach. Subsequently, the result was rederived using simple intuitive arguments.[11] The idea is that for large τ the noise fluctuates so slowly that the particle adiabatically follows the noise, i.e., its position $x(t)$ satisfies $V'(x)=\xi(t)$. This corresponds to neglecting $\dot{x}$ in the Langevin equation, which is justified *a posteriori* for large τ since $\dot{x}$ is of order τ^{-1}. The escape rate is the rate for ξ to reach its maximum value $V'(d)$, where d is the inflection point of the potential (see Fig. 1). Since ξ itself satisfies an Ornstein-Uhlenbeck process [Eq. (13) of paper I], this rate can be calculated by elementary methods,[11] and yields Eq. (20) below.

While the above approach is physically appealing and intuitive, it is not obvious how to extend it beyond the leading term. (A first attempt in this direction[12] gives results quite different from those derived below). The instanton approach, on the other hand, provides the basis for a systematic large-τ expansion. It turns out, however, that this expansion is more delicate than the analogous small-τ expansion, so we will limit our considerations to the large-τ limit and the leading correction to it. We emphasize that here, as elsewhere in this paper, the results obtained are valid in the limit $D\rightarrow 0$, this limit underlying the whole instanton method. In particular, it is implicit throughout that the small-D limit is taken before the large-τ limit.

Guided by the physical arguments above, which indicate that the instanton width will be of order τ for large τ, we first make the rescaling $t\rightarrow\tau t$. In terms of the variable $y\equiv\dot{x}$, this means $y\rightarrow y/\tau$. Then the action functional (10) becomes

$$S[y]=(\tau/4)\int_a^b dx[(V'^2/y+yV''^2)+2\tau^{-1}(V'+yy'V'') +\tau^{-2}(y+yy'^2)] . \qquad (18)$$

To leading order for large τ we retain only the $O(1)$ terms in the integrand. Extremizing with respect to y yields immediately

$$y(x)=\pm V'(x)/V''(x) .$$

The required solution, corresponding to path 1 in Fig. 1, has $y>0$ everywhere. Since $V''(x)$ changes sign (from positive to negative) as x passes through the inflection point d, the required solution is

$$y_\infty(x)=V'(x)/|V''(x)| , \qquad (19)$$

i.e., one takes the plus sign in the interval (a,d) and the minus sign in the interval (d,b). The subscript ∞ in (19) indicates the leading large-τ solution. The corresponding action, obtained by putting (19) into (18) [and retaining only the $O(1)$ terms in the integrand], is

$$S_\infty=(\tau/2)V'(d)^2 . \qquad (20)$$

For the quartic bistable potential (16), this becomes

$$S_\infty=\tfrac{2}{27}\tau=\tfrac{8}{27}\tau\Delta V . \qquad (21)$$

These leading order results agree with those obtained in Refs. 9 and 11.

The leading correction to (20) is naively obtained by using (19) for y in the $O(\tau^{-1})$ terms in the integrand of (18), which give (naively) an $O(1)$ contribution to the action. The resulting integrals, however, *diverge*, indicating that the large-τ behavior is more complicated than a simple expansion in powers of τ^{-1}. In fact, we shall show that the leading correction to (20) is a term of order $\tau^{1/3}$.

A heuristic argument serves both to illuminate the origin of the $\tau^{1/3}$ term, and to suggest a scheme for calculating its coefficient. The key point is that $y_\infty(x)$, given by (19), diverges at the inflection point $x=d$. Expanding around this point, via $x=d+\tilde{x}$, yields

$$y\simeq V'/|V'''\tilde{x}| \qquad (22)$$

for small $\tilde{x}$, where we have suppressed the subscript on y, and the derivatives of V are understood to be evaluated at $x=d$. Consider now the term $yy'V''$ in the second bracket in (18). For small $\tilde{x}$, this term is of order

$$yy'V''\sim(V'^2/|V'''|)\tilde{x}^{-2} . \qquad (23)$$

The integral of this term over $\tilde{x}$ therefore diverges at $\tilde{x}=0$. We may estimate its contribution to the action, however, by cutting off the divergent integral at the point where the expansion in powers of τ^{-1} starts to break down, i.e., when (23) becomes comparable with the leading-order terms

$$\tau V'(x)^2/y\sim\tau yV''(x)^2\sim\tau V'|V'''\tilde{x}| . \qquad (24)$$

Comparing (24) with (23) shows that the τ^{-1} expansion for y breaks down when

$$|\tilde{x}|\sim\tilde{x}_0\equiv(V'/|V'''|^2\tau)^{1/3} .$$

We estimate the leading correction to (20) by cutting off the divergent integral of (23) at $\tilde{x}_0$. This gives a contribution

$$\delta S\sim[V'(d)^5/|V'''(d)|]^{1/3}\tau^{1/3} \qquad (25)$$

to the action. As a check on the internal consistency of this approach, we compare the relative sizes of terms in the first and third brackets in (18). While the third bracket is nominally down by a further factor of τ^{-1} compared to the second bracket, the integrand is more divergent, $yy'^2\propto|\tilde{x}|^{-5}$. In fact, all three terms in (18) become comparable for $|\tilde{x}|\sim\tilde{x}_0$, and cutting off the divergent integral at this point gives another contribution of the same order as (25).

The above heuristic arguments suggest the following scaling treatment of the large-τ limit. The key observation is that the $O(\tau^{1/3})$ contribution is associated with a breakdown of (19) near the inflection point for large but finite τ. Near the inflection point, therefore, we rescale the variables according to

$$y=(V'/|V'''|\tilde{x}_0)g(\tilde{x}/\tilde{x}_0) . \qquad (26)$$

The condition that (22) be recovered for $\tilde{x}\gg\tilde{x}_0$ requires

$$g(z)\rightarrow 1/|z|, \quad z\rightarrow\pm\infty . \qquad (27)$$

Substituting the form (26) into (18), changing the integration variable to $z=\bar{x}/\bar{x}_0$, subtracting off the leading-order contribution (20), and expanding the remainder to $O(\tau^{1/3})$, gives

$$\delta S=(\tau^{1/3}/4)[V'(d)^5/|V'''(d)|]^{1/3} \times\int_{-\infty}^{\infty} dz\{1/g(z)+g(z)[g'(z)-z]^2-2|z|\} . \quad (28)$$

The extension of the limits to $\pm\infty$ is justified by the convergence of the integral.

Equation (28) has the same form as (25), but now the integral gives a universal (i.e., potential-independent) prefactor. The equation for $g(z)$ is obtained by substituting (26) into the general equation (8) (after the rescaling $y\to y/\tau$) and again expanding to leading order for large τ. The result is

$$2g^3g''+g^2g'^2-2g^3-z^2g^2+1=0 , \quad (29)$$

which is to be solved with the boundary conditions (27). Since, by inspection $g(z)=g(-z)$ satisfies both the differential equation and the boundary conditions, we solve (29) numerically for $z\geq 0$ only, using (27) for one boundary condition and $g'(0)=0$ for the other. The final result is, including the leading-order term (20),

$$S=(\tau/2)V'(d)^2\{1+\tau^{-2/3}\Lambda[V'(d)|V'''(d)|]^{-1/3} +\cdots\} , \quad (30)$$

$$\Lambda=2.046\,204\ldots , \quad (31)$$

where Λ is a universal number, independent of the details of the potential.

For the special case of the quartic bistable potential (16), we obtain $d=-1/\sqrt{3}$, $V'(d)=2/3\sqrt{3}$, $|V'''(d)|=2\sqrt{3}$, and

$$S=\tau\Delta V(\tfrac{8}{27}+\lambda\tau^{-2/3}+\cdots) , \quad (32)$$

where

$$\lambda=2(2^4/3^8)^{1/3}\Lambda=0.550\,844\ldots . \quad (33)$$

Equation (32) may be compared with the action obtained from the solution of the full differential equation (8). According to (32), $\lambda=\lim_{\tau\to\infty}(S-S_\infty)/\tau^{1/3}\Delta V$. The right-hand side is found to decrease steadily with increasing τ, approaching a constant value for large τ. For $\tau=10^6$ it has the value 0.5509. . . , consistent with (33). The numerical solution also allows us to estimate the next term in the expansion in (32). Anticipating an expansion in powers of $\tau^{-2/3}$, we form the quantity $[(S-S_\infty)/\Delta V-\lambda\tau^{1/3}]\tau^{1/3}$ and examine numerically its behavior for large τ. We indeed find that this quantity approaches a constant ($\simeq 0.37$), verifying that the next term in the braces in (32) is of order $\tau^{-4/3}$ and suggesting a systematic expansion in powers of $\tau^{-2/3}$.

V. NUMERICAL RESULTS FOR GENERAL τ

The results of solving (8) numerically, with boundary conditions (9), for the quartic bistable potential (16), are presented in Fig. 2. The equation was solved using the COLSYS package,[13] which requires an initial guess for the solution. It is the initial guess that distinguishes between uphill and downhill instanton solutions: these satisfy the same differential equation (8) and the same boundary conditions (9), which read $y(-1)=0=y(0)$ in this case.

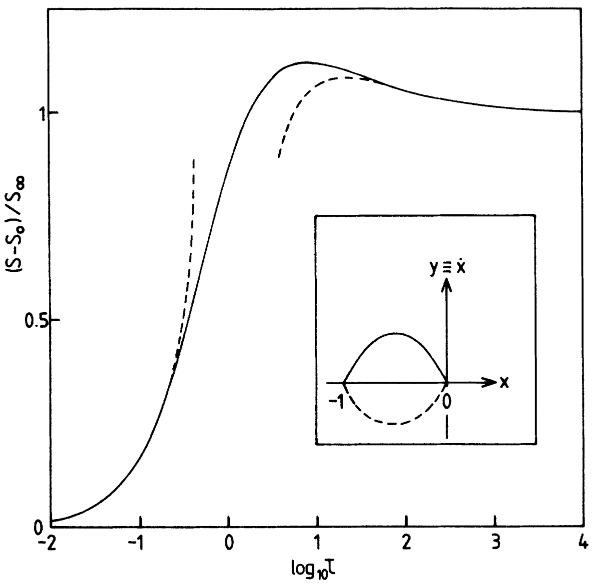

FIG. 2. Instanton action S as a function of the noise correlation time τ, plotted as $(S-S_0)/S_\infty$ vs $\log_{10}\tau$, for the "quartic bistable" potential (16), where S_0 and S_∞ are the $\tau=0$ and $\tau\to\infty$ results (13) and (21), respectively. The dashed curves are the small- and large-τ expansions (17) and (32). The inset shows the general form (schematic) of the uphill (continuous curve) and downhill (broken curve) instanton solutions in the $x-\dot{x}$ plane.

For the uphill solution of interest here, one has (see the inset in Fig. 2) $y>0$ everywhere, and the qualitative form of the solution suggests the parabolic initial guess $y^{(0)}(x)=-\lambda x(1+x)$. The value of λ was fixed by linearizing the differential equation near the boundary points $x=-1,0$ to determine the behavior of the solution near these points. This gives $y^{(0)}\simeq\lambda_{-1}(1+x)$ for x near -1, and $y^{(0)}\simeq-\lambda_0 x$ for x near 0, where $\lambda_{-1}=\min(2,1/\tau)$ and $\lambda_0=\min(1,1/\tau)$. In practice, the compromise value $\lambda=(\lambda_{-1}+\lambda_0)/2$ was used in the initial guess. Convergence to the final solution was very fast in all cases.

For convenience of presentation, we do not plot S directly, but rather the ratio $(S-S_0)/S_\infty$, where S_0 and S_∞ are the white-noise and large-τ results (13) and (21) respectively. Similarly, we use $\log_{10}\tau$ rather than τ as abscissa in order to present results for a large range of τ without unduly compressing the region $\tau=O(1)$, which contains most of the structure. The small- and large-τ expansions (17) and (32) are shown by the broken curves on the figure. We note that the "bridging formula" suggested in Ref. 11, connecting the $\tau=0$ and large-τ regimes, corresponds to $S=S_0+S_\infty$, i.e., to $(S-S_0)/S_\infty=1$. It is clear from Fig. 2 that this is a good approximation only for very large τ, where S_0 is in any case negligible compared to S_∞.

To check that the differential equation has indeed been solved, we look at the derivative of the action with respect to the "scale" of the instanton, i.e., the derivative with respect to the scale change $y_c(x)\to b^{-1}y_c(x)$.[14] In terms of $x_c(t)$, this corresponds to the time rescaling $x_c(t)\to x_c(bt)$. For an extremal path one has $d\ln S(b)/db|_{b=1}=0$. In practice we find that this quantity is typically of order 10^{-10}, indicating that the differential equation has been solved to rather high precision, i.e., the numerical results for S are essentially exact.

The reader may wonder what advantages this numerical approach has over, for example, direct numerical solution of the corresponding two-dimensional Fokker-Planck equation [Eq. (35) below]. The key point is that the small-D limit has been taken explicitly, through the use of the steepest-descent method, to obtain Eq. (11). The evaluation of the action S is then a comparatively trivial task, and may be accomplished with almost arbitrary precision. For $\tau=1$, for example, one obtains $S/\Delta V=1.251\,761\,6$, with an estimated error of 1 in the final place.

VI. STATIONARY DISTRIBUTIONS

In addition to calculating escape rates, the instanton approach may also be used to compute stationary distribution functions in the weak-noise limit. In Sec. VI A we derive a general formal expression for $P_{\rm st}(x,\dot{x})$, the stationary joint probability distribution function for the position and velocity of the particle. The stationary marginal distribution for the coordinate is then obtained from $P_{\rm st}(x)=\int_{-\infty}^{\infty}d\dot{x}\,P_{\rm st}(x,\dot{x})$. Here, as elsewhere in this paper, the results obtained are limited to exponentially correlated noise.

A. The joint probability distribution $P_{\rm st}(x,\dot{x})$

It is convenient to consider in the first instance the conditional probability density $P_{1|1}(x,\dot{x},t|a,0,t_0)$, corresponding to the particle being at rest at the local minimum a of the potential (see Fig. 1) at time t_0. For $t_0\to-\infty$, this conditional probability becomes independent of t and equal to, up to normalization, the required stationary probability density. Evaluating the path integral for P by the method of steepest descents gives, to leading order for small D,

$$P_{\rm st}(x,\dot{x})\sim\exp\{-S(x,\dot{x})/D\}\ . \tag{34}$$

Here $S(x,\dot{x})$ is the minimum of the action $S[x]$ over paths $x(t)$ such that $x(-\infty)=a$, $\dot{x}(-\infty)=0$, $x(0)=x$, $\dot{x}(0)=\dot{x}$, and we have taken the final time to be $t=0$ without loss of generality. The calculation of the preexponential factor in (34) is beyond the scope of the present paper. Equation (34) is the leading small-D result in the following sense:

$$\lim_{D\to 0}-D\ln P_{\rm st}(x,\dot{x})=S(x,\dot{x})\ .$$

Equation (34) can also be obtained from the Fokker-Planck equation for the two-dimensional Markov process which, for exponentially correlated noise, is equivalent to the one-dimensional non-Markov process considered here. The Fokker-Planck equation for $P(x,\dot{x},t)$ [Eq. (89) of paper I] reads

$$\frac{\partial P}{\partial t}=-\dot{x}\frac{\partial P}{\partial x}+\frac{1}{\tau}[1+\tau V''(x)]\frac{\partial}{\partial\dot{x}}(\dot{x}P)+\frac{V'(x)}{\tau}\frac{\partial P}{\partial\dot{x}}+\frac{D}{\tau^2}\frac{\partial^2 P}{\partial\dot{x}^2}\ . \tag{35}$$

The stationary solution is obtained by replacing the left-hand side by zero. In the limit $D\to 0$, we seek a solution of the form

$$P_{\rm st}(x,\dot{x})\sim\exp[-f(x,\dot{x})/D]\ . \tag{36}$$

Putting this form into (35), and retaining only the leading order [i.e., $O(D^{-1})$] terms for small D, yields

$$0=\frac{1}{\tau^2}\left[\frac{\partial f}{\partial\dot{x}}\right]^2-\frac{1}{\tau}\{V'(x)+\dot{x}[1+\tau V''(x)]\}\frac{\partial f}{\partial\dot{x}}+\dot{x}\frac{\partial f}{\partial x}\ . \tag{37}$$

Guided by Eq. (34), but choosing to work with the "velocity" $y(x)\equiv\dot{x}$ instead of $x(t)$, we make the ansatz

$$f(x_0,\dot{x}_0)=S(x_0,\dot{x}_0)\ , \tag{38}$$

where S is the minimum over paths $y(x)$ of the action functional

$$S[y]=\tfrac{1}{4}\int_a^{x_0}(dx/y)[y+V'+\tau y(y'+V'')]^2\ , \tag{39}$$

and the minimum has to be taken over paths for which $y(x_0)=\dot{x}_0$. [The subscript zero will be added to x and $\dot{x}$ where necessary to avoid possible confusion between the endpoint and other points $x(t)$ on the path.] The integrands in (39) and (10) differ by an additional "cross-term" present in (39) which can be integrated immediately to give

$$(\delta S)_{\rm surf}=\frac{\tau}{4}[\dot{x}_0+V'(x_0)]^2 \tag{40}$$

from the upper limit. The analogous contribution from the lower limit vanishes since, by assumption, $\dot{x}$ and $V'(x)$ both vanish at $x=a$. Equation (40) is just the contribution $(\tau/4)\xi^2(0)$ which arises from integrating over the noise for times $t>0$ [cf. Eq. (54) of paper I].

Finally we must check that the ansatz (38) satisfies (37). It is simple to show that

$$\frac{\partial S}{\partial\dot{x}_0}=(\tau/2)A(x_0,\dot{x}_0)\ , \tag{41}$$

$$\frac{\partial S}{\partial x_0}=\tfrac{1}{4}A(x_0,\dot{x}_0)\{1+[V'(x_0)/\dot{x}_0]-\tau[y_c'(x_0)-V''(x_0)]\}\ , \tag{42}$$

where

$$A(x_0,\dot{x}_0)=\{\dot{x}_0+V'(x_0)+\tau\dot{x}_0[y_c'(x_0)+V''(x_0)]\}\ , \tag{43}$$

and $y_c(x)$ is the extremal (or "classical") path (we reserve the word "instanton" for an extremal path connecting turning points of the potential). Substituting these results into Eq. (37) (with $f=S$), one finds that the equation is indeed satisfied. Thus the form (36), with $f(x,\dot{x})=S(x,\dot{x})$, has been obtained both directly from the path-integral representation and via the equivalent two-dimensional Fokker-Planck equation.

Numerical results for the quartic bistable potential (16) may be obtained by solving (8) on the interval $(-1,x_0)$ with boundary conditions $y(-1)=0$ and $y(x_0)=\dot{x}_0$. Since results have been obtained for only part of the $(x,\dot{x})$ plane, due to technical difficulties in the solution of (8) in certain part of the plane [where $y_c(x)$ is multivalued], numerical results for the joint probability density will not be presented here. Instead, we will concentrate on the marginal distribution $P_{st}(x)$, for which precise numerical results are readily attainable. Before doing so, however, we should discuss an important point of principle in the calculation of $S(x,\dot{x})$.

First note that for any even potential $V(x)=V(-x)$, only one sign of x (say, $x<0$) need be considered: results for positive x follow from the inversion symmetry $P_{st}(-x,-\dot{x})=P_{st}(x,\dot{x})$. It follows that $S(-x,-\dot{x})=S(x,\dot{x})$. A corollary of this symmetry argument is the following observation. To calculate $P_{st}(x,\dot{x})$ for points in the right-hand potential well, one should use the minimum at $x=1$ as a reference point, rather than the minimum at $x=-1$. This means that Eq. (8) should be solved in the interval $(x_0,1)$, with boundary conditions $y(1)=0$, $y(x_0)=\dot{x}_0$, corresponding to a classical path which starts at $x=1$ and ends at $x=x_0$. More generally, the correct prescription is to calculate separately the action for classical paths starting at each local minimum of the potential, and ending at the desired point with the desired velocity. The dominant contribution to P_{st} for small D is associated with the *smallest* action.

This discussion generalizes to asymmetric potentials where the two (or more) minima have different depths, as in Fig. 1. Then the stationary probability densities for the two minima will, in general, be different. One can use, however, as a common normalization point, the probability density to find the particle with zero velocity at the top of the barrier (point b in Fig. 1). Relative to this point, the probability densities to find the particle with zero velocity at the minima a and c of Fig. 1 are, up to prefactors, $\exp(S_{ab}/D)$ and $\exp(S_{cb}/D)$, respectively, where S_{ab} and S_{cb} are actions for instantons starting at a and c, respectively, and ending at b. To calculate $P_{st}(x_0,\dot{x}_0)$ for the asymmetric case, the actions $S_a(x_0,\dot{x}_0)$ and $S_c(x_0,\dot{x}_0)$, corresponding to classical paths starting at a and c, respectively, are calculated. In the weak noise limit, $D\to 0$, the path which gives the largest contribution to P_{st} dominates. In this limit, therefore, $P_{st}\sim\exp[-S(x_0,\dot{x}_0)/D]$, with $S=\min[S_a(x_0,\dot{x}_0)-S_{ab},S_c(x_0,\dot{x}_0)-S_{cb}]$.

B. The marginal probability distribution $P_{st}(x)$

The marginal distribution is obtained by integrating the joint distribution over all velocities. For $D\to 0$, this integral can be evaluated by the method of steepest descents, i.e., it is dominated by the velocity for which the action is minimal:

$$P_{st}(x)=\int_{-\infty}^{\infty}d\dot{x}\,P_{st}(x,\dot{x})\sim\int_{-\infty}^{\infty}d\dot{x}\,\exp[-S(x,\dot{x})/D]\sim\exp\{-S(x,0)/D\}\,, \tag{44}$$

where we have used the fact (see below) that the action is minimal for $\dot{x}=0$. The preexponential factors for $P_{st}(x,\dot{x})$ and $P_{st}(x)$ differ by the contribution to the latter from the Gaussian integral over $\dot{x}$ around $\dot{x}=0$. Consistent with the spirit of this paper we consider only the dominant exponential term: calculation of the prefactors is beyond the scope of the present work. The function $S(x,0)$ is sometimes called the "weak-noise potential."

We now argue that $S(x,\dot{x})$ is minimized by $\dot{x}=0$. First, we demonstrate that the function $A(x,\dot{x})$, Eq. (43), vanishes for $\dot{x}=0$, so that S is stationary with respect to $\dot{x}$, through Eq. (41). Then we consider whether the stationary point is a maximum or a minimum.

To prove stationarity we set $\dot{x}_0=0$ in (43), but retain the term in $\dot{x}_0 y_c'(x_0)\equiv y_c(x_0)y_c'(x_0)$ since it will turn out to be nonzero. Thus, dropping the subscripts for brevity,

$$A(x,0)=V'(x)+\tau(yy')|_{y=0}\,. \tag{45}$$

To express the right-hand side in terms of x alone, we use Eq. (8) for $y(x)$. Setting $y=0$ explicitly, but keeping the terms which contain derivatives of y, gives

$$-V'(x)^2=\tau^2(2y^3y''+y^2y'^2)\,, \tag{46}$$

for $y\to 0$. Since the left-hand side in nonzero in general, the product yy' must approach a constant, say C, for $y\to 0$. Then $y''\to -(C/y^2)y'\to -C^2/y^3$, and (46) becomes,

$$-V'(x)^2=-\tau^2C^2=-\tau^2[(yy')|_{y=0}]^2\,. \tag{47}$$

The negative square root being required on physical grounds (see the inset in Fig. 2), one has $(yy')|_{y=0}=-V'(x)$ and hence $A(x,0)=0$ from (45). This shows that $\dot{x}=0$ makes $S(x,\dot{x})$ stationary for fixed x.

It remains to show that the stationary point corresponds to a minimum, rather than a maximum. Unfortunately, we have as yet been unable to do this by analytical means, except when x is near a turning point of the potential (see, for example, Sec. VI D below). Explicit numerical solution of (8), however, has shown $\dot{x}=0$ to be a local minimum of $S(x,\dot{x})$ in all cases studied. We will henceforth *assume* that the global minimum of $S(x,\dot{x})$ for fixed x occurs at $\dot{x}=0$.

C. Numerical results

Numerical results were obtained for the quartic bistable potential (16), for $x_0<0$, by solving Eq. (8) with boundary conditions $y(-1)=0=y(x_0)$, using the COLSYS package[13] with an initial guess corresponding to an uphill path (i.e., with $y>0$ if $0>x_0>-1$ and $y<0$ if

$x_0 < -1$). The corresponding action $S(x_0,0)$ is calculated from Eq. (39). Results for $x_0 > 0$ are obtained immediately from the symmetry $P_{st}(-x)=P_{st}(x)$, i.e., $S(-x_0,0)=S(x_0,0)$, where it is implicit that for $x_0>0$ the boundary conditions $y(x_0)=0=y(1)$ would be used.

The results are presented in Fig. 3, as plots of $S(x,0)$ versus x for various τ. By construction, $S(-1,0)=0=S(1,0)$, reflecting the fact that the minima of the potential are the reference points for the calculation of probability. For white noise, $\tau=0$, one obtains $S(x,0)=V(x)-V(\pm 1)$, equivalent to the Boltzmann distribution through (44). For this case, $S(x,\dot{x})$ is independent of $\dot{x}$: the classical path follows the instanton trajectory $y=V'(x)$. If a final velocity inconsistent with this trajectory is imposed, the noise can make a sudden jump to the value $\xi_0=\dot{x}_0+V'(x_0)$ at the final instant, since for white noise the action contains no time derivatives of the noise.

The data show, with increasing τ, an increasing probability of finding the particle close to the minima of the potential. The most distinctive feature of the data, however, is the "plateau," centered on $x=0$, that develops for $\tau>1$. The length of the plateau is an increasing function of τ. For $\tau\to\infty$, the plateau covers the entire region $-1/\sqrt{3}<x<1/\sqrt{3}$ between the two inflection points of the potential. The reason is as follows. For the instanton path $y_c(x)$, the large-τ solution (19) implies that the leading action (20), as calculated from (39), arises entirely from the interval $(-1,-1/\sqrt{3})$ for $\tau\to\infty$. The interval $(-1/\sqrt{3},0)$ contributes a negligible amount to the action.

The $\tau=2$ data can be compared with that of Hanggi,[15] obtained by solving the equivalent two-dimensional Fokker-Planck equation. Hanggi calculates directly $P_{st}(x)$, for $\tau=2$ and $D=0.1$, for the potential (16), and plots his data as $\ln P_{st}(x)$ against x. Multiplying his data by $-D$ gives data which can be compared with our data for $S(x,0)$. While the comparison reveals qualitative similarity, the rather pronounced plateau evident in our $\tau=2$ data is absent in Hanggi's. This difference probably reflects the nonzero value of D used in Hanggi's calculation: the limit $D\to 0$ is implicit in all our results. A quantitative measure of the discrepancy is through the difference $S(0,0)-S(\pm 1,0)$. For $\tau=2$, we find this difference to be $\simeq 0.40$, whereas the value inferred from Hanggi's data is $\simeq 0.64$.

The striking, and unexpected, appearance of the plateau in the data for $\tau>1$ has prompted us to investigate analytically the behavior of the weak-noise potential for small x. This we do in Sec. VI D. We find that a change in behavior occurs at $\tau=1$ [more generally when $1+\tau V''(0)=0$]. For $\tau<1$, $S(x,0)-S(0,0)$ is quadratic in x for small x. For $\tau=1$, however, the coefficient of the quadratic term vanishes, and for $\tau>1$ a new behavior emerges in which $S(x,0)-S(0,0)$ is of order $|x|^{1+\tau}$. This remarkable result accounts for the extreme "flatness" of $S(x,0)$ for small x and large τ.

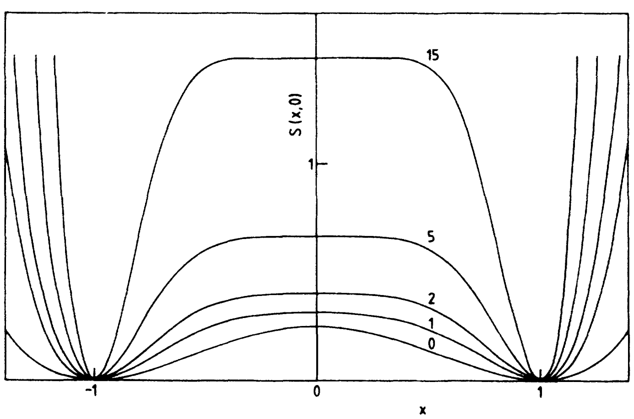

FIG. 3. Classical action $S(x,0)$ (or "weak-noise potential"), which determines (see text) the stationary probability density via $P_{st}(x)\sim\exp[-S(x,0)/D]$, for the quartic bistable potential (16). The curves are labeled by the value of τ. The $\tau=0$ (white noise) curve is (apart from a constant shift which can be absorbed into the normalization) just the potential, $S(x,0)=V(x)-V(\pm 1)$, and for $\tau=0$ $P_{st}(x)\propto\exp[-V(x)/D]$, the usual canonical distribution.

D. The weak-noise potential near the top of the barrier

The analysis starts from Eq. (37) for $f(x,\dot{x})$ [$=S(x,\dot{x})$]. Near the unstable maximum $x=0$, we approximate the potential by the inverted parabola $V(x)=-x^2/2$ [using units for x such that $V''(0)=-1$, as in (16)]. In this regime we anticipate that (37) will have the solution

$$f(x,\dot{x})=f(0,0)+(\alpha/2)x^2+(\beta/2)\dot{x}^2 \ . \tag{48}$$

Substituting this into (37) [with $V(x)=-x^2/2$] verifies that this form is a possible solution, and yields the equations

$$\beta^2=\tau(1-\tau)\beta$$

$$\alpha=-\beta/\tau$$

for the coefficients α and β. Hence either

$$\beta=\tau(1-\tau)\ , \quad \alpha=\tau-1\ , \tag{49}$$

or $\alpha=0=\beta$. For $\tau<1$, the nontrivial solution (49) is physically reasonable: $\beta>0$ and $\alpha<0$ imply that small velocities are more probable than large, and that $f(x,0)$ has a local maximum at $x=0$ [corresponding to a local minimum in the distribution $P_{st}(x)$ at the top of the barrier]. For $\tau>1$, however, the signs of α and β for the nontrivial solution are reversed. In particular, $f(x,0)$ now has a local minimum at $x=0$, contrary both to physical intuition and to the numerical data presented in Fig. 3. We conclude that for $\tau > 1$ the trivial solution $\alpha = 0 = \beta$ must be adopted: if the difference $|f(x,0) - f(0,0)|$ decreases as a power of x for $x \to 0$, the power must be greater than 2.

Guided by the above, we try a solution for $\delta f(x,\dot{x}) \equiv f(x,\dot{x}) - f(0,0)$ of the form

$$\delta f(x,\dot{x}) = \dot{x}^p G(x/\dot{x}) ,$$

a homogeneous function of x and $\dot{x}$ of order $p > 2$. Simple power counting on (37) shows that the term nonlinear in f is negligible for $\dot{x} \to 0$ at fixed $z = x/\dot{x}$. The resulting linear equation [with $V(x) = -x^2/2$] can be exactly solved. In the regime $x \le 0$, $\dot{x} \ge 0$ (which we consider for later convenience) the solution is

$$\delta f(x,\dot{x}) = \text{const} \times [(\dot{x} - x)^\tau (\tau\dot{x} + x)]^{p/(1+\tau)} , \quad (50)$$

where the exponent p is still undetermined. The arbitrary constant prefactor is a consequence of solving a linear equation.

The exponent p can be determined by a simple argument. The factor $(\tau\dot{x} + x)$ in (50) vanishes along the line $x = -\tau\dot{x}$ in the $(x,\dot{x})$ plane. On physical grounds, this should be a simple zero of δf, which requires $p = 1 + \tau$. Alternatively, the same result can be derived by considering special points $(x,\dot{x})$ which lie on the instanton trajectory, i.e., on the extremal path $y_c(x)$ connecting the turning points $(-1,0)$ and $(0,0)$ of the potential. Introducing, as before, subscripts zero to distinguish the endpoint of the path from other points on the path, we will consider special points $(x_0,\dot{x}_0)$ with $\dot{x}_0 = y_c(x_0)$. Subtracting from the expression (39) (with $a = -1$) for the action the same expression with $x_0 = 0$, and using $f \equiv S$, gives for small x_0

$$\delta f = -\tfrac{1}{4} \int_{x_0}^{0} (dx/y_c)[y_c - x + \tau y_c(y_c' - 1)]^2 , \quad (51)$$

where we have again used the form $V(x) = -x^2/2$ for the potential near the top of the barrier. Using this form also in (8), and solving for $x \to 0$, gives for $\tau > 1$ the result

$$y_c = -x/\tau + K(-x)^\tau + \cdots , \quad (52)$$

where K is an arbitrary constant. Substituting this form into (51), and evaluating the integral yields, for $x_0 \to 0$,

$$\delta f = -\tfrac{1}{2} K^2 \tau^2 (-x_0)^{2\tau} . \quad (53)$$

The dependence on K is a consequence of the fact that the leading-order terms cancel inside the square brackets in (51).

We emphasize that (53) holds only for special points $(x_0,\dot{x}_0)$ for which $\dot{x}_0 = y_c(x_0)$. However, the result enables us to determine the exponent p by comparing (53) with the general result (50) evaluated at these same special points. Using (52) in (50), the leading-order terms again cancel to give, for $x_0 \to 0$

$$\delta f \sim (-x_0)^{2p\tau/(1+\tau)} . \quad (54)$$

Comparison of (53) and (54) yields $p = 1 + \tau$. In addition, the const term in (50) is related to the amplitude K in (52). Its value is not determined by the form of the potential in the vicinity of the unstable maximum, but depends on the whole function $V(x)$.

It is instructive to apply a similar analysis of δf to the case $\tau < 1$. Instead of (52) one obtains, for small x,

$$y_c = -x + K'(-x)^{1/\tau} + \cdots . \quad (55)$$

The difference between (52) and (55) can be understood as a switch of dominance, at $\tau = 1$, of the two exponentials in the asymptotic behavior $x_c(t) = A\exp(-t) + B\exp(-t/\tau)$ of the instanton path in the original (x,t) variables. Putting (55) into (51), one finds that now the leading-order terms in (51) do not cancel, and the leading small-x_0 behavior is

$$\delta f = -\tfrac{1}{2}(1-\tau)^2 x_0^2 . \quad (56)$$

This time the amplitude is determined, because K' does not enter the leading-order result. Furthermore, (56) agrees precisely with (48) [with α and β given by (49)] evaluated at the special points $\dot{x} = -x = -x_0$.

The above analysis confirms that the change in behavior, at $\tau = 1$, of the weak-noise potential near the top of the barrier, inferred originally on physical grounds, is indeed correct. We summarize the result for $\dot{x} = 0$:

$$\delta S(x,0) = \begin{cases} -\tfrac{1}{2}(1-\tau)x^2 + \cdots , & \tau < 1 \\ -\text{const} \times |x|^{1+\tau} + \cdots , & \tau > 1 . \end{cases} \quad (57)$$

The result explains the appearance in the data, for $\tau > 1$, of a "plateau" which becomes more pronounced with increasing τ. A numerical study of the behavior near the maximum confirms the $|x|^{1+\tau}$ dependence. Similarly, a study of the $\dot{x}$ dependence of $S(x,\dot{x})$ at fixed $x = 0$ confirms the expected variation as $|\dot{x}|^{1+\tau}$.

VII. GENERALIZATIONS OF THE INSTANTON APPROACH

In this section we consider various generalizations of the Langevin equation (1) to which path-integral and instanton methods can be applied.

A. Multiplicative noise

The methods applied above to the "additive" noise problem (1) can be simply extended to the Langevin equation with "multiplicative" noise:

$$\dot{x} = -V'(x) + g(x)\xi(t) . \quad (58)$$

Perhaps the simplest approach[16] is to convert the multiplicative noise into additive noise via the change of variable $u = \int^x dx'/g(x')$, which requires that $g(x)$ nowhere vanish, to obtain

$$\begin{aligned} \dot{u} &= -\tilde{V}'(u) + \xi(t) , \\ \tilde{V}'(u) &= \tilde{V}'(x(u))/g(x(u)) . \end{aligned} \quad (59)$$

It is simple enough, however, to apply path-integral methods directly to (58), to calculate, for example, the escape rate over a potential barrier. The analog of Eq. (5) for the action is obtained by substituting $\xi = (\dot{x} + V')/g$ in Eq. (3). In terms of the variable $y \equiv \dot{x}$, one has

$\xi=(y+V')/g$, and using $\dot{\xi}=y\,d\xi/dx$, the action for the "uphill" instanton path reads

$$S[y]=\frac{1}{4}\int_a^b \frac{dx}{y}\left[\left[\frac{y+V'}{g}\right]^2 + \tau^2 y^2\left[\frac{y'+V''}{g}-\frac{(y+V')g'}{g^2}\right]^2\right], \tag{60}$$

instead of (10). Clearly one can derive, via $\delta S[y]/\delta y(x)=0$, a differential equation for the instanton $y_c(x)$, and solve it for any given functions $V(x)$ and $g(x)$. Here we content ourselves with calculating the white-noise result and the leading (order τ^2) correction to it. To this order one can use the white-noise instanton solution $y_c=V'$ (which is unchanged by the multiplicative noise) to obtain

$$S=\int_a^b dx\frac{V'}{g^2}+\tau^2\int_a^b dx\, V'\left[\frac{V''}{g}-\frac{V'g'}{g^2}\right]^2+O(\tau^4), \tag{61}$$

consistent with the result obtained by a different method in Ref. 10.

For large τ, the calculation proceeds as in Sec. IV. As far as the leading-order result is concerned, the only modification is the replacement of V' by V'/g. Instead of (20) one finds

$$S_\infty=(\tau/2)[\max_x(V'/g)]^2 . \tag{62}$$

Since our primary purpose in this section is to demonstrate the utility of path-integral methods, rather than to present results, we will pursue multiplicative noise no further.

B. Decay of a metastable state

In this subsection we show that the methods used in this paper can be generalized from the dynamics of a single particle to the dynamics of a field theory. As a concrete example, we calculate the decay rate for a scalar (Euclidean) field theory, initially prepared in a metastable minimum of the potential, to reach the stable state. A physical realization could be an Ising ferromagnet in the equilibrium state with, say, negative magnetization. If a weak positive magnetic field is switched on, one is interested in the rate for the system to reach the stable equilibrium state, which has positive magnetization. We will consider white noise only.

For a simple dynamics with no conservation laws, the Langegin equation reads,

$$\dot{\phi}(\mathbf{x})=-\delta\mathcal{H}/\delta\phi(\mathbf{x})+\xi(\mathbf{x},t) , \tag{63}$$

where

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle=2D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') . \tag{64}$$

We will take $\mathcal{H}[\phi]$ to have the conventional form

$$\mathcal{H}[\phi]=\int d^dx\,[\tfrac{1}{2}(\nabla\phi)^2+V(\phi)] , \tag{65}$$

with $V(\phi)$ having two (asymmetrical) minima, much as in Fig. 1, located at $\phi_\pm$ with the stable minimum at ϕ_+. The analog of (5) for the action follows immediately from the probability weight

$$P[\xi]=\mathcal{N}\exp\left[-\int_{-\infty}^{\infty}dt\int d^dx\,\xi(\mathbf{x},t)^2/4D\right]$$

for the noise on expressing the noise in terms of ϕ via Eq. (63):

$$S[\phi]=\frac{1}{4}\int_{-\infty}^{\infty}dt\int d^dx\,[\dot{\phi}-\nabla^2\phi+V'(\phi)]^2 . \tag{66}$$

The instanton solution which gives the dominant contribution to the escape rate for weak noise connects the metastable minimum $\phi=\phi_-$ of $\mathcal{H}$ with the saddle point $\phi_S(\mathbf{x})$ which is a nontrivial solution of the extremal equation $\delta\mathcal{H}/\delta\phi=0$, i.e.,

$$0=-\nabla^2\phi_S+V'(\phi_S) , \tag{67}$$

with boundary conditions $\phi(0)\simeq\phi_+$, and $\phi(\mathbf{x})\to\phi_-$ for $|\mathbf{x}|\to\infty$. The function $\phi_S(\mathbf{x})$ describes the "critical droplet" (centered on $\mathbf{x}=0$) that must be nucleated to reach the stable state.

The equation for the instanton is $\delta S[\phi]/\delta\phi=0$, i.e.

$$0=-\ddot{\phi}+\nabla^4\phi-\nabla^2V'(\phi)-V''(\phi)\nabla^2\phi+V''(\phi)V'(\phi) . \tag{68}$$

By inspection of Eq. (66), one solution of (68) must correspond to

$$\dot{\phi}=\nabla^2(\phi)-V'(\phi) , \tag{69}$$

since this gives S its smallest possible value of zero. The extremal solutions $\phi_\pm$ and ϕ_S are all stationary solutions of this equation. Equation (69) is in fact the "downhill," zero-action, instanton solution [the analog of (12), with the minus sign] that takes the system from the saddle point ϕ_S to either of the local minima $\phi_\pm$. It is easy to check that (69) satisfies (68). The corresponding "uphill" solution can be guessed by analogy with (12): it is

$$\dot{\phi}=-\nabla^2(\phi)+V'(\phi) . \tag{70}$$

Again, it is easily verified that (70) satisfies (68). Equation (70) has the same stationary solutions as (69). The action for the path which connects the metastable minimum ϕ_- to the saddle point ϕ_S is obtained by setting $\phi=\phi_c$ in (66), where $\phi_c(\mathbf{x},t)$ satisfies (70):

$$\begin{aligned} S&=\int_{-\infty}^{\infty}dt\int d\mathbf{x}\,\dot{\phi}_c[-\nabla^2\phi_c+V'(\phi_c)]\\ &=\int d\mathbf{x}\int_{\phi_-}^{\phi_S}d\phi_c(x)(\delta\mathcal{H}/\delta\phi)_{\phi=\phi_c}\\ &=\mathcal{H}[\phi_S]-\mathcal{H}[\phi_-] . \end{aligned} \tag{71}$$

This action will be a finite number, since $\phi_S(\mathbf{x})$ differs significantly from ϕ_- only over a finite region, i.e., within the critical droplet. The rate for the system to reach the stable state, by nucleation of a critical droplet, is given by the usual result $\Gamma\sim\exp(-S/D)$. As expected on physical grounds (one speaks of a nucleation rate *per unit volume*), the prefactor is proportional to the volume of the system. In the instanton approach, this is a consequence of the translational invariance (in space) of the instanton (whose center can be anywhere in the volume), in addition to the usual time-translational invariance. While these results have been obtained before by other methods,[17] the instanton approach outlined above is more physically transparent than earlier derivations.

Further generalizations of the path-integral approach, involving a more general correlator than (2), have been discussed elsewhere.[3] If the noise is not exponentially correlated, the probability weight for the noise contains higher time derivatives than appear in (3): in general an infinite number of terms appear, involving arbitrarily high-order derivatives, as discussed in paper I. If the correlator has the scaling form $\langle \xi(t)\xi(t') \rangle = (D/\tau)C(|t-t'|/\tau)$, involving the single timescale τ, then the higher derivative terms which modify (3) also contain higher powers of τ, so that a small-τ expansion is still possible.[3] For general τ, a compact form for the action may still be derived, by means of auxiliary variable methods,[9] but the instanton equation then becomes an integrodifferential equation and much of the simplicity of exponentially correlated noise is lost.

A different type of generalization is to non-Gaussian noise, corresponding to (for example) higher powers of ξ appearing in Eq. (3).[3] This poses no additional problems of principle, and generalizations of Eqs. (6)–(8) may readily be derived, and solved numerically.

VIII. SUMMARY

The path-integral formalism developed in an earlier paper has been used to discuss the behavior of a heavily damped particle driven by (exponentially correlated) colored noise. This approach is especially fruitful in the weak-noise limit, where a steepest-descent evaluation of the path integrals is possible. In this limit we have obtained essentially exact leading-order results for escape rates over a barrier, and stationary distributions. For the escape rate we find $\Gamma \sim \exp(-S/D)$, i.e., a generalization of the usual Arrhenius result, with S no longer simply the height of the barrier. The stationary distribution has the form $P_{st}(x) \sim \exp[-f(x)/D]$. The results are "leading order" in the sense that we calculate the arguments of the exponentials but not the prefactors. While these can be calculated in principle, the computations are technically difficult and will be presented in a future publication.[5] Various generalizations, including nonadditive, nonexponentially correlated, and non-Gaussian noise have been briefly discussed. The extension to field theory is straightforward, and provides a physically intuitive method to calculate the lifetime of a metastable state.

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