RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 41, NUMBER 11 1 JUNE 1990

Coherence and chaos in a quantum optical system

G. J. Milburn

Department of Physics, The University of Queensland, St. Lucia, Queensland 4067, Australia

(Received 27 December 1989; revised manuscript received 6 March 1990)

A nonlinear, kicked quantum optical system, which classically exhibits regions of regular and chaotic motion, is proposed as a possible experimental test of quantum chaos. The mean photon number is shown to undergo regular collapse and revival in the regular region of phase space and irregular revivals in the chaotic region.

The quantum behavior of periodically kicked, nonlinear Hamiltonian systems can reflect the classical dynamics be it regular or chaotic. Early work focused on the change in quasienergy statistics of the quantum problem as the corresponding classical realization moved into the chaotic regime.[1] More recently, it has been shown that the classical transition to chaos is reflected in the appearance of irregular collapses and revivals in the time evolution of certain moments,[2] as opposed to a regular collapse and revival sequence in the nonchaotic case. However, in many ways it is becoming clear that classically chaotic behavior is drastically modified if not eliminated by quantization. This is very evident in the quantum suppression of energy diffusion, for example in the kicked rotor,[3,4] due to complicated interference effects reminiscent of Anderson localization.[5] In this phenomenon we see the signature of the most characteristic feature of quantum mechanics, coherent superposition states, a feature at the heart of the less intuitive aspects of the theory.

The quantum suppression of chaotic diffusion is the one aspect of quantum chaos which has been subjected to experimental test.[6] The microwave ionization of hydrogen can be closely modeled by the combined effects of the Coulomb field and an oscillating electric field. Numerical results indicate that the observed threshold for ionization follows the classical threshold for the appearance of a chaotic instability in a suitable parameter range. Bayfield *et al.*[6] have recently presented results which clearly indicate the localization phenomenon discussed above. The search is now on for other systems which may enable yet more direct tests of the unusual quantum dynamics of nonintegrable systems.

Most recently Prange and Fishman[7] have suggested that the nonlinear behavior of fields in optical fibers may provide simple realizations of the dynamics of kicked nonlinear systems. In their proposal, the field is not quantized but the nonlinear wave interactions in the fiber are analogous to a quantum kicked system. In this Rapid Communication I wish to present a relatively simple, fully quantum model, which may enable a direct experimental test of a kicked nonlinear system.

Consider a single-mode field propagating in a medium with an intensity-dependent refractive index. This could be an optical fiber or a nonlinear material placed in a cavity. The single-mode field is well described in terms of the dynamics of a simple harmonic oscillator. The effect of the nonlinearity is then to induce an energy-dependent phase shift of the oscillators' complex amplitude, that is a rotational sheer in the complex plane.[8] The mode is also periodically "kicked" by parametric amplification which for a very short time turns the origin in phase space into a hyperbolic fixed point, thus stretching and shrinking the phase plane in orthogonal directions. Such devices produce squeezed states and now operate successfully in a number of laboratories.[9] The periodic amplification could be produced by placing a chain of parametric amplifiers along a nonlinear fiber, or in the cavity configuration a pulsed pump field could be used to turn the parametric amplifier placed inside the cavity on and off.

The classical dynamics of this system exhibits a rich structure of regular and chaotic motion, with the amplifier gain being the control parameter. The phase space is divided into two regions, bounded but possibly chaotic motion around the origin and unbounded motion at some distance from the origin as trajectories escape to infinity. When the system is quantized one finds, for example, that the mean photon number undergoes a collapse and revival sequence for initial states localized in the bounded region of phase space. This sequence is periodic for initial states in a regular region and irregular for states in a chaotic region. Similar behavior has been found by Haake and coworker in the dynamics of a kicked nonlinear top.[10]

During the period of free evolution between the kicks, the system dynamics is determined by the Hamiltonian[8]

$$H_{\mathrm{NL}}=\frac{\chi}{2}(a^{\dagger})^2a^2\,, \tag{1}$$

where χ is proportional to the third-order nonlinear susceptibility and $a,a^{\dagger}$ are the complex amplitude operators for the field. These operators obey the commutation relation $[a,a^{\dagger}]=1$. In terms of dimensionless "position" and "momentum" operators (or quadrature phase operators) $(\hat{X}_1,\hat{X}_2)$, $a=\hat{X}_1+i\hat{X}_2$.

The Heisenberg equation of motion for a is

$$\frac{da}{dt}=-i\chi a^{\dagger}aa \tag{2}$$

(units have been chosen such that $\hbar=1$). As the energy operator $a^{\dagger}a$ is a constant of motion, Eq. (2) has the solution

$$a(t)=e^{-i\mu a^{\dagger}a}a(0) \tag{3}$$

where $\mu=\chi L/c$, with L the interaction length of the nonlinear medium and c the speed of light. Equation (3) describes an energy-dependent phase shift.

The parametric kicks are described by the Hamiltonian[9]

$$H_K = i\hbar\frac{\kappa}{2}(a^{\dagger 2} - a^2)\,. \tag{4}$$

In writing this Hamiltonian it is assumed that the pump carrier frequency is twice that of the field oscillator and a transformation to an interaction picture has been made. The coupling constant κ is the product of the pump field amplitude and the second-order nonlinear susceptibility in the parametric gain medium. Thus κ is increased by increasing the pump amplitude.

The Heisenberg equation of motion following from Eq. (4) is

$$\frac{da}{dt} = \kappa a^{\dagger} \tag{5}$$

(and a corresponding Hermitian conjugate equation). The solution is

$$a(t) = a(0)\cosh(r) + a^{\dagger}(0)\sinh(r)\,, \tag{6}$$

where $r = \kappa t$. (In the case of a pulsed pump field r is determined by the integrated time-dependent amplitude of the pump.) Equation (6) may be written in terms of the Hermitian phase-space operators:

$$\hat{X}_1(t) = e^{r}\hat{X}_1(0) \equiv g\hat{X}_1(0)\,, \tag{7a}$$

$$\hat{X}_2(t) = e^{-r}\hat{X}_1(0) \equiv \frac{1}{g}\hat{X}_2(0)\,, \tag{7b}$$

where $g \equiv e^{r}$ is the parametric gain.

To define the corresponding classical equations we replace the operators $(a,a^{\dagger})$ and $(\hat{X}_1,\hat{X}_2)$ by classical commuting phase-space variables (a,a^*) and (X_1,X_2).[11] The classical analogues of Eqs. (3) and (6) are then

$$a(t) = e^{-i\mu|a|^2}a(0)\,, \tag{8}$$

and

$$a(t) = a(0)\cosh(r) + a^*(0)\sinh(r), \tag{9}$$

where $a = X_1 + iX_2$. Combining Eqs. (8) and (9) the classical map for the variables (X_1,X_2) may be written

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & 1/g \end{bmatrix} \begin{bmatrix} \cos(\mu R^2) & \sin(\mu R^2) \\ -\sin(\mu R^2) & \cos(\mu R^2) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{10}$$

where $R^2 = X_1^2 + X_2^2$.

In Figs. 1(a) and 1(b) phase portraits of the system for two values of the gain g are shown. Clearly evident in Fig. 1(a) are two period-one fixed points near the origin. (There must be two as the map is symmetric under $X_1 \to -X_1$; $X_2 \to -X_2$.) Various island chains of odd and even order occur. Further from the origin is a period-two fixed point beyond which chaotic unbounded trajectories occur. Regions of chaotic behavior are clearly evident in Fig. 1(b).

The quantum map is better specified in terms of the change in the state vector rather than the system operators (i.e., in the Schrödinger picture rather than the Heisenberg picture). The change in state after free evolution and a kick is then given by

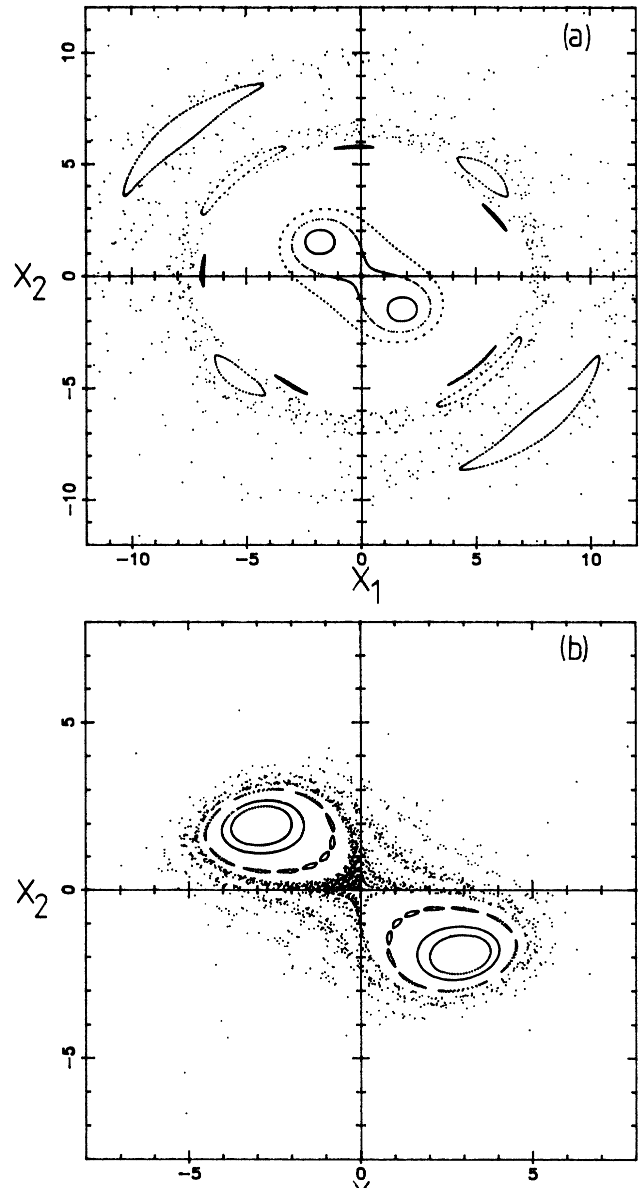

FIG. 1. Classical phase portrait of the map in Eq. (10); (a) $g = 1.2$, $\mu = 0.01\pi$; (b) $g = 1.5$, $\mu = 0.01\pi$.

$$|\psi'\rangle = U|\psi\rangle\,,$$

where the quantum description is provided by the unitary operator

$$U = \exp\left[\frac{r}{2}[(a^{\dagger})^2 - a^2]\right]\exp\left[-i\frac{\mu}{2}(a^{\dagger})^2a^2\right]\,. \tag{11}$$

The initial states for the quantum analysis were chosen to be coherent states $|\alpha\rangle$, which may be expanded in terms of the energy eigenstates of a free oscillator as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{12}$$

The quantum dynamics was determined by writing the unitary operator in Eq. (11) in the number basis $\{|n\rangle\}$ with, of course, some suitable truncation. In this basis the matrix elements of U are

$$\langle n|U|m\rangle = \begin{cases} S\beta^{-(R+1/2)}\lambda^{(Q-R)/2}\exp\left[-i\frac{\mu}{2}(m^2-m)\right]\sum_{p=0}^{N}(-1)^p\frac{(\beta\lambda)^{2p}}{p!}\frac{[m!n!]^{1/2}}{(R-2p)![p+\frac{1}{2}(Q-R)]!}\,, & n-m \text{ is even}\,, \\ 0, \quad n-m \text{ is not even}\,, \end{cases} \tag{13}$$

where $\beta = \cosh r$, $\lambda = \frac{1}{2}\tanh r$, $R = \min(n,m)$, $Q = \max(n,m)$, and

$$S = \begin{cases} +1, & n \geq m\,, \\ (-1)^{(m-n)/2}, & n < m\,, \end{cases} \tag{14}$$

$$N = \begin{cases} R/2, & \text{if } n \text{ and } m \text{ are even}\,, \\ R-1/2, & \text{if } n \text{ and } m \text{ are odd}\,. \end{cases} \tag{15}$$

The initial state in Eq. (12) is also truncated and the system evolved numerically by repeated application of the truncated matrix. The truncation is chosen sufficiently large to ensure that the state after every kick is very closely normalized to unity. (In practice, $n = 51$ was found to be satisfactory.)

In Figs. 2(a) and 2(b) the mean photon number versus kick number are shown. The parameters in each case are chosen as in Figs. 1(a) and 1(b), respectively. In Fig. 2(a) the initial state was taken to be the vacuum state ($\alpha = 0$). Classically one expects an initial distribution of

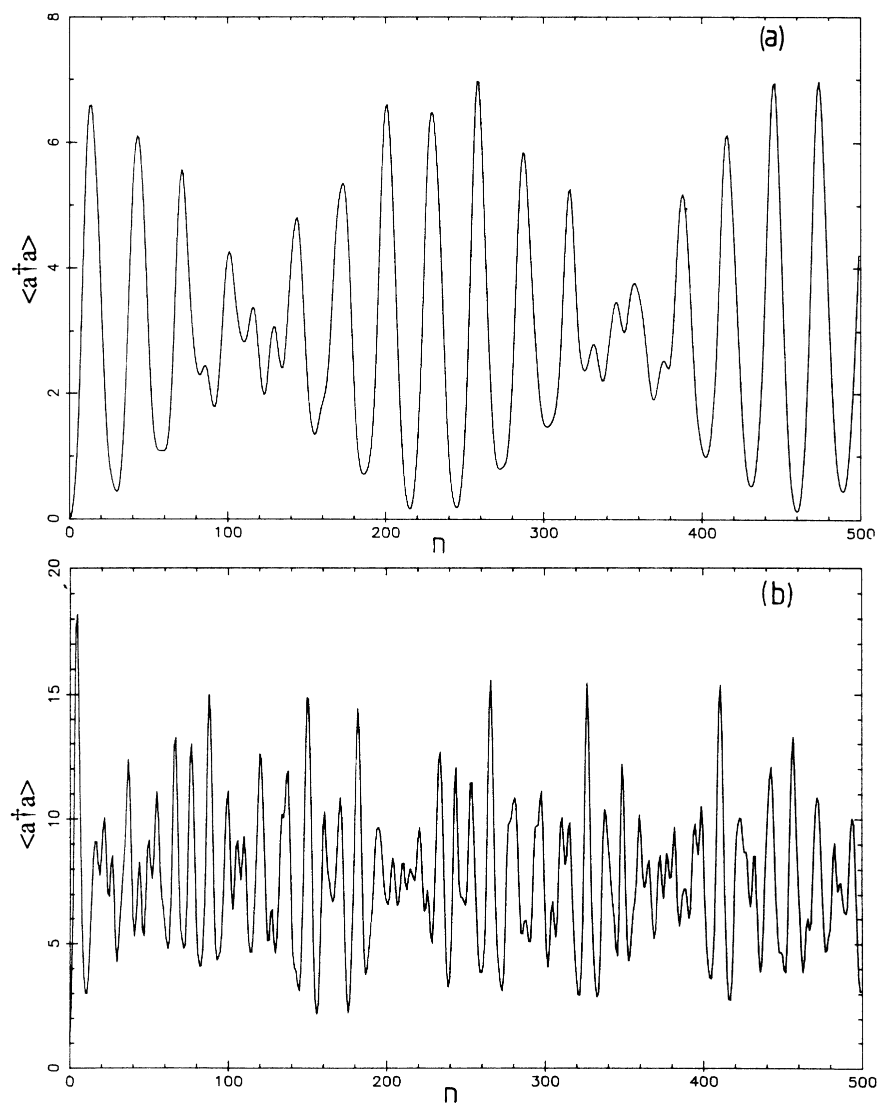

FIG. 2. Mean photon number $\langle a^\dagger a\rangle$ vs kick number for an initial coherent state $|\alpha\rangle$; (a) $\alpha = 0$, $g = 1.2$, $\mu = 0.01\pi$; (b) $\alpha = 1.0$, $g = 1.5$, $\mu = 0.01\pi$.

points centered on the origin to split, moving along the hyperbolic orbits about the origin, eventually becoming evenly distributed around each period one fixed point, leading to a steady-state mean photon number. The quantum result, however, clearly indicates regular collapses and revivals, as expected for a regular region. For initial states in regular regions further from the origin (e.g., at the four-island chain) regular collapse and revival is still observed with a very wide separation in time (not shown). In contrast, in Fig. 2(b) an initial coherent state was chosen in the chaotic band surrounding the two fixed points in Fig. 1(b). The mean photon number is now seen to undergo a very irregular collapse and revival sequence centered on a mean photon number of approximately ten. Such an irregular recurrence sequence was also reported in Ref. 2.

Is this system practical? The gain parameters used in Figs. 1(a) and 1(b), 1.2 and 1.5, respectively, are quite modest corresponding to a squeezing reduction of vacuum fluctuations of 30% and 56%, respectively. These values have been achieved in experiment.[9] The nonlinear phase-shift parameter can be scaled out of the equations by $x_i = \mu^{1/2} X_i$ for $i = 1,2$. Thus decreasing μ produces the same phase-space structure but at larger scales, i.e., larger mean photon number. We can estimate a photon number scale as follows. In the units used here, μ is given by[8]

$$\mu = \frac{3\hbar\omega^2}{4\epsilon_0^2 V}\chi^{(3)}\frac{L}{c} ,$$

V the interaction volume, and $\chi^{(3)}$ the third-order susceptibility. In SI units a very modest value for $\chi^{(3)}$ is 10^{-31} (corresponding to about 10^{-12} esu). At optical frequencies we then find $\mu \sim 3\times 10^{-17}$. The mean photon number scale then is proportional to μ^{-1}, i.e., $\sim 10^{16}$ photons. This is at the upper end of a cw laser scale but easily achievable in a pulsed experiment. Of course a higher third-order nonlinearity would make the experiment easier. In conclusion, it seems that a practical realization of the scheme described in this paper is easily within the reach of current technology.

The real problem in searching for coherent quantum structure, however, is dissipation. The crucial question is the time scale of quantum recurrences compared to the dissipative decay rates. In the cavity configuration one could use very high Q cavities, as the mean photon number inside the cavity may be monitored by determining depletion of the pulsed pump field of the parametric amplifiers and the pumpfield does not couple to the cavity through the cavity mirrors. Including small dissipation, the map turns fixed points into attracters but chaos can still be found for certain parameter ranges. This work will be published elsewhere. If the decay is not too large some coherent recurrence features may still be observed. Thus one requires low loss materials with a high third-order optical susceptibility.

I would like to thank B. Sanders for useful discussions.

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