Path integrals and non-Markov processes. I. General formalism

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We develop the path-integral formalism as applied to non-Markov stochastic processes in order to allow us to study the effects of colored external noise on a physical system. The system we initially consider consists of a Langevin equation $\dot{x}=-V'(x)+\xi$, where ξ is a Gaussian noise with zero mean and correlator $\langle\xi(t)\xi(t')\rangle=(D/\tau)C(|t-t'|/\tau)$, τ being the noise correlation time. Starting from the Langevin equation, we obtain a path-integral representation for probability density functions on the infinite time interval $-\infty<t<\infty$, and show how in certain cases a simple representation also exists in terms of a sum over paths on a finite time interval. The weighting factor for paths in this latter case consists of an exponential factor which is a generalization of that originally found by Onsager and Machlup but also contains nontrivial boundary terms depending on the initial preparation of the system.

I. INTRODUCTION

When investigating the effects of noise upon a system described by a macroscopic deterministic equation, it is usual to distinguish between the cases where the noise is external and where it is internal.[1,2] In the former case the noise is due to an external source and is not influenced by the system itself. By contrast in the latter case it is intrinsically tied up with the evolution of the system. External noise[3] is typically nonwhite, i.e., colored, and as a consequence many of the techniques developed for the study of Markov processes[2,4,5] are not applicable. This paper is the first in a series concerned with the application of path-integral techniques to the study of the effects of external (and hence generally nonwhite) noise.[6] In it we restrict ourselves to the derivation of various path-integral representations for probability density functions. In subsequent papers we shall try to convince the reader that the application of steepest-descent techniques to these path integrals provides an efficient way of calculating interesting physical quantities in the weak noise limit.

The system we consider is described by a single variable $x(t)$. There is no reason why the formalism we shall set up, and the methods we will later discuss, should not be applicable to systems described by more than one variable, but for simplicity and clarity of exposition we will not consider such systems here. We assume that the deterministic macroscopic equation of motion has the simple form

$$\dot{x}=f(x)\ . \tag{1}$$

The effect of external noise on the system can be modeled by constructing the appropriate Langevin equation

$$\dot{x}=f(x)+g(x)\xi(t)\ , \tag{2}$$

where $\xi(t)$ is the noise and $g(x)$ is a given function. For definiteness, it is useful to think of the system as a particle moving in a potential $V(x)$ and subject to a damping force $-\alpha\dot{x}$. The macroscopic equation for such a particle of mass m is $m\ddot{x}=-\alpha\dot{x}-V'(x)$. Scaling time by α, it reduces to the form (1) in the overdamped limit $m/\alpha^2\to 0$, with $f(x)=-V'(x)$. The finite m/α^2 problem can also be studied using path-integral techniques; the results of such a study will be reported elsewhere.[7]

The noise will be taken to have zero mean, and correlator

$$\langle\xi(t)\xi(t')\rangle=(D/\tau)C(|t-t'|/\tau)\ , \tag{3}$$

where τ is the noise correlation time and C is an arbitrary function, apart from restrictions to be imposed below. It will be normalized so that

$$\int_0^\infty ds\, C(s)=1\ , \tag{4}$$

and hence D is a measure of the strength of the noise. For the moment we will assume that the noise is Gaussian, so that all higher cumulants vanish and hence the stochastic process $\xi(t)$ has been completely specified. It will be shown below that these techniques apply to non-Gaussian noise as well. It is, however, better for our purposes to define such processes through the probability density functional, rather than by specifying higher cumulants, and hence we postpone the discussion of this point.

The simplest and most widely studied stochastic process involving colored noise is the case where the noise is exponentially correlated:

$$C(|t-t'|/\tau)=\exp(-|t-t'|/\tau)\ . \tag{5}$$

This reduces to the white-noise result, $C(s)=2\delta(s)$, in the limit $\tau\to 0$. The finite correlation time τ in (5) makes the process non-Markov in contrast with the case of white noise. It is perhaps worth remarking that if the noise were internal and nonwhite, we would expect to have to modify the Langevin equation (2) to include a memory kernel. No such term is required in the case of external noise.

It remains to specify $g(x)$ in (2). If $g(x)$ is constant the noise is said to be additive; otherwise it is said to be multiplicative. The distinction is somewhat artificial, since in many cases a change of variable will transform a Langevin equation with multiplicative noise into one with additive noise. There is some discussion of systems with multiplicative noise in the second paper[8] in the series (to be referred to as paper II from now on); for the rest of this paper we set $g(x)=1$. We shall for the most part leave the potential $V(x)$ unspecified. However, when carrying out calculations we shall assume that it has the generic form shown in Fig. 1. The reason for this is as follows. The generality and power of our approach is most obvious when studying problems involving escape over a potential barrier, e.g., the calculation of mean first passage times. As we plan to discuss in a third paper,[9] the regime in which our methods are applicable is one in which a coarse-grained conditional probability satisfies a two-state master equation. For a multidimensional potential with n wells, an n-state master equation would apply. One can therefore view the calculation of escape rates described in later papers as a method of calculating transition probabilities per unit time between wells, these then being used as input into the appropriate master equation. Thus there is really no loss of generality in assuming that $V(x)$ is a double-well potential.

Calculation of probability density functions, mean first passage times, etc., have traditionally been accomplished by the solution of the Fokker-Planck equation.[2,4,5] This is because in many cases of interest the system is acted upon by white noise and hence such an equation exists. When the noise is colored, no simple Fokker-Planck equation in x and t exists. As a consequence, most of the previous approaches to the study of colored noise have been concerned with the derivation of approximate Fokker-Planck equations. There are a vast number of papers on this topic. We mention an early review by van Kampen[10] and refer the reader to a selection of papers[11–28] written on this topic in the last decade. Many of these papers contain conflicting results and recently a number of authors have tried to clarify the situation by comparing the methodology and results of various groups.[29–32] Numerical work on the effects of colored noise has also been carried out;[12,33–38] Ref. 3 has a number of papers reviewing the current situation. We do not follow any of these approaches and bypass the use of Fokker-Planck equations as a method of calculation. Instead, our starting point will be expressions giving probability density functions as path integrals. Such representations are well known for Markov processes,[39,40] and various non-Markov generalizations have been derived.[41,42] Some authors, notably Fox,[17,18] have used functional integral methods to derive approximate Fokker-Planck equations; we will stress the usefulness of path integrals as a calculational tool and not just as a formal device. The evaluation of the path integrals is made possible by the observation that in the weak noise limit ($D \to 0$) the integral is dominated by the appropriate instanton solution of the theory. In other words, the path integral is evaluated by the method of steepest descent, the instanton being a saddle point in the function space. The instanton calculus allowing, for instance, the calculation of small D corrections to the leading behavior is well developed,[43–45] and has been applied to the study of Markov stochastic processes.[46–48] The method comes into its own, however, when the processes are non-Markov, the lack of a simple alternative being all too obvious from the literature cited above. A preliminary account of our work has already appeared.[49,50] Luciani and Verga[51,52] and Förster and Mikhailov[53] have also studied colored noise by applying the method of steepest descent to a path integral. These authors, however, express the path integral in a form which is not very useful when studying exponentially correlated noise [Eq. (5)] or performing a small τ^2 expansion. In addition the results of Luciani and Verga are mainly restricted to piecewise linear forces.

The outline of the paper is as follows. We begin in Sec. II by giving a precise specification of the type of noise we will be considering. Rather than defining the process through the cumulants of the noise, it is more convenient for our purposes to define it through the probability density functional. We give expressions for various probability density functions as path integrals over an infinite time interval in Sec. III and discuss how these are equivalent to path integrals over a finite time interval in Sec. IV. Finally, in Sec. V, the path-integral representations given in Sec. IV are rederived by considering the Markov processes into which the non-Markov processes under consideration have been embedded.

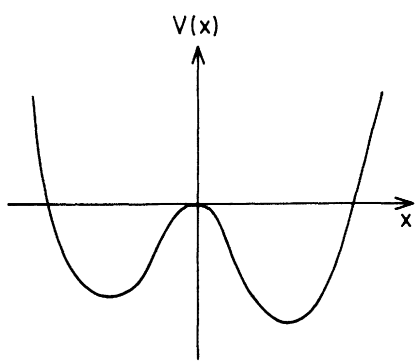

FIG. 1. Typical potential considered in this paper.

II. SPECIFICATION OF THE STOCHASTIC PROCESS

If the noise is Gaussian with zero mean, then it is completely specified by (3). The process is defined over the infinite time interval $-\infty < t < \infty$ and it is therefore convenient to work with the Fourier transform of the correlator defined in (3):

$$\langle \xi(\omega)\xi(\omega') \rangle = D 2\pi\delta(\omega+\omega')\tilde{C}(\omega\tau) , \tag{6}$$

where

$$\tilde{C}(\omega\tau) = \int_{-\infty}^{\infty} ds \exp(i\omega\tau s) C(|s|) = \int_0^{\infty} ds \exp(i\omega\tau s) C(s) + \text{c.c.} \tag{7}$$

Assuming that the integrals

$$\mu_n \equiv \int_0^\infty ds\, s^{2n} C(s) \tag{8}$$

exist, we can expand $\tilde{C}(\omega\tau)$ as a power series in $(\omega\tau)^2$:

$$\tilde{C}(\omega\tau) = 2 \sum_{n=0}^{\infty} (-1)^n (\omega\tau)^{2n} \mu_n / (2n)! \;. \tag{9}$$

In the case of white noise $C(|s|) = 2\delta(s)$, hence $\tilde{C} = 2$. Therefore, to regain the white-noise result when $\tau \to 0$ we normalize $\tilde{C}(\omega\tau)$ by taking $\mu_0 = 1$, which is (4).

The probability density functional $P[\xi]$ is Gaussian by assumption. Using $\langle \xi(\omega) \rangle = 0$ and (6) we can write it as

$$P[\xi] = \mathcal{N} \exp\left[-\frac{1}{2D} \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} \xi(-\omega) [\tilde{C}(\omega\tau)]^{-1} \xi(\omega) \right] , \tag{10}$$

where $\mathcal{N}$ is a normalization constant. From the power-series expansion (9) we have

$$\begin{aligned} [\tilde{C}(\omega\tau)]^{-1} &= \tfrac{1}{2}(1 - \mu_1 \tau^2\omega^2/2! + \mu_2 \tau^4 \omega^4/4! - \cdots)^{-1} \\ &= \tfrac{1}{2}(1 + \kappa_1 \tau^2 \omega^2 + \kappa_2 \tau^4 \omega^4 + \cdots) , \end{aligned} \tag{11}$$

where

$$\begin{aligned} \kappa_1 &= \mu_1/2 \\ \kappa_2 &= \mu_1^2/4 - \mu_2/24, \ldots . \end{aligned} \tag{12}$$

If the function C is such that some of the integrals (8) do not exist then an expansion such as (9) cannot be made, and the methods we will develop do not apply. An extreme example of this is the case where $C(s)$ is Lorentzian: $C(s) = \pi^{-1}(1+s^2)^{-1}$, so that $\tilde{C}(\omega\tau) = 2\exp -|\omega|\tau$. In this case, the dependence of $\tilde{C}(\omega\tau)$ on $|\omega|$ prevents us from taking the Fourier transform of (10) to obtain a simple expression for the probability of a path $\xi(t)$, and our approach is not appropriate.

For white noise $\mu_n = 0 = \kappa_n$, $n > 0$, and (10) reduces to the familiar Gaussian white-noise functional. When the noise correlator has the form (5), $\mu_n = (2n)!$ hence $\kappa_n = 0$, $n > 1$ and the sum (11) terminates at the second term. Exponentially correlated noise is in this sense the simplest non-Markov process. A hierarchy of processes can be defined by continuing in this way; we will define an mth-order process as one for which $\kappa_n = 0$, $n > m$. In paper II we will be concerned, amongst other things, with carrying out perturbation expansions in τ^2. When working to order τ^{2m}, terms of order τ^{2m+2} and higher which appear in (11) are ignored; this is equivalent to assuming that the process is an mth-order one. For this reason, such processes arise naturally when perturbing about the Markov limit. It is useful to view them in another way by embedding them into higher-dimensional processes which are Markov.

To illustrate this for the exponentially correlated $(m=1)$ case, consider the Ornstein-Uhlenbeck process

$$\dot{\xi} = -(\kappa_1\tau^2)^{-1/2}\xi + (\kappa_1\tau^2)^{-1/2}\eta \;, \tag{13}$$

where η is a Gaussian white noise with zero mean and strength D. The process is defined on the interval $-\infty < t < \infty$. Taking the Fourier transform of (13) yields

$$\xi(\omega) = \frac{\eta(\omega)}{[1 - i\omega(\kappa_1\tau^2)^{1/2}]} \;, \tag{14}$$

with $\xi(t=-\infty) = \xi(t=\infty) = 0$. A process with a correlation time τ will have no "memory" of the value of ξ, imposed at an initial time, after a period of the order of several τ's. Hence the imposition of an initial condition in the infinitely distant past is irrelevant for the process under consideration. Similarly for a final condition set in the infinitely distant future.

Given that η is Gaussian with zero mean, we see from (14) that ξ is a Gaussian noise with zero mean but with

$$\langle \xi(\omega)\xi(\omega') \rangle = \frac{2D\,2\pi\delta(\omega+\omega')}{1+\kappa_1\tau^2\omega^2} \;, \tag{15}$$

in agreement with (6) and (11), if the process is a first-order one. Hence we have shown the correspondence between the one-dimensional process consisting of the Langevin equation (2) with the noise specified by (15), and the two-dimensional process consisting of the Langevin equation (2) and the subsidiary Langevin equation (13) where η is white noise. In other words, we have shown explicitly how the one-dimensional non-Markov process can be embodied into a two-dimensional Markov process.

To generalize this to the mth-order process $\kappa_n = 0$, $n > m$ is straightforward. Consider the process defined by

$$\begin{aligned} \dot{\xi} &= -(\alpha_1\tau)^{-1}\xi + (\alpha_1\tau)^{-1}\zeta_1 \;, \\ \dot{\zeta}_1 &= -(\alpha_2\tau)^{-1}\zeta_1 + (\alpha_2\tau)^{-1}\zeta_2 \;, \\ \dot{\zeta}_2 &= -(\alpha_3\tau)^{-1}\zeta_2 + (\alpha_3\tau)^{-1}\zeta_3 \;, \\ &\cdots \\ \dot{\zeta}_{m-1} &= -(\alpha_m\tau)^{-1}\zeta_{m-1} + (\alpha_m\tau)^{-1}\eta \;, \end{aligned} \tag{16}$$

where η is a Gaussian white noise with zero mean and strength D. The α_n, $n = 1, 2, \ldots, m$, are complex numbers satisfying $\mathrm{Re}(\alpha_n) > 0$ and the process is again defined on the interval $-\infty < t < \infty$. Taking the Fourier transform of (16) with ξ, ζ_n $(n = 1, 2, \ldots, m-1)$ equal to zero at $t = \pm\infty$ yields

$$\begin{aligned} \xi(\omega) &= \frac{\zeta_1(\omega)}{1 - i\alpha_1\omega\tau} \;, \\ \zeta_{n-1}(\omega) &= \frac{\zeta_n(\omega)}{1 - i\alpha_n\omega\tau}, \quad n = 2, \ldots, m-1 \\ \zeta_{m-1}(\omega) &= \frac{\eta(\omega)}{1 - i\alpha_m\omega\tau} \;. \end{aligned} \tag{17}$$

The initial and final conditions are equivalent to demanding that ξ and its first $m-1$ derivatives vanish at both end points. Once again this choice will be irrelevant for the physics in any finite time interval. Using the properties of η we see from (17) that ξ is a Gaussian noise with zero mean and with correlator

$$\langle \xi(\omega)\xi(\omega')\rangle = \frac{2D2\pi\delta(\omega+\omega')}{(1+\alpha_1^2\omega^2\tau^2)(1+\alpha_2^2\omega^2\tau^2)\cdots(1+\alpha_m^2\omega^2\tau^2)} . \quad (18)$$

Comparison with (6) shows that the process is an mth-order one, and from (11) we see that we require

$$\sum_{n=0}^{m} \kappa_n \omega^{2n}\tau^{2n} = \prod_{n=1}^{m} (1+\alpha_n^2\omega^2\tau^2), \quad \kappa_0 = 1 \quad (19)$$

to make contact with the previous formulation. If the κ_n are given, then the α_n are the m solutions of

$$\alpha^{2m} - \kappa_1\alpha^{2(m-1)} + \kappa_2\alpha^{2(m-2)} - \cdots + (-1)^m\kappa_m = 0 \quad (20)$$

with $\text{Re}(\alpha_n) > 0$. The one-dimensional non-Markov process defined by (2), (3), and (11), with $\kappa_n = 0$, $n > m$, has been embedded into the $(m+1)$-dimensional Markov process in the variables $(x, \xi, \zeta_1, \zeta_2, \ldots, \zeta_{m-1})$.

Let us now return to the probability density functional (10) and substitute in the expression for $[\tilde{C}(\tau\omega)]^{-1}$. One finds

$$\begin{aligned} P[\xi] &= \mathcal{N}\exp\left[-\frac{1}{4D}\int_{-\infty}^{\infty} d\omega \frac{1}{2\pi}\xi(-\omega)[1+\kappa_1\tau^2\omega^2+\kappa_2\tau^4\omega^4+\cdots]\xi(\omega)\right] \\ &= \mathcal{N}\exp\left[-\frac{1}{4D}\int_{-\infty}^{\infty} dt\, \xi(t)[\xi(t)-\kappa_1\tau^2\ddot{\xi}(t)+\kappa_2\tau^4\ddddot{\xi}(t)-\cdots]\right] , \\ &= \mathcal{N}\exp\left[-\frac{1}{4D}\int_{-\infty}^{\infty} dt\, [\xi^2(t)+\kappa_1\tau^2\dot{\xi}^2(t)+\kappa_2\tau^4\ddot{\xi}^2(t)+\cdots]\right] . \end{aligned} \quad (21)$$

We have used the fact that ξ and its first $(m-1)$ derivatives vanish at $t = \pm\infty$ for the process under consideration, if it is mth order. In general, the sum in (11) will not terminate, the process will be an infinite order one, and we will specify that ξ and all its derivatives vanish at the end points $t = \pm\infty$.

Finally we can generalize (21) in the case where ξ is not Gaussian by including higher powers of ξ and its derivatives:

$$\begin{aligned} P[\xi] = \mathcal{N}\exp\Bigg[-\frac{1}{4D}\int_{-\infty}^{\infty} dt\, [&\xi^2(t)+\kappa_1\tau^2\dot{\xi}^2(t) \\ &+\kappa_2\tau^4\ddot{\xi}^2(t)+\cdots \\ &+w_0\xi^3(t)+\cdots \\ &+u_0\xi^4(t)+\cdots]\Bigg] . \end{aligned} \quad (22)$$

This expression allows us to investigate the effects of noise, of a very general character, on a simple physical system such as (1).

III. PATH-INTEGRAL REPRESENTATION OF PROBABILITY DENSITY FUNCTIONS

In Sec. I we introduced the Langevin equation describing the time development of the system acted upon by external Gaussian noise as

$$\dot{x} = -V'(x) + \xi . \quad (23)$$

This equation, together with an initial condition $x(t_0) = x_0$, can be viewed as defining a mapping $\xi \mapsto x$ from the noise to the coordinate. Averaging over the noise can be carried out by using the identity

$$\begin{aligned} 1 &= \int \prod_{t>t_0} [dx(t)\delta(x(t)-x_\xi(t))] \\ &= \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t)\right] \text{Det}(\delta\xi/\delta x) \\ &\quad \times \prod_{t>t_0} \delta(\dot{x}+V'(x)-\xi) . \end{aligned} \quad (24)$$

Here $x_\xi(t)$ is the solution of (23) for some particular realization $\xi(t)$ of the process. Note that since $x(t_0)$ has been specified, there is a unique solution for a given ξ, and hence no problems arise because of multiple solutions, as sometimes happens in the theory of disordered systems.[54,55]

Averaging over the background noise can now be achieved by multiplying (24) by the quantity to be averaged and using (22). This leads to

$$\begin{aligned} \langle F[x]\rangle &= \int \left[\prod_t d\xi(t)\right] P[\xi]F[x_\xi] \\ &= \int \left[\prod_t d\xi(t)\right] \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t)\right] \text{Det}[\delta\xi/\delta x] \left[\prod_{t>t_0} \delta(\dot{x}+V'(x)-\xi)\right] P[\xi]F[x] , \end{aligned} \quad (25)$$

where we have dropped the subscript ξ on x, since the δ-function constrains x to be a solution of (23) for $t > t_0$. The ξ integration from times in the infinitely distant past to $t = t_0$ can be carried out independently of x, since the latter quantity has only been defined for $t \geq t_0$. This gives

$$\langle F[x] \rangle = \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \left[\prod_{t \geq t_0} d\xi(t) \right] \mathrm{Det}[\delta\xi/\delta x] \prod_{t>t_0} \delta(\dot{x} + V'(x) - \xi) \tilde{P}[\xi] F[x] , \quad (26)$$

where $\tilde{P}[\xi]$ is the probability density functional for $\xi(t)$, $t \geq t_0$, which, in general, will include boundary terms coming from the integrations carried out for $t < t_0$. These will be discussed in more detail in Sec. IV. The remaining $\xi(t)$ can be integrated out using the δ function, and we find

$$\langle F[x] \rangle = \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \tilde{P}[x] F[x] , \quad (27)$$

where

$$\tilde{P}[x] \equiv \mathrm{Det}[\delta\xi/\delta x] \tilde{P}[\xi] |_{\xi = \dot{x} + V'(x)} . \quad (28)$$

The evaluation of the Jacobian factor has been extensively discussed in the literature.[40,56] It is given by

$$\mathrm{Det}[\delta\xi/\delta x] = C \exp\left[\frac{1}{2} \int_{t_0}^{\infty} dt\, V''(x) \right] , \quad (29)$$

where C is a constant. For colored noise the factor in the exponential is definitely $\frac{1}{2}$, but for white noise it can be any number between 0 and 1. This ambiguity is due to the singular nature of white noise; the choice we have made in (29) corresponds to the Stratonovich prescription. It arises naturally when white noise is viewed as the $\tau \to 0$ limit of colored noise. Hence we adopt it in this paper.

The probability density functional $\tilde{P}[x]$ takes on a particularly simple form if $t_0 \to -\infty$, which, as we shall discuss shortly, is also a physically interesting limit. We can now drop the tilde since $P[\xi]$ contains no boundary terms and is the probability density functional over the infinite range $-\infty < t < \infty$. Assuming that the noise is Gaussian, and using (21), (28), and (29), we have

$$P[x] = \mathcal{N} \exp(-S[x]/D) , \quad (30)$$

where $\mathcal{N}$ is a normalization constant and

$$S[x] = \frac{1}{4} \int_{-\infty}^{\infty} dt \{ [\dot{x} + V'(x)]^2 + \kappa_1 \tau^2 [\ddot{x} + \dot{x} V''(x)]^2 + \kappa_2 \tau^4 [\dddot{x} + \ddot{x} V''(x) + \dot{x}^2 V'''(x)]^2 + \cdots \} - \frac{D}{2} \int_{-\infty}^{\infty} dt\, V''(x) \quad (31)$$

is the generalized Onsager-Machlup functional for the system. In subsequent work we will frequently refer to the first term in (31) as the "action" and incorporate the second term coming from the Jacobian into the prefactor.

Probability density functions can now be expressed as path integrals by using (27). For example

$$\begin{aligned} P_1(x_1,t_1) &\equiv P(x_1,t_1|x_0,t_0) \\ &= \langle \delta(x(t_1)-x_1) \rangle |_{x(t_0)=x_0} \\ &= \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \tilde{P}[x] \delta(x(t_1)-x_1) , \end{aligned} \quad (32)$$

and more generally

$$\begin{aligned} P_n(x_n,t_n;\ldots;x_1,t_1) &\equiv P(x_n,t_n;\ldots;x_1,t_1|x_0,t_0) \\ &= \langle \delta(x(t_1)-x_1) \cdots \delta(x(t_n)-x_n) \rangle_{x(t_0)=x_0} \\ &= \int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \tilde{P}[x] \delta(x(t_1)-,x_1) \cdots \delta(x(t_n)-x_n) . \end{aligned} \quad (33)$$

(In our notation, a numerical subscript on a probability distribution function indicates that the dependence on the preparation of the system has been suppressed.) Conditional probability densities are defined using Bayes theorem, in the usual way;

$$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{P_2(x_2,t_2;x_1,t_1)}{P_1(x_1,t_1)} = \frac{\int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \tilde{P}[x] \delta(x(t_1)-x_1) \delta(x(t_2)-x_2)}{\int_{x(t_0)=x_0} \left[\prod_{t>t_0} dx(t) \right] \tilde{P}[x] \delta(x(t_1)-x_1)} . \quad (34)$$

For a Markov process $P_{1|1}(x_2,t_2|x_1,t_1) = P(x_2,t_2|x_1,t_1;x_0,t_0)$ is independent of the initial condition on the Langevin equation. However, this is not true in general; for non-Markov processes, *the specification of the initial condition is a physical choice not a mathematical one.* As we will see in paper II, for our purposes this choice is unimportant and we will frequently assume that the initial condition on x is set in the infinitely distant past. Then, for example,

$$P(x_1,t_1)=\int_{\rm IC} d[x]P[x]\delta(x(t_1)-x_1) \tag{35}$$

and

$$P(x_2,t_2|x_1,t_1) = \frac{\int_{\rm IC} d[x]P[x]\delta(x(t_1)-x_1)\delta(x(t_2)-x_2)}{\int_{\rm IC} d[x]P[x]\delta(x(t_1)-x_1)}, \tag{36}$$

where we have introduced the notation

$$d[x]=\prod_t dx(t) \tag{37}$$

and where IC stands for the initial condition $x(-\infty)=x_0$. This choice corresponds to ensuring that the system has equilibrized before anything is calculated. With this choice probability density functions only depend on time differences and in particular Eq. (35) defines the stationary probability density $P(x_1)$.

Equations (35) and (36) are the starting point for the calculations to be described in paper II.

IV. PATH INTEGRALS OVER A FINITE TIME INTERVAL

The path-integral representation for the conditional probability (36) was obtained from the Langevin equation and is defined over the infinite time interval $(-\infty,\infty)$. But, as is well known, another form is possible if the process is Markov. One can start from the Fokker-Planck equation and express $P_{1|1}(x_1,t_2|x_1,t_1)$ as a path integral in exactly the same way as in quantum mechanics, where one starts from the Schrödinger equation and expresses the transition amplitude $\langle x_2,t_2|x_1,t_1\rangle$ as a path integral. Equivalently, one can transform the Fokker-Planck equation into a Schrödinger-like equation (in imaginary time) and use the quantum-mechanical formalism directly. Either way, one finds, using the Stratonovich prescription,[57]

$$P_{1|1}(x_2,t_2|x_1,t_1)=\int_{x(t_1)=x_1} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2), \tag{38}$$

where

$$\mathcal{P}[x]\equiv \exp\left[-\frac{1}{4D}\int_{t_1}^{t_2} dt\,[\dot{x}+V'(x)]^2\right] \times \exp\left[\frac{1}{2}\int_{t_1}^{t_2} dt\, V''(x)\right]. \tag{39}$$

This result is formally derived by subdividing the interval (t_1,t_2) into N time segments of equal duration ϵ. Then

$$\mathcal{D}x \equiv \lim_{\substack{N\to\infty\\ \epsilon\to 0}} \prod_{l=1}^{N} \left[\frac{1}{4\pi D\epsilon}\right]^{1/2} dx(t^{(l)}),\quad N\epsilon \text{ fixed} \tag{40}$$

where $x(t^{(0)})=x_1$ and $x(t^{(N)})=x_2$. The representation (38) gives $P_{1|1}(x_2,t_2|x_1,t_1)$ as an integral over all paths starting at x_1 at time t_1 and ending at x_2 at time t_2. We will now derive (38) from (36) and show how, in some cases, a non-Markov version of (38) can be obtained by following the same steps.

Let us begin with the more general expression (34), where we have not taken $t_0\to-\infty$. If the noise is white, then specifying $x(t_1)=x_1$ means that $\tilde{P}[x]$ *factorizes* into a part for $t\le t_1$ (which we denote by $\tilde{P}_<$ or $\tilde{P}_{\rm I}$) and a part for $t>t_1$ (denoted by $P_>$). This is due to the fact that $\tilde{P}[x]$ contains no time derivatives of x higher than the first. Writing $\tilde{P}[x]=\tilde{P}_<[x]P_>[x]$, Eq. (34) becomes

$$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{\int_{x(t_1)=x_1}\left[\prod_{t>t_1} dx(t)\right]P_>[x]\delta(x(t_2)-x_2)}{\int_{x(t_1)=x_1}\left[\prod_{t>t_1} dx(t)\right]P_>[x]} \tag{41}$$

The integrals over paths for $t\le t_1$ have cancelled top and bottom, as expected for a Markov process. A further factorization in the numerator of (41) yields

$$P_{1|1}(x_2,t_2|x_1,t_1)=\frac{\int_{x(t_1)=x_1}\left[\prod_{t_1<t\le t_2} dx(t)\right]P_{\rm II}[x]\delta(x(t_2)-x_2)\int_{x(t_2)=x_2}\left[\prod_{t>t_2} dx(t)\right]P_{\rm III}[x]}{\int_{x(t_1)=x_1}\left[\prod_{t>t_1} dx(t)\right]P_>[x]}. \tag{42}$$

In the above, region II is $t_1<t\le t_2$ and region III is $t>t_2$. Integrating an analogous expression to (24), with x_0 and t_0 replaced by x_2 and t_2, respectively, and with weighting $P_{\rm III}[\xi]$, gives

$$\int \left[\prod_{t\geq t_2} d\xi(t)\right] P_{\rm III}[\xi] = \int_{x(t_2)=x_2} \left[\prod_{t\geq t_2} d\xi(t)\right]\left[\prod_{t>t_2} dx(t)\right] {\rm Det}(\delta\xi/\delta x) \prod_{t>t_2}\delta(\dot{x}+V'(x)-\xi)P_{\rm III}[\xi]$$
$$= \int_{x(t_2)=x_2}\left[\prod_{t>t_2} dx(t)\right] P_{\rm III}[x] \;. \tag{43}$$

This shows that the right-hand side of (43) is independent of the value of x_2, as is the denominator of (42). Moreover, $P[\xi]$ factorizes for white noise and so

$$\int_{x(t_2)=x_2}\left[\prod_{t>t_2} dx(t)\right] P_{\rm III}[x] = \prod_{t\geq t_2}\int_{-\infty}^{\infty} d\xi(t) P(\xi(t)) \;. \tag{44}$$

Hence we may write (42) as

$$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{\displaystyle\int_{x(t_1)=x_1}\left[\prod_{t_1<t\leq t_2} dx(t)\right] P_{\rm II}[x]\delta(x(t_2)-x_2)}{\displaystyle\prod_{t_1<t\leq t_2}\int_{-\infty}^{\infty} d\xi(t)P(\xi(t))}$$
$$= \int_{x(t_1)=x_1} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2) \;, \tag{45}$$

in the notation of (39) and (40). We have therefore arrived at the form (38), but starting from the Langevin equation instead of the Fokker-Planck equation.

Let us now attempt the above derivation when the noise is not white, but exponentially correlated. We again begin with (34), but now $\tilde{P}[x]$ does not factorize as $\tilde{P}_<[x]P_>[x]$ since it contains second derivatives. However, if both $x(t_1)$ and $\dot{x}(t_1)$, and not just $x(t_1)$, were specified then it would factorize. We can arrange for this to happen by introducing the identity

$$1 = \int d\dot{x}_1 \delta(\dot{x}(t_1)-\dot{x}_1) \tag{46}$$

into the integrand of the numerator of (34). This latter quantity now has the form

$$\int d\dot{x}_1 \int_{x(t_0)=x_0}\left[\prod_{t_0<t\leq t_1} dx(t)\right]\tilde{P}_<[x]\delta(\dot{x}(t_1)-\dot{x}_1)\delta(x(t_1)-x_1)\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]\delta(x(t_2)-x_2)$$

$$= \int d\dot{x}_1 \int_{x(t_0)=x_0}\left[\prod_{t_0<t\leq t_1} dx(t)\right]\tilde{P}_<[x]\delta(\dot{x}(t_1)-\dot{x}_1)\delta(x(t_1)-x_1)$$

$$\times \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]\left[\frac{\displaystyle\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]\delta(x(t_2)-x_2)}{\displaystyle\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]}\right]$$

$$= \int d\dot{x}_1 P(\dot{x}_1,x_1,t_1|x_0,t_0)\left[\frac{\displaystyle\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]\delta(x(t_2)-x_2)}{\displaystyle\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}}\left[\prod_{t>t_1} dx(t)\right]P_>[x]}\right] \;. \tag{47}$$

We will now show that the quantity in curly brackets has the simple form

$$\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2)\ , \tag{48}$$

where the appropriate measure is now

$$\mathcal{D}x \equiv \lim_{\substack{N\to\infty\\ \epsilon\to 0}} \prod_{l=1}^{N} \left[\frac{\tau^2}{4\pi D\epsilon^3}\right]^{1/2} dx(t^{(l)}),\quad N\epsilon \text{ fixed} \tag{49}$$

and $\mathcal{P}[x]$ will be defined below. The denominator of (47) is

$$\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \left[\prod_{t>t_1} dx(t)d\xi(t)\right] \mathrm{Det}[\delta\xi/\delta x] \times \prod_{t>t_1} \delta(\dot{x}+V'-\xi)P_>[\xi]$$

$$= \int_{\xi(t_1)=\xi_1} \left[\prod_{t>t_1} d\xi(t)\right] P_>[\xi] \tag{50}$$

using (24)–(28). In the numerator we can follow similar steps, to find that the integrals for $t>t_2$ reduce to

$$\int_{\xi(t_2)\text{ fixed}} \left[\prod_{t>t_2} d\xi(t)\right] P_{\mathrm{III}}[\xi]\ . \tag{51}$$

In this case $\xi(t_2)$ is fixed since we have chosen to carry out the $\xi(t)$ $(t>t_2)$ integrals first, and all the variables defined for $t_1<t\le t_2$ are kept constant. Only $\xi(t_2)$ is relevant for the evaluation of (51), however.

It is convenient in what follows to imagine an upper time cutoff t', which we will subsequently take to infinity. The normalization constant in $P_>[x]$ multiplying the exponential cancels between the numerator and denominator. In the same way, we can replace $\prod dx(t)$ by $\mathcal{D}x$, defined by (49), over the appropriate time interval. The integrals (50) and (51) become

$$\lim_{t'\to\infty} \int_{\xi(t_i)\text{ fixed}} \mathcal{D}\xi \exp\left[-\frac{1}{4D}\int_{t_i}^{t'} dt(\xi^2+\tau^2\dot{\xi}^2)\right]\ ;\qquad i=1,2\ . \tag{52}$$

This is a Gaussian integral which can be evaluated by standard techniques. The result consists of a term which is exponentially small in D multiplied by the prefactor

$$\{4\pi D\tau \sinh[(t'-t_i)/\tau]\}^{-1/2}\ . \tag{53}$$

As $t'\to\infty$ the ratio of the prefactors between (50) and (51) tends to $\exp[(t_2-t_1)/2\tau]$. The exponential factor multiplying (53) is simply $\exp[-(\tau/4D)\xi(t_i)^2]$, and so the curly bracket in (47) equals (48) with

$$\mathcal{P}[x] = \exp\left[-\frac{1}{4D}\int_{t_1}^{t_2} dt(\xi^2+\tau^2\dot{\xi}^2)\right] \times\exp\left[-\frac{\tau}{4D}[\xi(t_2)]^2\right]\exp\left[\frac{\tau}{4D}[\xi(t_1)]^2\right] \times\exp\left[\tfrac{1}{2}\int_{t_1}^{t_2} dt[V''(x)+\tau^{-1}]\right]\ , \tag{54}$$

where $\xi=\dot{x}+V'(x)$ and $\dot{x}(t_1)=\dot{x}_1$, $\dot{x}(t_2)=\dot{x}_2$. A neater expression for $\mathcal{P}[x]$ is

$$\mathcal{P}[x] = \exp\left[-\frac{1}{4D}\int_{t_1}^{t_2} dt(\xi+\tau\dot{\xi})^2\right]\exp\left[\tfrac{1}{2}\int_{t_1}^{t_2} dt[V''(x)+\tau^{-1}]\right]\Bigg|_{\xi=\dot{x}+V'(x)}$$

$$= \exp\left[-\frac{1}{4D}\int_{t_1}^{t_2} dt\{\dot{x}+V'(x)+\tau[\ddot{x}+\dot{x}V''(x)]\}^2\right]\exp\left[\tfrac{1}{2}\int_{t_1}^{t_2} dt[V''(x)+\tau^{-1}]\right]\ . \tag{55}$$

Dividing (47) by $P(x_1,t_1|x_0,t_0)$ and using (48) we finally find that

$$P_{1|1}(x_2,t_2|x_1,t_1) = \int d\dot{x}_1 \frac{P(\dot{x}_1,x_1,t_1|x_0,t_0)}{P(x_1,t_1|x_0,t_0)} \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2)\ , \tag{56}$$

with $\mathcal{P}[x]$ given by (55). This is a considerably more complicated equation than the corresponding white-noise result (45). We stress again that the conditional probability will, in general, depend on the initial condition $x(t_0)=x_0$. It should also be noted that the derivation of (56) assumed that the noise was switched on in the infinitely distant past; however it will be shown in Sec. V that the same expression is valid independently of the preparation of the system.

The expression for $P_{1|1}$ simplifies in two cases:

(i) $t_0\to-\infty$. The initial condition is now irrelevant and (56) becomes

$$P_{1|1}(x_2,t_2|x_1,t_1) = \int d\dot{x}_1 \frac{P(\dot{x}_1,x_1,t_1)}{P(x_1,t_1)} \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2)\ . \tag{57}$$

(ii) $t_0=t_1$. Equation (56) applies if the initial condition is set at a time $t_0<t_1$. If it is set at time t_1 then

$$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{\int_{x(t_1)=x_1} \mathcal{D}x\, \mathcal{P}'[x]\delta(x(t_2)-x_2)}{\int_{x(t_1)=x_1} \mathcal{D}x\, \mathcal{P}'[x]}\ , \tag{58}$$

where $\mathcal{P}'[x]$ is given by

$$\mathcal{P}'[x]=\exp\left[-\frac{1}{4D}\int_{t_1}^{t_2}dt\,(\xi^2+\tau^2\dot{\xi}^2)\right]\Bigg|_{\xi=\dot{x}+V'(x)}\exp\left[-\frac{\tau}{4D}\{[\xi(t_1)]^2+[\xi(t_2)]^2\}\right]\Bigg|_{\xi=\dot{x}+V'(x)}$$
$$\times\exp\left[\tfrac{1}{2}\int_{t_1}^{t_2}dt\,[V''(x)+\tau^{-1}]\right] . \tag{59}$$

The simplest way to see this is to start with the basic definition

$$P_{1|1}(x_2,t_2|x_1,t_1)=P(x_2,t_2|x_1,t_1)$$
$$=\langle\delta(x(t_2)-x_2)\rangle|_{x(t_1)=x_1}=\int_{x(t_1)=x_1}\left[\prod_{t>t_1}dx(t)\right]\tilde{P}[x]\delta(x(t_2)-x_2) . \tag{60}$$

Here $\tilde{P}$ contains the result of integrating $\xi(t)$ from minus infinity to t_1, which is

$$\int_{\xi(t_1)\text{ fixed}}\left[\prod_{t<t_1}d\xi(t)\right]P_<[\xi] . \tag{61}$$

The expression (61) has the same form as (51) and can be evaluated in the same way. To avoid problems with normalization we divide by

$$1=\int_{x(t_1)=x_1}\left[\prod_{t>t_1}dx(t)\right]\tilde{P}[x] . \tag{62}$$

The prefactor, given by (53) with $(t'-t_i)$ replaced by (t_1-t'), cancels between numerator and denominator as $t'\to-\infty$. Hence (61) may be replaced by $\exp[-(\tau/4D)\xi(t_1)^2]$ and we have

$$P_{1|1}(x_2,t_2|x_1,t_1)=\frac{\int_{x(t_1)=x_1}\left[\prod_{t>t_1}dx(t)\right]P_>[x]\exp\left[-\frac{\tau}{4D}\xi(t_1)^2\right]\delta(x(t_2)-x_2)}{\int_{x(t_1)=x_1}\left[\prod_{t>t_1}dx(t)\right]P_>[x]\exp\left[-\frac{\tau}{4D}\xi(t_1)^2\right]} , \tag{63}$$

where $\xi(t_1)=\dot{x}(t_1)+V'(x_1)$. This expression is very similar to the one appearing in curly brackets in (47). Proceeding as we did in that case, we obtain (58). Up to normalization, this result coincides with the expression given in Ref. 42.

It is clear that all of the above formalism carries over to a correlator for which $\tilde{C}^{-1}$ terminates at order τ^{2m}. If the initial condition is set at $t_0<t_1$ then we need to introduce a generalization of (46); namely

$$1=\int d\dot{x}_1 d\ddot{x}_1\cdots dx_1^{(m)}\delta(\dot{x}(t_1)-\dot{x}_1)\delta(\ddot{x}(t_1)-\ddot{x}_1)\cdots\delta(x^{(m)}(t_1)-x_1^{(m)}) . \tag{64}$$

Following the same steps as for $m=1$ we arrive at

$$P_{1|1}(x_2,t_2|x_1,t_1)=\int d\dot{x}_1\cdots dx_1^{(m)}\frac{P(x_1^{(m)},\ldots,\dot{x}_1,x_1,t_1|x_0,t_0)}{P(x_1,t_1|x_0,t_0)}\int_{\{x\}}\mathcal{D}x\,\mathcal{P}[x]\delta(x(t_2)-x_2) , \tag{65}$$

where $\{x\}$ stands for $\{x(t_1)=x_1,\dot{x}(t_1)=\dot{x}_1,\ldots,x^{(m)}(t_1)=x_1^{(m)}\}$. In this case, the measure is defined as

$$\mathcal{D}x\equiv\lim_{\substack{N\to\infty\\ \epsilon\to 0}}\prod_{l=1}^{N}\left[\left[\prod_{n=1}^{m}\frac{\alpha_n\tau}{\epsilon}\right]\left[\frac{1}{4\pi D\epsilon}\right]^{1/2}dx(t^{(l)})\right],\quad N\epsilon\text{ fixed} \tag{66}$$

while

$$\mathcal{P}[x]=\exp\left\{-\frac{1}{4D}\int_{t_1}^{t_2}dt\left[\xi+\left[\sum_{n=1}^{m}\alpha_n\right]\dot{\xi}+\left[\sum_{n<n'}^{m}\alpha_n\alpha_{n'}\right]\ddot{\xi}+\cdots\right]^2\right\}\exp\left[\frac{1}{2}\int_{t_1}^{t_2}dt\left[V''(x)+\sum_{n=1}^{m}(\alpha_n\tau)^{-1}\right]\right] . \tag{67}$$

The boundary terms have been incorporated into the cross terms in the first part of the exponent in (67). We have also used the fact that

$$\lim_{t'\to\infty}\int_{\{\xi\}}\mathcal{D}\xi\exp-\frac{1}{4D}\int_{t_i}^{t'}dt\left[\xi+\left[\sum_{n=1}^{m}\alpha_n\right]\dot{\xi}+\left[\sum_{n<n'}^{m}\alpha_n\alpha_{n'}\right]\ddot{\xi}+\cdots\right]^2 \tag{68}$$

depends on t_i, but not on the boundary values $\{\xi\}=\{\xi(t_1)=\xi_1,\ldots,\xi^{(m)}(t_1)=\xi_1^{(m)}\}$.

If the initial condition is set at $t=t_0$, then the result (58) still holds, but with $\mathcal{P}'[x]$ now given by

$$\mathcal{P}'[x]=\exp\left[-\frac{1}{4D}\int_{t_1}^{t_2}dt\,[\xi^2+\kappa_1\tau^2\dot{\xi}^2+\cdots+\kappa_m\tau^{2m}(\xi^{(m)})^2]\right]$$
$$\times\exp\left[-\frac{1}{4D}(B_1+B_2)\right]\exp\left[\frac{1}{2}\int_{t_1}^{t_2}dt\left[V''(x)+\sum_{n=1}^{m}(\alpha_n\tau)^{-1}\right]\right]$$
$$=\exp\left\{-\frac{1}{4D}\int_{t_1}^{t_2}dt\left[\xi+\left[\sum_{n=1}^{m}\alpha_n\right]\dot{\xi}+\left[\sum_{n<n'}^{m}\alpha_n\alpha_{n'}\right]\ddot{\xi}+\cdots\right]^2\right\}$$
$$\times\exp(-B_1/2D)\exp\left[\frac{1}{2}\int_{t_1}^{t_2}dt\left[V''(x)+\sum_{n=1}^{m}(\alpha_n\tau)^{-1}\right]\right], \tag{69}$$

where B_1 and B_2 are the boundary terms obtained by integrating out the noise from minus infinity to t_1 and from t_2 to infinity, respectively. We shall not give explicit expressions for the B_i, as they are not particularly simple.

As m increases the factors depending on $\dot{x}_1,\ddot{x}_1,\ldots,x_1^{(m)}$ in (65) and (69) become unwieldy, and the usefulness of defining path integrals on a finite time interval becomes questionable.

V. PATH-INTEGRAL REPRESENTATION FROM THE EQUIVALENT MARKOV PROCESS

In Sec. II we explained how the non-Markov colored noise process of interest to us in this paper could be embedded in a higher-dimensional Markov process. This correspondence suggests an alternative method of deriving the results of Sec. IV. We proceed by setting up the equivalent Markov process as a path integral, and then projecting the resulting expression onto the one-dimensional subspace of interest.

Again, this is most simply illustrated in the case of exponentially correlated noise. Here the Markov process is two dimensional, the auxiliary equation being given by (13) (we take $\kappa_1=1$). Differentiating the Langevin equation (23) with respect to time and using (13) gives

$$\tau[\ddot{x}+\dot{x}V''(x)]+[\dot{x}+V'(x)]=\eta\ , \tag{70}$$

where η is white noise. To set up the path-integral representation for this process starting from the Langevin equation (70), we proceed as we did for (23) *et seq.* The main difference is that in order for the mapping $\eta\mapsto x$ to be well defined we must specify $\dot{x}(t_0)$ as well as $x(t_0)$. Thus the analogous expression to (24) reads

$$1=\int_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}}\left[\prod_{t>t_0}dx(t)\right]\mathrm{Det}[\delta\eta/\delta x]\times\prod_{t>t_0}\delta((\dot{x}+V')+\tau(\ddot{x}+\dot{x}V'')-\eta)\ . \tag{71}$$

Ensemble averages are calculated from

$$\langle F[x]\rangle=\int_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}}\left[\prod_{t>t_0}dx(t)\right]\tilde{P}[x]F[x]\ , \tag{72}$$

where

$$\tilde{P}[x]=\mathrm{Det}[\delta\eta/\delta x]\tilde{P}[\eta]|_{\eta=[\dot{x}+V']+\tau[\ddot{x}+\dot{x}V'']}\ . \tag{73}$$

The tilde on $\tilde{P}$ again symbolizes that the noise has been integrated out for $t<t_0$ and incorporated into this factor. This is trivial in the case of white noise: the integration just gives a constant since there are no boundary terms for white noise. The Jacobian factor again has the form (29), up to a constant, since

$$\mathrm{Det}[\delta\eta/\delta x]=\mathrm{Det}[\delta\eta/\delta\xi]\mathrm{Det}[\delta\xi/\delta x]$$
$$=C\exp\left[\frac{1}{2}\int_{t_0}^{\infty}dt\,[V''(x)+\tau^{-1}]\right]. \tag{74}$$

An expression for the constant C can be obtained by taking the continuous limit of discrete representations of the functions $\eta(t)$ and $x(t)$. This expression can then be used to derive the measure (49).

For the two-dimensional process under consideration here, it is natural to define the probability density functions

$$P_1(\dot{x}_1,x_1,t_1)\equiv P(\dot{x}_1,x_1,t_1|\dot{x}_0,x_0,t_0)=\langle\delta(\dot{x}(t_1)-\dot{x}_1)\delta(x(t_1)-x_1)\rangle|_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}} \tag{75}$$

and more generally

$$P_n(\dot{x}_n,x_n,t_n;\ldots;\dot{x}_1,x_1,t_1)\equiv P(\dot{x}_n,x_n,t_n;\ldots;\dot{x}_1,x_1,t_1|\dot{x}_0,x_0,t_0)=\langle\delta(\dot{x}(t_n)-\dot{x}_n)\cdots\delta(x(t_1)-x_1)\rangle_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}}\ . \tag{76}$$

The conditional probability density $P_{1|1}(\dot{x}_2,x_2,t_2|\dot{x}_1,x_1,t_1)$ is therefore given by

$$\frac{\int_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}} \left[\prod_{t>t_0} dx(t)\right] \tilde{P}[x]\delta(\dot{x}(t_1)-\dot{x}_1)\delta(x(t_1)-x_1)\delta(\dot{x}(t_2)-\dot{x}_2)\delta(x(t_2)-x_2)}{\int_{\substack{x(t_0)=x_0\\ \dot{x}(t_0)=\dot{x}_0}} \left[\prod_{t>t_0} dx(t)\right] \tilde{P}[x]\delta(\dot{x}(t_1)-\dot{x}_1)\delta(x(t_1)-x_1)} . \tag{77}$$

Since both $x(t_1)$ and $\dot{x}(t_1)$ are specified, $\tilde{P}[x]$ factorizes, and following the discussion of Sec. IV we see that

$$P_{1|1}(\dot{x}_2,x_2,t_2|\dot{x}_1,x_1,t_1)=\frac{\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \left[\prod_{t>t_1} dx(t)\right] P_>[x]\delta(\dot{x}(t_2)-\dot{x}_2)\delta(x(t_2)-x_2)}{\int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \left[\prod_{t>t_1} dx(t)\right] P_>[x]} . \tag{78}$$

The cancellation of all contributions for $t<t_1$ confirms that the process is Markov. Proceeding as in (42) *et seq.* we arrive at the result

$$P_{1|1}(\dot{x}_2,x_2,t_2|\dot{x}_1,x_1,t_1) = \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(\dot{x}(t_2)-\dot{x}_2)\delta(x(t_2)-x_2)\ , \tag{79}$$

where $\mathcal{D}x$ is defined by (49) and

$$\mathcal{P}[x]=\exp\left[-\frac{1}{4D}\int_{t_1}^{t_2} dt\,[(\dot{x}+V')+\tau(\ddot{x}+\dot{x}V'')]^2\right] \times\exp\left[\frac{1}{2}\int_{t_1}^{t_2} dt\,(V''+\tau^{-1})\right] . \tag{80}$$

Given the Markov nature of the process these formulas could have also been obtained from the relevant Fokker-Planck equation, which we give below.

The condition probability $P_{1|1}$ is now easily found using (79). The condition on $\dot{x}_2$ can be integrated out directly, while that on $\dot{x}_1$ can be integrated out using

$$P(x_2,t_2|x_1,t_1;\dot{x}_0,x_0,t_0)=\int d\dot{x}_1 P(x_2,t_2|\dot{x}_1,x_1,t_1) \times\frac{P(\dot{x}_1,x_1,t_1|\dot{x}_0,x_0,t_0)}{P(x_1,t_1|\dot{x}_0,x_0,t_0)} . \tag{81}$$

Notice how the dependence on events that occur at times $t<t_1$, characteristic of non-Markov processes, enters through (81). From (79) and (81) we find that

$$P(x_2,t_2|x_1,t_1;\dot{x}_0,x_0,t_0) = \int d\dot{x}_1 \frac{P(\dot{x}_1,x_1,t_1|\dot{x}_0,x_0,t_0)}{P(x_1,t_1|\dot{x}_0,x_0,t_0)} \times \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2) . \tag{82}$$

Provided that the preparation of the system has been specified [in other words, the initial distribution $P_1(\dot{x}_0,x_0,t_0)$ is known] then the conditional probabilities which appear in (82) can be expressed as ratios of joint probabilities. If the resulting equation is then multiplied by the factor

$$\frac{P_2(x_1,t_1;\dot{x}_0,x_0,t_0)}{P_2(x_1,t_1;x_0,t_0)}$$

one obtains

$$\frac{P_3(x_2,t_2;x_1,t_1;\dot{x}_0,x_0,t_0)}{P_2(x_1,t_1;x_0,t_0)} = \int d\dot{x}_1 \frac{P_2(\dot{x}_1,x_1,t_1;\dot{x}_0,x_0,t_0)}{P_2(x_1,t_1;x_0,t_0)} \times \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2) . \tag{83}$$

Integrating over $\dot{x}_0$ then gives

$$\frac{P_3(x_2,t_2;x_1,t_1;x_0,t_0)}{P_2(x_1,t_1;x_0,t_0)} = \int d\dot{x}_1 \frac{P_2(\dot{x}_1,x_1,t_1;x_0,t_0)}{P_2(x_1,t_1;x_0,t_0)} \times \int_{\substack{x(t_1)=x_1\\ \dot{x}(t_1)=\dot{x}_1}} \mathcal{D}x\, \mathcal{P}[x]\delta(x(t_2)-x_2)\ , \tag{84}$$

which has the same form as (56) when written in terms of conditional probabilities. Note that this derivation holds independently of the preparation of the system, whereas the derivation of (56) was valid only for systems in which the noise had been switched on in the infinitely distant past. In this sense, Eq. (84) is a generalization of Eq. (56).

It is worth contrasting the approach used here with that of Sec. IV, in relation to the setting of boundary conditions. In Sec. IV, the noise $\xi(t)$ was treated on a different footing to the coordinate $x(t)$. Conditions on $\xi(t)$ at $t=\pm\infty$ were imposed in order to obtain the Fourier transform of the noise. At times of the order of several τ from these endpoints, the system has "forgotten" these conditions. This contrasts with the initial condition which has to be set on x at some time t_0, in order for the problem to be well defined. In this section both $x(t)$ and $\xi(t)$ satisfy Langevin equations and are treated on the same footing. Consequently, initial conditions on

$x(t)$ and $\xi(t)$ [or equivalently, $x(t)$ and $\dot{x}(t)$] need to be set.

To obtain the generalizations of (79) and (80) for the mth-order process is straightforward. We need only note that, from (16),

$$\eta(t)=\left[(\alpha_1\tau)\frac{d}{dt}+1\right]\left[(\alpha_2\tau)\frac{d}{dt}+1\right]\cdots \times\left[(\alpha_m\tau)\frac{d}{dt}+1\right]\xi(t)\,, \tag{85}$$

where $\xi=\dot{x}+V'(x)$, and also that

$$\begin{aligned}\mathrm{Det}[\delta\eta/\delta x]&=\mathrm{Det}[\delta\eta/\delta\zeta_{m-1}]\mathrm{Det}[\delta\zeta_{m-1}/\delta\zeta_{m-2}]\cdots\\&\quad\times\mathrm{Det}[\delta\xi/\delta x]\\&=C\exp\left[\frac{1}{2}\int_{t_0}^{\infty}dt\left[V''(x)+\sum_{n=1}^{m}(\alpha_n\tau)^{-1}\right]\right].\end{aligned} \tag{86}$$

Then

$$\begin{aligned}P_{1|1}(x_2^{(m)},\dots,x_2,t_2|x_1^{(m)},\dots,x_1,t_1)\\=\int_{\{x\}}\mathcal{D}x\,\mathcal{P}[x]\delta(x^{(m)}(t_2)-x_2^{(m)})\cdots\delta(x(t_2)-x_2)\end{aligned} \tag{87}$$

with $\mathcal{D}x$ and $\mathcal{P}[x]$ given by (66) and (67), respectively.

We have remarked already that path integrals such as (79) and (87) could have been obtained directly from the appropriate Fokker-Planck equation. We conclude by writing down these differential equations. Their derivations are standard[2,4,5] and will not be given here. For the two-dimensional system consisting of (13) and (23) with $\kappa_1=1$,

$$\frac{\partial Q}{\partial t}=-\frac{\partial}{\partial x}\{[-V'(x)+\xi]Q\}+\frac{1}{\tau}\frac{\partial}{\partial\xi}\left[\xi Q+\frac{D}{\tau}\frac{\partial Q}{\partial\xi}\right], \tag{88}$$

where Q is a function of x, ξ, and t. Transforming to new variables x, $\dot{x}$, and t, (88) becomes

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(\dot{x}P)+\frac{\partial}{\partial\dot{x}}\left[[V''(x)+\tau^{-1}](\dot{x}P)+\tau^{-1}V'(x)P+\frac{D}{\tau^2}\frac{\partial P}{\partial\dot{x}}\right]. \tag{89}$$

For the mth-order process,

$$\begin{aligned}\frac{\partial Q}{\partial t}=&-\frac{\partial}{\partial x}\{[-V'(x)+\xi]Q\}-\frac{\partial}{\partial\xi}[(\alpha_1\tau)^{-1}(-\xi+\zeta_1)Q]\\&-\sum_{n=1}^{m-2}\frac{\partial}{\partial\zeta_n}[(\alpha_{n+1}\tau)^{-1}(-\zeta_n+\zeta_{n+1})Q]\\&-\frac{\partial}{\partial\zeta_{m-1}}[(\alpha_m\tau)^{-1}(-\zeta_{m-1})Q]+\frac{D}{\alpha_m^2\tau^2}\frac{\partial^2 Q}{\partial\zeta_{m-1}^2}\,.\end{aligned} \tag{90}$$

The corresponding equation in the variables $\{x\}$ and t can be derived from (90), although it does not have an especially simple form.

VI. CONCLUSION

In developing the formalism described in this paper, we had in mind applications to the study of systems under the influence of external colored noise. However, the path-integral representation of non-Markov processes has not received much attention in the past and we hope that our work will stimulate some interest in this topic. For example, action functionals containing higher time derivatives are worthy of studying in their own right.

In the following papers in this series, these results will be combined with standard path-integral techniques to derive various quantities of physical interest in the weak noise limit.

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