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Non-Markovian stochastic jump processes. I. Input field analysis

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A non-Markovian model of correlated phase jumps is introduced for phase fluctuations of an electromagnetic field. This generalized jump model (GJM) treats phase jumps of arbitrary size, occurring at random times; but in contrast to previous work, the jumps are allowed to be fully correlated, partially correlated, or uncorrelated. The degree of correlation is defined by a single parameter derived from the theory. The familiar phase-diffusion model, telegraph-noise model, Burshtein model, and Brownian-motion-like model are all obtained from the GJM in the proper limits. The standard way of characterizing the spectrum of a laser has been the assignment of a single parameter—the linewidth. However, in experiments where the details of the fluctuations are important, or where exact line shapes are measured, this single-parameter characterization might be insufficient. This GJM describes most cases by a set of three stochastic parameters: the degree of correlation between the jumps, the characteristic jump size, and the mean time between jumps. In this paper expressions are derived for the correlation function and the spectrum of a stochastic field in terms of these three stochastic parameters. In addition to analytical work, detailed numerical simulations are presented for the various limiting cases of the model, and the agreement between theory and simulation is excellent. Since the stochastic parameters are not *a priori* known, a procedure is described for extracting the stochastic parameters from measurable quantities such as the field correlation function or spectrum. Since correlated fluctuations are very common in optics (any stabilization feedback procedure involves anticorrelation), the questions of relevance of the present model to problems of current interest in optical communication and nonlinear optics are also discussed.

I. INTRODUCTION

The interaction of resonant or nearly resonant strong laser light with an absorbing medium provides the basis for a very wide range of phenomena within the even wider field of nonlinear optics. In many applications the laser light may be assumed to be monochromatic, and indeed, the bulk of the theoretical treatment of nonlinear optical phenomena is based on this assumption. As an example, the Bloembergen approach to nonlinear susceptibilities generally assumes a δ function for the laser frequency, while still allowing for (phenomenological) relaxation processes in the material system. As experiments become more sophisticated, the details of resonance lines rather than their mere existence are being investigated. Thus, the stochastic character of the laser fields should be considered in detail.

Several authors, using different approaches, have dealt with the topic of laser phase fluctuations in various nonlinear optical phenomena. They considered phenomena such as saturation,[1–7] resonance fluorescence,[2,5–18] multiphoton resonance processes,[4,19–30] double optical resonance,[4,31–36] four-wave mixing,[27,37–39] Hanle resonances,[39–40] diffraction of atoms by a standing wave,[41] autoionization in a strong field,[42] photon echoes,[43] population fluctuations,[44] etc.[45–47] The most common assumption about stochastic laser fields has been that each field may be described by a constant amplitude, a fixed frequency, and a phase that undergoes a diffusionlike (Wiener-Levy) process.[12,14,19–21,23,24,26,27,33,37–39,44,47] This phase-diffusion model (PDM), has been treated mostly for mathematical convenience, as the statistical properties of diffusion processes are well known,[48] even though in most cases, *there is no a priori good reason to assume that real lasers obey the Markovian assumptions of the* PDM. The Lorentzian laser line shape predicted by the PDM is not generally found, and thus, more general treatment of stochastic processes is warranted.

A possible way to extend the PDM to the case of non-Lorentzian line shapes is to consider a Gaussian-Markovian (Ornstein-Uhlenbeck, Brownian motion) process[49] of frequency fluctuations. This model for an intensity stabilized laser field, which is similar to the Anderson-Kubo treatment of material energy level fluctuations,[50–52] was successfully applied to a number of nonlinear optical problems.[3,7,8,15,17,22,25,28,32,34,35,40,53]

A different approach to the problem of phase fluctuations was adopted by Burshtein and co-workers.[1,2,11] These authors developed a jump model, where the phase of the field is piecewise constant, with jumps of random size occurring at random times. The Burshtein model is Markovian, namely, the jumps are not correlated, and the random jump time distribution is Poissonian. With these assumptions, exact equations for the field correlation function, the interaction with a two level system, and many other physical processes were derived and solved.[1,2,7,11,13,28,31,41,44–46] The Burshtein equations are valid for uncorrelated jumps of any size, with the PDM recovered in the limit of small jumps. The Burshtein work while very elegant, is still restricted to the Markovian assumption and Lorentzian line shapes.

Another jump model discussed by Eberly and co-workers is the phase telegraph-noise model.[5] In this model the phase is allowed to jump at random times between two fixed values. Eberly and others obtained spectra and correlation functions,[5,9,18,33,41] but again, the model is restricted by an unphysical assumption that the phase can only have one of two values.

Since in our opinion, fluctuations in nature are generally correlated, in the present paper we extend the Burshtein method to a non-Markovian generalized jump model (GJM) for phase fluctuations. *The model treats phase jumps of arbitrary size occurring at random times, but the jumps are allowed to be fully correlated, partially correlated, or uncorrelated, where the degree of correlation is defined by a single parameter derived from the theory.* Results are obtained for any degree of correlation and all jump sizes. In the appropriate limits, the present model recovers the results of the theories discussed above.

In what follows, the field is written as

$$E(t)=E_0 e^{-i[\omega_L t+\phi(t)]}+\text{c.c.} \tag{1.1}$$

where ω_L is the nominal frequency of the laser and E_0 is the amplitude. The random phase $\phi(t)$ is assumed to undergo sudden jumps at random times. The phase jumps β are described by a distribution $f(\beta)$. Each phase jump β depends on the preceding jump β' according to the conditional probability density $h(\beta',\beta)$. In this work, the assumption is made that the memory of the system extends only to the jump immediately preceding the current jump. Possible relaxation of this assumption to allow for longer memory effects will not be discussed in this paper. A correlation parameter γ is introduced with the following properties. When $h(\beta',\beta)$ is independent of β' (an uncorrelated process), $\gamma=0$; when the current jump is likely to be of the same sign as the last jump (correlated jumps), $\gamma>0$; and when the current jump is likely to be of the opposite sign to the last jump (anticorrelated jumps), $\gamma<0$. In the limits, this model reduces to well-known cases. The obvious limits can be pointed out immediately: For $\gamma=0$, the jumps are uncorrelated, and the model reduces to the Burshtein case. For $\gamma=-1$, the jumps are fully anticorrelated ($\beta=-\beta'$), and for any realization of the stochastic process the phase is jumping between two values the telegraph-noise case. In the small-jump limit, phase diffusion is described, with a diffusion constant that depends on the degree of correlation. The traditional PDM is recovered in the limit of zero correlation. In the nearly fully correlated small-jump limit the GJM approximates a continuous Markovian frequency fluctuation process, yielding as a special case the Gaussian Markovian frequency fluctuation model.

The GJM describes most cases by a set of three stochastic parameters: the correlation parameter γ, the characteristic jump size B, and the weighted average of the mean time between jumps τ_{av}. In this paper, expressions are derived for the correlation function and the spectrum of a stochastic field in terms of these three stochastic parameters. Since the stochastic parameters are not *a priori* known, a procedure is described for extracting these parameters from such measurable quantities as the field correlation function or spectrum.

In all experiments, linear or nonlinear, the laser field interacts with some medium. When the interaction is linear (e.g., linear absorption) the spectrum of the outgoing light is identical to that of the incoming light. In a nonlinear experiment, however, the situation is more complex, and the details of the stochastic nature of the input field might be needed in order to predict the exact output line shape. In the following paper[54] (hereafter referred to as paper II) we derive equations for the nonlinear interaction of the present stochastic field with a two-level system, and apply them to the interesting case of resonance fluorescence. In paper II, we show that two fields with identical linewidths may give rise to very different resonance fluorescence spectra, depending on their stochastic parameters as defined in the present paper. In addition to analyzing the observed spectrum, we also propose a way to derive the stochastic parameters from the observed resonance fluorescence spectrum, a method that is complementary to the one which is based on the analysis of the input field correlation function. It is also shown in paper II that in some cases the procedure proposed here for determining the stochastic parameters from the input field correlation function may not differentiate between different possibilities, while the procedure based on the output analysis may indicate which stochastic parameters better describe a particular fluctuating field.

The organization of the paper is as follows. In Sec. II the basic formalism is introduced. In Sec. III we introduce several parameters which are used throughout the paper and discuss their physical significance. In Sec. IV we apply the formalism to the different regions in parameter space and obtain specific results for the different regimes. For ease of reading, most of the mathematical derivations were placed in appendices, but they are quite significant, and should not be overlooked. The major results, however, are reproduced in the main body of the paper. Throughout the paper we use the Kielson-Storer model (KSM) for phase fluctuation (to be defined below) as an illustration for our results, even though in most cases our results are more general. It is stated clearly in each section what are the validity limits of the derivation in that section. Thus, in Sec. IV we discuss, among others, the Born approximation for small jumps (IV B), the case of large jumps (IV C), the case of highly correlated jumps (IV D), and the case of highly anticorrelated jumps (IV E). Section V includes a discussion of the numerical results in the different regions. In addition, independent computer simulations which were performed to verify the analytical results are described. The paper ends (Sec. VI) with the proposed procedure for extracting the stochastic parameters from experimentally measurable quantities. All aspects of the nonlinear interaction of the stochastic field with a two-level system will be discussed in paper II.

II. FORMALISM

For the purpose of this paper, we use the following rather general definition of a Markovian vector process. A vector $V(t)$ is Markovian if its conditional probability $f(V',t';V,t)$ of having the value V at time t, given that it had the value V' at time t', obeys the equation

$$\frac{\partial f(V',t';V,t)}{\partial t}=L_V f \tag{2.1}$$

where L_V is a linear instantaneous operator which does not depend on previous times. Here the initial condition is $f(V',t';V,t')=\delta(V-V')$. The stochastic operator L_V can be shown to have the following properties:

$$\int L_V g(V)dV=0 , \tag{2.2a}$$

$$L_V f(V)=0 \tag{2.2b}$$

where $g(V)$ is an arbitrary function and $f(V)$ is the distribution function of V that is assumed to be constant in time. Moreover, if L_V is also time independent, $V(t)$ is a stationary process and $f(V',t';V,t)=f(V',0;V,t-t')=f(V';V,t-t')$.

The process under consideration involves phase jumps with a random temporal distribution, where the value of the present jump β depends on the value of the last jump β' according to the conditional probability density $h(\beta',\beta)$. The field phase $\phi(t)$ as defined in Eq. (1.1) is depicted in Fig. 1(a). At time t_{n-1} a phase jump of size β_{n-1} occurred, at time t_n a jump β_n occurred, etc. In Fig. 1(b) the jump process $\beta(t)$ is defined. The function $\beta(t)$ has the value of the last jump. In this paper, the stepwise random function $\beta(t)$ is a purely discontinuous Markov process.[55] The mean time between two successive jumps $\tau_0(\beta)$ may, in general, depend on the current value of β, and the probability for the time of the next jump is given by the Poissonian distribution

$$P(\tau_{n+1})=\frac{1}{\tau_0(\beta_n)}\exp\left[-\frac{\tau_{n+1}}{\tau_0(\beta_n)}\right] \tag{2.3}$$

with $\tau_{n+1}=t_{n+1}-t_n$. The conditional probability $f(\beta_0;\beta,t)$ obeys the Kolmogorov-Feller equation[55,56]

$$\frac{\partial f}{\partial t}=-\frac{f}{\tau_0(\beta)}+\int f(\beta_0;\beta',t)\frac{h(\beta',\beta)}{\tau_0(\beta')}d\beta'\equiv L_\beta f \tag{2.4}$$

with the initial condition $f(\beta_0;\beta,0)=\delta(\beta-\beta_0)$. The first term on the right-hand side of the equation describes jumps away from the value β and the second term gives the integral over all previous values β'. Equations (2.2) are equivalent in this case to the following normalization and stationarity conditions on $h(\beta',\beta)$:

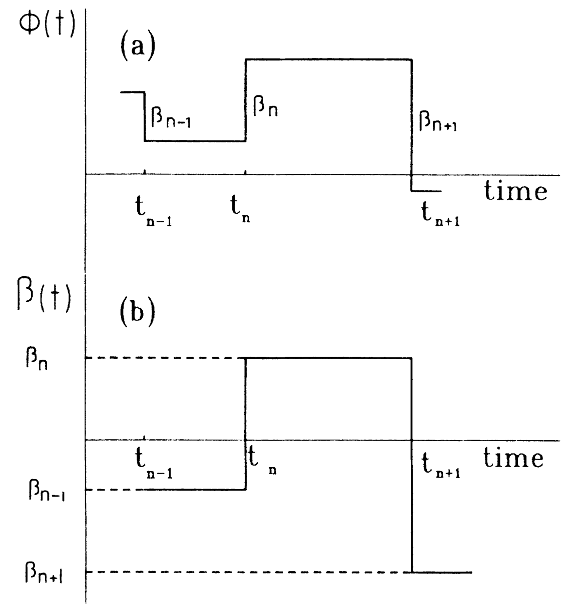

FIG. 1. A schematic drawing of the phase jump process as a function of time showing (a) the phase $\phi(t)$ and (b) the corresponding phase jumps $\beta(t)$.

$$\int h(\beta',\beta)d\beta=1 , \tag{2.5a}$$

$$\int\frac{f(\beta')}{\tau_0(\beta')}h(\beta',\beta)d\beta'=\frac{f(\beta)}{\tau_0(\beta)} . \tag{2.5b}$$

While the phase $\phi(t)$ is a non-Markovian random variable, as it depends on the previous jump, the vector $V(t)=[\phi(t),\beta(t)]$ can be shown to be a jump Markovian vector. More specifically, one can write the Kolmogorov-Feller equation for the conditional probability $f(\phi_0,\beta_0;\phi,\beta,t)$:

$$\frac{\partial f}{\partial t}=\frac{-f}{\tau_0(\beta)}+\int f(\phi_0,\beta_0;\phi-\beta,\beta',t)\frac{h(\beta',\beta)}{\tau_0(\beta')}d\beta'\equiv L_V f \tag{2.6}$$

with the initial condition $f(\phi_0,\beta_0;\phi,\beta,0)=\delta(\phi-\phi_0)\delta(\beta-\beta_0)$. The field is characterized by the normalized correlation function

$$k(t)=\langle e^{-i[\phi(t)-\phi_0]}\rangle \tag{2.7}$$

and by the normalized intensity spectrum

$$J(\omega-\omega_L)=\frac{1}{\pi}\text{Re}\int_0^\infty k(t)e^{i(\omega-\omega_L)t}dt . \tag{2.8}$$

Equation (2.7) can be written as

$$k(t)=\int d\beta_0 d\phi\, d\beta\, f(\beta_0)f(\phi_0,\beta_0;\phi,\beta,t)e^{-i(\phi-\phi_0)}$$
$$=\int d\beta\, r(\beta,t) . \tag{2.9}$$

From the Kolmogorov-Feller equation (2.6) one derives an integro-differential equation for the partially averaged correlation function $r(\beta,t)$:

$$\dot{r}=-\frac{r}{\tau_0(\beta)}+e^{-i\beta}\int r(\beta',t)\frac{h(\beta',\beta)}{\tau_0(\beta')}d\beta' \quad (2.10)$$

with the initial condition $r(\beta,0)=f(\beta)$. Equation (2.10) can be cast as

$$\dot{r}=(B_\beta+L_\beta)r \quad (2.11)$$

where L_β is defined by Eq. (2.4) and

$$B_\beta r(\beta,t)=(e^{-i\beta}-1)\int\frac{h(\beta',\beta)}{\tau_0(\beta')}r(\beta',t)d\beta' . \quad (2.12)$$

We assume that $f(\beta)$ and $\tau_0(\beta)$ are even functions of β, and the function $f(\beta)$ is fully characterized by its even moments: $\langle\beta^{2k}\rangle\sim B^{2k}$ where B characterizes a jump size. In particular $\langle\beta^2\rangle=B^2$. Unless otherwise specified, all the results obtained below are valid for arbitrary distribution functions $f(\beta)$ and $h(\beta',\beta)$ which satisfy these conditions together with the stationarity condition (2.5b). In a previous paper[10] we have discussed the Kielson-Storer model, which is also used here as an illustrative example that satisfies the general conditions. As is clear from Eq. (2.8), the spectrum is determined by the correlation function $k(t)$. This function has a different functional dependence in the different limits, and in Sec. IV analytical solutions are obtained for $k(t)$ for all the important limiting cases.

III. CHARACTERIZATION OF THE JUMP PROCESS

The discussion of a stochastic process of the kind being considered here is quite complicated, and involves the introduction of many parameters which take different values in the various limits. Thus, the mean jump size, the mean time between jumps, the degree of correlation between successive jumps, or the parameters defining boundaries between different limiting cases need all be defined.

The first quantity of physical significance is the weighted average of the mean time between jumps τ_{av}. The mean time between jumps $\tau_0(\beta)$ introduced in Eq. (2.5) may depend, in general, on the current value of β. The weighted average of these times is

$$(\tau_{av})^{-1}=B^{-2}\int\frac{\beta^2 f(\beta)}{\tau_0(\beta)}d\beta . \quad (3.1)$$

For a constant $\tau_0(\beta)$, $\tau_{av}=\tau_0$.

A second quantity of physical interest is $\epsilon(t)$, the fluctuating frequency deviation averaged over the time $\tau_0(\beta)$: $\epsilon(t)=\beta(t)/\tau_0[\beta(t)]$. The quantity $\epsilon(t)$ has the dimension of frequency, and it is almost the instantaneous frequency deviation. (Almost, because strictly speaking, in this model the frequency fluctuations are δ functions.) The autocorrelation function of this quantity is given by

$$k_\epsilon(t)=\langle\epsilon(0)\epsilon(t)\rangle$$
$$=\int\int d\beta_0 d\beta\,\epsilon(\beta_0)\epsilon(\beta)f(\beta_0)f(\beta_0;\beta,t) \quad (3.2)$$

where $\epsilon(\beta)=\beta/\tau_0(\beta)$. If $\tau_0(\beta)$ were constant $k_\epsilon(t)$ would be proportional to the autocorrelation function of the jumps $\langle\beta(0)\beta(t)\rangle$.

Another very important time scale is ν_β^{-1}, the jump process memory time, given by the correlation time of the fluctuating frequency deviation $\epsilon(t)$,

$$(\nu_\beta)^{-1}=B^{-2}\tau_{av}^2\int_0^\infty k_\epsilon(t)dt . \quad (3.3)$$

For constant $\tau_0(\beta)$, ν_β^{-1} is the correlation time of the process $\beta(t)$.

For the case of uncorrelated jumps [$h(\beta',\beta)$ independent of β'], the "jump process memory time" ν_β^{-1} is τ_{av} (Sec. IV A). In general, however, the jump memory time may be longer or shorter than τ_{av}. A (positively) correlated process is one where the next jump is likely to be of the same sign as the previous one, and an anticorrelated process is the opposite. This intuitive description leads to the generalized definition of the correlation parameter γ:

$$\gamma=1-\nu_\beta\tau_{av} \quad (3.4)$$

and the generalized definitions of positive and negative correlations:

$\gamma=0$, uncorrelated ,

$\gamma>0$, (positively) correlated ,

$\gamma<0$, anticorrelated .

In order to clarify these somewhat abstract definitions, consider the well-known Kielson-Storer model,[57] introduced first for velocity changing collisions. In this model $\tau_0(\beta)=\tau_0=$const, and the distribution functions are given by

$$f(\beta)=\frac{1}{(2\pi B^2)^{1/2}}\exp\left[-\frac{\beta^2}{2B^2}\right] , \quad (3.5)$$

$$h(\beta',\beta)=h(\beta-\gamma\beta')$$
$$=\frac{1}{[2\pi(1-\gamma^2)B^2]^{1/2}}\times\exp\left[-\frac{(\beta-\gamma\beta')^2}{2(1-\gamma^2)B^2}\right], \quad (-1\le\gamma\le 1) . \quad (3.6)$$

Strictly speaking, a different symbol should have been used for the γ in the above equation, but it can be shown that the parameter γ in the KSM satisfies the definition of the generalized correlation parameter γ, and thus the same symbol is used for both. By inspection, for $\gamma=0$, the KSM function $h(\beta',\beta)$ is independent of β', and the jumps are uncorrelated. The average value of β after the jump β' is given by $\gamma\beta'$, leading to the obvious meaning of positive and negative correlation.

The characteristic jump size B, the correlation parameter γ and the weighted average time τ_{av} describe the main features of the stochastic field. Of these, τ_{av} is a (time) scaling parameter, while B and γ determine the other parameters of the problem in terms of τ_{av}. In the general case of $\tau_0(\beta)$ not being a constant, the details of this function are needed to fully characterize the field.

Since there are many formulas in this paper, Table I lists all of the important mathematical expressions with their definitions, a brief description of their physical significance, and the place in the paper where first introduced.

IV. FIELD ANALYSIS

A very convenient way to consider the various cases discussed in this paper is to plot the phase-space diagram for the parameters B and γ. The range of these parameters are $0<B<\infty$ and $-1\le\gamma\le 1$. Figure 2 depicts the phase space, and the different regions are marked. They are labeled according to the letter denoting the corresponding subsection.

Before discussing the specifics of the various regions, a general statement should be made regarding the short-time asymptotic behavior of the correlation function. In all regions of parameter space, for short times $t \ll \tau_{av}$, $r(\beta,t)\approx r(\beta,0)=f(\beta)$. Inserting this equation into the right-hand side of Eqs. (2.10), considering Eq. (2.5b), and integrating both sides of the resulting equation over β yields the following short-time asymptotic form of the correlation function:

$$k(t)=1-\nu_1 t\ , \tag{4.1}$$

where

$$\nu_1=\int \frac{1-e^{-i\beta}}{\tau_0(\beta)} f(\beta)d\beta\ . \tag{4.2}$$

The far wings fall off quadratically[50] with $\omega-\omega_L$,

$$J(\omega-\omega_L)=\frac{\nu_1}{\pi(\omega-\omega_L)^2}\ . \tag{4.3}$$

The value of the far wing width parameter ν_1 depends on the particular region, but Eqs. (4.1) and (4.3) are valid in all regions. For example,

TABLE I. A description of the important parameters used in the paper, a general formula, and the place in the paper where first mentioned.

Symbol	Parameter description	General formula	First introduced
ω_L	The laser field nominal frequency		Eq. (1.1)
$\phi(t)$	The fluctuating phase of the field		Eq. (1.1)
$\beta(t)$	The stepwise random function equal to the latest phase jump		Fig. 1
$f(\beta)$	The unconditional distribution of β		After Eq. (1.1)
$h(\beta',\beta)$	The conditional distribution of β		After Eq. (1.1)
$\tau_0(\beta)$	The mean time between the phase jumps		Before Eq. (2.3)
B	A characteristic width of $f(\beta)$	$(\langle\beta^2\rangle)^{1/2}$	After Eq. (2.12)
τ_{av}	A weighted average of the time between jumps	$B^2/\langle\beta^2/\tau_0(\beta)\rangle$	Eq. (3.1)
$\epsilon(t)$	The effective frequency deviation	$\beta(t)/\tau_0(\beta(t))$	Before Eq. (3.2)
$k_\epsilon(t)$	The correlation function of the random process $\epsilon(t)$	$\langle\epsilon(0)\epsilon(t)\rangle$	Eq. (3.2)
ν_β	The reciprocal jump process memory time	$(B/\tau_{av})^2/\int_0^\infty k_\epsilon(t)dt$	Eq. (3.3)
γ	The correlation parameter $(-1\le\gamma\le 1)$	$1-\nu_\beta\tau_{av}$	Eq. (3.4)
$k(t)$	The laser field correlation function	$\langle e^{-[\phi(t)-\phi_0]}\rangle$	Eq. (2.7)
$J(\omega)$	The laser field intensity spectrum	$\frac{1}{\pi}\mathrm{Re}\int_0^\infty k(t)e^{i(\omega-\omega_L)t}dt$	Eq. (2.8)
$r(\beta,t)$	The partially averaged field correlation function		Eq. (2.9)
L_β	The integral operator entering the Kolmogorov-Feller equation	$-\frac{1}{\tau_0(\beta)}+\int d\beta' \frac{h(\beta',\beta)}{\tau_0(\beta')}$	Eq. (2.4)
B_β	The integral operator	$(e^{-i\beta}-1)\int d\beta' \frac{h(\beta',\beta)}{\tau_0(\beta')}$	Eq. (2.12)
$\bar{\tau}$	A characteristic time between the jumps	$\left[\int d\beta\, f(\beta)/\tau_0(\beta)\right]^{-1}$	Eq. (4.4b)
ν_1	The short-time damping rate	$\int d\beta\, f(\beta)\frac{1-e^{-i\beta}}{\tau_0(\beta)}$	Eqs. (4.2), (4.4)
ν	The reciprocal field correlation time for $B^2\ll(1-\gamma)^2$	$\frac{1+\gamma}{1-\gamma}\frac{B^2}{2\tau_{av}}$	Eq. (4.22)
ν_a	A characteristic long-time decay rate of $k(t)$ for $1+\gamma\ll 1$ (in the KSM case)	$\frac{(1+\gamma)B^2}{4\tau_0}$	After Eq. (4.61)

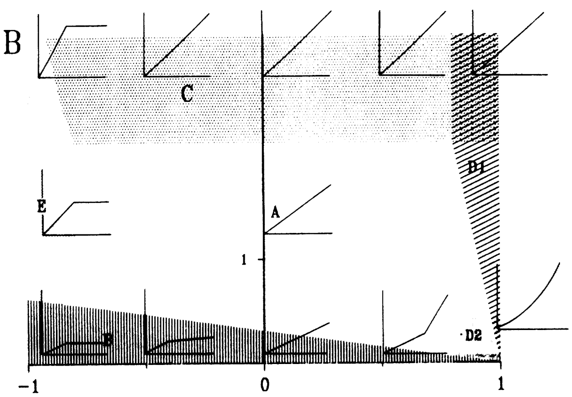

FIG. 2. (B,γ) phase space, the regions where specific solutions are obtained, are shaded and marked by the letter corresponding to that region. The plots describe schematically the negative logarithm of the correlation functions against time delay.

$$\nu_1 = \begin{cases} B^2/2\tau_{av}, \quad B^2 \ll 1 & (4.4a) \\ \int d\beta f(\beta)/\tau_0(\beta) \equiv 1/\bar{\tau}, \quad B^2 \gg 1 \ . & (4.4b) \end{cases}$$

In what follows, various regions of the B,γ parameter space will be explored, resulting in specific predictions for correlation functions and spectra.

A. Uncorrelated jumps

If successive values of $\beta(t)$ are uncorrelated, $h(\beta',\beta)=h(\beta)$ is independent of β'. In this case (the kangaroo process[58]) Eq. (2.5b) yields

$$h(\beta)=f(\beta)\bar{\tau}/\tau_0(\beta) \ . \tag{4.5}$$

Inserting $h(\beta)$ into the Kolmogorov-Feller equation (2.4), one can obtain the Laplace transform of the conditional probability

$$F(\beta_0,\beta) \equiv \int_0^\infty f(\beta_0;\beta,t)e^{-st}dt$$
$$= \frac{1}{s+\tau_0^{-1}(\beta_0)} \left[\delta(\beta-\beta_0) + \frac{f(\beta)}{s\tau_0(\beta_0)[s\tau_0(\beta)+1]} \left[\int d\beta \frac{f(\beta)}{s\tau_0(\beta)+1} \right]^{-1} \right] \ . \tag{4.6}$$

Using this expression in the Laplace transform of Eq. (3.2) and taking the inverse transform of the result yields

$$k_\epsilon(t) = \int d\beta f(\beta)[\beta/\tau_0(\beta)]^2 e^{-t/\tau_0(\beta)} \ . \tag{4.7}$$

By inserting Eq. (4.5) into Eq. (2.10) one can obtain the Laplace transform of the correlation function of the field (2.9) for the kangaroo process

$$K(s) = \int_0^\infty k(t)e^{-st}dt = J_{00} + \bar{\tau}J_{01}J_{11}/(1-\bar{\tau}J_{12}) \tag{4.8}$$

where

$$J_{mn} = \int d\beta \frac{f(\beta)e^{-im\beta}}{[\tau_0(\beta)]^n[s+\tau_0^{-1}(\beta)]} \ . \tag{4.9}$$

Equation (4.8) directly yields the spectrum $J(\omega-\omega_L)=\pi^{-1}\mathrm{Re}K(-i(\omega-\omega_L))$ and allows one to get, by means of the inverse Laplace transform, the correlation function $k(t)$. We note that the correlation function $k(t)$ is nonexponential unless $\tau_0(\beta)$ is a constant τ_0. For that special case treated by Burshtein $k(t)=e^{-\nu_1 t}$ and the spectrum will be a Lorentzian with a half width at half maximum (HWHM) of ν_1. Specifically, for a Gaussian distribution $f(\beta)$, the spectral width of the field is given by

$$\nu_1 = [1-\exp(-B^2/2)]/\tau_0 \ .$$

The most interesting result of this section is obtained by using Eq. (4.7) in the definition of the jump process memory ν_β Eq. (3.3). In this case, one obtains the surprising result $\nu_\beta\tau_{av}=1$, namely, $\gamma=0$ for arbitrary functional forms of $\tau_0(\beta)$ and $f(\beta)$. *A priori* one might have expected that correlations may be introduced either through the conditional probability $h(\beta',\beta)$ or through a nonconstant $\tau_0(\beta)$, but this result indicates that correlations are introduced only through the conditional probability.

B. The Born approximation (small jumps)

When dealing with a phase fluctuating field, one may, for all practical purposes, always approximate a continuously varying phase by a succession of many small jumps, whether correlated or not. Thus, a good point to start a discussion of the generalized jump process is the analysis of the small-jump limit.

In this section, the GJM is analyzed in the region of the (B,γ) space where the PDM proves to be approximately valid. The diffusion coefficient is calculated and the results for the entire time and frequency domains (i.e., beyond the limits of the PDM) are obtained. For $B^2 \ll (1-\gamma)^2$ a generalized cumulant expression was used to solve Eq. (2.11). The details of the method are described in Appendix A, and the validity condition is derived in Appendix B.

In Appendix A the following equation is derived:

$$\dot{k} = \left[-\nu_1 + \int_0^t \theta(t')dt' \right] k \tag{4.10}$$

where

$$\theta(t) = \int \int d\beta \, d\beta' b(\beta')f(\beta')B_\beta f(\beta';\beta,t) - \nu_1^2 \tag{4.11}$$

and $b(\beta)=(e^{-i\beta}-1)/\tau_0(\beta)$. As follows from Eq. (2.4)

$$B_\beta f(\beta';\beta,t)=(e^{-i\beta}-1)[\dot f(\beta';\beta,t)+f(\beta';\beta,t)/\tau_0(\beta)] \ . \quad (4.12)$$

Combining Eq. (4.11) and Eq. (4.12) yields

$$\theta(t)=-\dot k_1(t)-k_2(t)-\nu_1^2 \ , \quad (4.13)$$

where

$$k_1(t)=-\int\int d\beta_0 d\beta\, b(\beta_0)(e^{-i\beta}-1)f(\beta_0)f(\beta_0;\beta,t) \ , \quad (4.14)$$

$$k_2(t)=-\int\int d\beta_0 d\beta\, b(\beta_0)b(\beta)f(\beta_0)f(\beta_0;\beta,t) \ . \quad (4.15)$$

The condition $B^2 \ll (1-\gamma)^2$ means that the phase jumps are small, simplifying Eqs. (4.2), (4.14), and (4.15) to yield $\nu_1=B^2/2\tau_{\rm av}$,

$$k_1(t)=\langle \epsilon(0)\beta(t)\rangle \ , \quad (4.16)$$

and $k_2(t)=\langle\epsilon(0)\epsilon(t)\rangle=k_\epsilon(t)$. The third term on the right-hand side of Eq. (4.13) is of fourth order in B exceeding the accuracy of $k_1(t)$ and $k_2(t)$ which are of order B^2. Thus, we may approximate $\theta(t)$ by $\theta_0(t)$ where

$$\theta_0(t)=-\dot k_1(t)-k_\epsilon(t) \ . \quad (4.17)$$

The solution of Eq. (4.10) now yields the correlation function of the field in the form

$$k(t)=\exp\left[-\nu_1 t+\int_0^t dt(t-t')\theta_0(t')dt'\right] \ . \quad (4.18)$$

This equation can be rewritten in another form which shows more explicitly the long-time asymptotics,

$$k(t)=\exp[-\nu t+q(t)-q(0)] \ . \quad (4.19)$$

Here

$$\nu=\nu_1-\int_0^\infty \theta_0(t)dt=\int_0^\infty k_\epsilon(t)dt-\nu_1 \ , \quad (4.20)$$

$$q(t)=\int_t^\infty (t'-t)\theta_0(t')dt' \ . \quad (4.21)$$

Using the definition of γ, and after algebraic simplification, Eq. (4.20) can be written as

$$\nu=\frac{1+\gamma}{1-\gamma}\frac{B^2}{2\tau_{\rm av}}=\frac{1+\gamma}{1-\gamma}\nu_1 \ . \quad (4.22)$$

We note that $q(t)\sim B^2/(1-\gamma)^2$ and is therefore a small correction. Expanding Eq. (4.19) yields

$$k(t)=e^{-\nu t}[1-q(0)]+q(t) \ . \quad (4.23)$$

The function $q(t)$ decays on a time scale ν_β^{-1} where $\nu_\beta \gg \nu$. Therefore, for $\nu_\beta t \gg 1$, $k(t)$ is approximately $e^{-\nu t}$. Thus for long times, the GJM is equivalent to a phase-diffusion model in which the conditional probability for the phase obeys a diffusion equation with the diffusion constant ν, which is a function of γ.

For the uncorrelated case ($\gamma=0$) one can show with the help of Eq. (4.6) that $\theta_0(t)=0$. Hence, Eq. (4.10) yields an exponential correlation function $[k(t)=e^{-\nu_1 t}]$ for all times, irrespective of the functional form of $\tau_0(\beta)$. Thus, for $\gamma=0$ and $B^2 \ll 1$, the standard PDM holds.

The easiest way to obtain the power spectrum of the field is to use the convolution form of the truncated cumulant expression (see Appendix A). The relevant equation is

$$\dot k=-\nu_1 k+\int_0^t \theta_0(t-t')k(t')dt' \ . \quad (4.24)$$

Using Laplace transform, and neglecting higher-order terms, one can show that

$$J(\omega-\omega_L)=\frac{1}{\pi}\frac{\nu_1+\mu(\omega-\omega_L)}{(\omega-\omega_L)^2+\nu^2} \quad (4.25)$$

where

$$\mu(\omega)=-\int_0^\infty \theta_0(t)\cos(\omega t)dt \ . \quad (4.26)$$

For the central portion of the spectrum $|\omega-\omega_L| \ll \nu_\beta$, $\mu(\omega-\omega_L)\approx\mu(0)=\nu_1-\nu$ and the spectrum looks like a Lorentzian with width ν. Since $\nu\tau_{\rm av}\sim B^2 \ll 1$ this part of the spectrum accounts for almost the entire intensity of the field. The far wings $|\omega-\omega_L| \gg \nu_\beta$ decay like a Lorentzian of width ν_1, as expected from the short-time asymptotes.

Rearranging Eq. (4.22) one obtains an expression for γ

$$\gamma=(\nu-\nu_1)/(\nu+\nu_1) \ . \quad (4.27)$$

Since both ν and ν_1 are non-negative quantities, the correlation parameter obeys $-1\le\gamma\le 1$.

Summarizing the above results we note that in the Born approximation, the behavior of the correlation function for short and long times and the shapes of the central and peripheral parts of the spectrum are independent of the details of the statistics of the underlying noise process $\beta(t)$, and are characterized by the two parameters ν [Eq. (4.22)] and ν_1 [Eq. (4.4a)]. Note also that these results are valid even though $\tau_0(\beta)$ is not a constant.

The power of the technique may be illustrated using the Kielson-Storer model, where many of the functions can be calculated explicitly. In this model, a constant $\tau_0(\beta)=\tau_0$ is assumed. Multiplying both sides of Eq. (2.4) by $(\beta_0\beta/\tau_0^2)$, integrating over β_0, β results in an expression for $k_\epsilon(t)$:

$$k_\epsilon(T)=(B/\tau_0)^2 e^{-\nu_\beta t}, \quad \nu_\beta=(1-\gamma)/\tau_0 \ . \quad (4.28)$$

Using a constant $\tau_0(\beta)$ in Eq. (4.17) yields

$$\theta_0(t)=-\frac{\gamma B^2}{\tau_0^2}e^{-\nu_\beta t} \ . \quad (4.29)$$

From Eq. (4.19)

$$k(t)=\exp[-\nu_1 t-(1/\nu_\beta)(\nu-\nu_1)\times(\nu_\beta t-1+e^{-\nu_\beta t})] \quad (4.30)$$

while from Eq. (4.23) one obtains an alternative result,[10]

$$k(t)=\left[1+\frac{\gamma B^2}{(1-\gamma)^2}\right]e^{-\nu t}-\frac{\gamma B^2}{(1-\gamma)^2}e^{-\nu_\beta t} \ . \quad (4.31)$$

The spectrum of the field is

$$J(\omega-\omega_L)=\frac{1}{\pi}\frac{\nu_1+(\nu-\nu_1)[(\omega-\omega_L)^2/\nu_\beta^2+1]^{-1}}{(\omega-\omega_L)^2+\nu^2} \quad (4.32)$$

C. Large jumps $[B^2\gg 1/(1+\gamma)]$

For very large jumps $[B^2\gg 1/(1+\gamma)]$ each jump destroys the field coherence irrespective of the degree of correlation between the jumps. Indeed, since only relative phases are important [cf. Eq. (2.10)] and the phase is measured modulo 2π, for very large jumps one intuitively expects that it is immaterial whether the current jump is or is not correlated to the previous jump. In this case one can neglect the second term on the right-hand side of Eq. (2.10), yielding

$$k(t)=\int \exp[-t/\tau_0(\beta)]f(\beta)d\beta \ . \quad (4.33)$$

The correlation function is generally nonexponential with a monotonically decreasing decay rate. The short-time asymptotes described above are valid for $t\ll\bar{\tau}$ with $\nu_1=1/\bar{\tau}$ [cf. Eq. (4.4b)]. In the case $\tau_0(\beta)=\tau_0=\text{const}$, however, the function (4.33) is exponential:

$$k(t)=e^{-t/\tau_0} , \quad (4.34)$$

yielding a Lorentzian spectrum with the HWHM width τ_0^{-1}. This result agrees with the case of uncorrelated jumps where $\nu_1=\tau_0^{-1}$ for $B\gg 1$.

D. Highly correlated jumps $(\gamma\approx 1)$

The case when successive phase jumps differ insignificantly from each other $(\gamma\approx 1)$ is designated as the highly correlated jumps region. The derivation of the correlation function splits into two overlapping cases: (i) the slowly varying jump case, where β is approximately constant for each stochastic realization, and (ii) the frequency fluctuation case, where $B\ll 1$. In both cases, one can consider the width of $h(\beta',\beta)$ as a function of β' to be much smaller than B.

1. The slowly varying jump region

In this region $h(\beta',\beta)$ is approximated by $\delta(\beta-\beta')$ [i.e., the phase jump $\beta(t)$ is constant] in both terms of Eq. (2.14) yielding

$$k(t)=\int d\beta f(\beta)\exp[-(1-e^{-i\beta})t/\tau_0(\beta)] \ . \quad (4.35)$$

For $B\gg 1$, Eq. (4.34) is regained. For $B\sim 1$, no explicit expressions for $k(t)$ have been found, and the integral must be evaluated numerically. The integral (4.35) can be substituted by a series which is more convenient for numerical calculations, e.g., for constant $\tau_0(\beta)=\tau_0$:

$$k(t)=e^{-t/\tau_0}\sum_{n=0}^{\infty}(m_n/n!)(t/\tau_0)^n \quad (4.36)$$

where $m_n=\int d\beta f(\beta)e^{-in\beta}$. For the Gaussian $f(\beta)$ [Eq. (3.5)]

$$m_n=\exp(-n^2B^2/2) \ . \quad (4.37)$$

For $B\ll 1$, approximate solutions are found in the discussion of the frequency fluctuation region.

2. The frequency fluctuation region (generalized Kubo oscillator)

When successive phase jumps are both small and highly correlated, the phase $\phi(t)$ may be approximated by a continuous stochastic process. A good approximation for $\phi(t)$ is then

$$\phi(t)=\int_0^t \epsilon(t')dt'+\phi_0 \quad (4.38)$$

where the fluctuating frequency deviation $\epsilon(t)=\beta(t)/\tau_0[\beta(t)]$ is Markovian. The quality of this approximation is not *a priori* known, but the problem is now equivalent to the case of the familiar Kubo oscillator.[51,52] One can calculate $k(t)$ using Kubo's treatment and obtain the equation

$$\dot{r}=-i\epsilon(\beta)r+L_\beta r \ . \quad (4.39)$$

A more rigorous derivation of Eq. (4.39) can be obtained from Eq. (2.11) by replacing $h(\beta',\beta)$ by $\delta(\beta'-\beta)$ in the operator B_β resulting in

$$\dot{r}=[L_\beta+(e^{-i\beta}-1)/\tau_0(\beta)]r \ . \quad (4.40)$$

When $B^2\ll 1$, the exponential can be expanded, dropping terms of order β^2 and higher to obtain Eq. (4.39). In this formulation, higher-order terms may be included to generalize the Kubo result. Note that this approximation may be inaccurate for very short times $[t\le\tau_{av}$, when $k(t)\approx 1]$ and for very long times $[t\sim\tau_{av}/B^2$ when $k(t)\approx 0]$. It should be stressed that the random process $\epsilon(t)$ in Eq. (4.38) cannot be arbitrary, but approximates a Markovian continuous nondifferentiable process.

The generalized cumulant expansion (Appendix A) truncated after second order can be used to solve Eq. (4.39). In the long-time limit $(\nu_\beta t\gg 1)$ $k(t)$ decays exponentially[50,51,52] with a damping rate $\nu=\int_0^\infty k_\epsilon(t)dt$, regaining the "motional narrowing limit" of the Kubo oscillator. The validity condition for this Born approximation is $\nu\ll\nu_\beta$ or $B^2\ll(1-\gamma)^2$.

For the case where $1-\gamma\ll B\ll 1$, (corresponding to Kubo's quasistatic limit), the slowly varying jump limit applies, and the L_β term in Eq. (4.39) can be neglected to obtain

$$k(t)=\int e^{-i\epsilon(\beta)t}f(\beta)d\beta \quad (4.41)$$

where $\epsilon(\beta)=\beta/\tau_0(\beta)$. The field spectrum is

$$J(\omega-\omega_L)=\tilde{f}(\omega-\omega_L) \quad (4.42)$$

where $\tilde{f}(\epsilon)$ is the distribution function of the frequency deviation ϵ:

$$\tilde{f}(\epsilon)=f(\beta(\epsilon))\frac{d\beta(\epsilon)}{d\epsilon} \ . \quad (4.43)$$

Here $\beta(\epsilon)$ is the inverse function of $\epsilon(\beta)$. The spectrum now mimics $\tilde{f}(\epsilon)$ and is, thus, inhomogeneously broadened. In particular, for $\tau_0(\beta)=\tau_0$, $J(\omega-\omega_L)=\tau_0 f((\omega-\omega_L)\tau_0)$. Thus, the experimental determination of the spectrum yields direct information on the probability density $f(\beta)$.

A more detailed description of the correlation function and the spectrum of the laser field can be obtained for KSM. In this case Eq. (4.39) takes the form[59]

$$\dot{r}=\frac{-i\beta r}{\tau_0}+\nu_\beta\left[r+\beta\frac{\partial r}{\partial\beta}+B^2\frac{\partial^2 r}{\partial\beta^2}\right] . \quad (4.44)$$

The solution of Eq. (4.44) yields[50,59]

$$k(t)=\exp\left[-\frac{B^2}{(1-\gamma)^2}(\nu_\beta t-1+e^{-\nu_\beta t})\right] . \quad (4.45)$$

We note that except for very short times ($t\leq\tau_0$) when $\nu_1 t$ cannot be neglected, this result is the same as the expression derived in the Born approximation Eq. (4.30). For $B\ll 1-\gamma$, expanding $\exp(-\nu_\beta t)$ in (4.45) in powers of t yields

$$k(t)\approx\exp(-B^2t^2/2\tau_0^2) \quad [\tau_0\ll t\ll\tau_0/(1-\gamma),\ \tau_0/B^2] \quad (4.46)$$

which coincides with the slowly varying jump approximation, Eq. (4.41).

E. Highly anticorrelated jumps ($\gamma\approx-1$)

Consider the case where for each realization of the stochastic process $\beta(t)$ the absolute values of successive phase jumps are nearly constant with alternating signs. This region is designated as the highly anticorrelated jumps region. In this region, the function $h(\beta',\beta)$ is sharply peaked around the value $\beta=-\beta'$. In the treatment of this case we first consider completely anticorrelated jumps, where $\gamma=-1$ and $h(\beta',\beta)=\delta(\beta'+\beta)$, and then allow for incomplete anticorrelations where $\gamma\approx-1$.

1. Fully anticorrelated jumps (generalized telegraph noise)

Under conditions of full anticorrelation, each realization of the stochastic field is a random telegraph signal, where the phase jumps between the values of $\pm\beta$. Equation (2.10), reduces to a system of two equations for $r(\pm\beta,t)$,

$$\dot{r}(\beta,t)=-[r(\beta,t)-e^{-i\beta}r(-\beta,t)]/\tau_0(\beta) \quad (4.47)$$

and the equation which is obtained from Eq. (4.47) by the substitution $\beta\leftrightarrow-\beta$. The solution is

$$r(\beta,t)=(\tfrac{1}{2})f(\beta)[1+e^{-i\beta}+(1-e^{-i\beta})e^{-2t/\tau_0(\beta)}] . \quad (4.48)$$

Integration of this expression over β yields

$$k(t)=\int[e^{-2t/\tau_0(\beta)}\sin^2(\tfrac{1}{2}\beta)+\cos^2(\tfrac{1}{2}\beta)]f(\beta)d\beta . \quad (4.49)$$

Equation (4.49) shows that $\lim_{t\to\infty}k(t)$ is nonzero. Correspondingly, the spectrum consists of a δ function and a nonsingular component,

$$J(\omega-\omega_L)=\langle\cos^2(\tfrac{1}{2}\beta)\rangle\delta(\omega-\omega_L)+\frac{1}{2\pi}\int d\beta\frac{\sin^2(\frac{1}{2}\beta)\tau_0(\beta)f(\beta)}{[(\omega-\omega_L)\tau_0(\beta)/2]^2+1} . \quad (4.50)$$

Although the spectrum is clearly non-Lorentzian, the wings of the spectrum ($|\omega-\omega_L|\tau_{av}\gg1$) fall off as $\nu_1/[\pi(\omega-\omega_L)^2]$. For the case $\tau_0(\beta)=\tau_0=\text{const}$, the second term is a Lorentzian of the width $2/\tau_0$, but the δ function remains.

The usual telegraph model[5] follows from this result if only one value of $|\beta|$ is realized:

$$f(\beta)=\tfrac{1}{2}[\delta(\beta-\beta_0)+\delta(\beta+\beta_0)] . \quad (4.51)$$

In the generalized telegraph-noise model, however, each individual realization of the stochastic process is a telegraph-noise process, and the average is over an ensemble of such processes. In this paper, the distribution function $f(\beta)$ is characterized by a single parameter B and as defined above is an even function of β with values centered and continuously spread about $\beta=0$ [e.g., the case of a Gaussian distribution $f(\beta)$]. For $\tau_0(\beta)=\tau_0=\text{const}$ Eq. (4.49) simplifies to

$$k(t)=\langle\sin^2(\tfrac{1}{2}\beta)\rangle e^{-2t/\tau_0}+\langle\cos^2(\tfrac{1}{2}\beta)\rangle \quad (4.52)$$

or in the case of a Gaussian distribution $f(\beta)$

$$k(t)=e^{-B^2/4}[e^{-2t/\tau_0}\sinh(\tfrac{1}{4}B^2)+\cosh(\tfrac{1}{4}B^2)] \quad (4.53)$$

with a spectrum

$$J(\omega-\omega_L)=e^{-B^2/4}\left[\cosh(\tfrac{1}{4}B^2)\delta(\omega-\omega_L)+\sinh(\tfrac{1}{4}B^2)\frac{2/(\pi\tau_0)}{(\omega-\omega_L)^2+(2/\tau_0)^2}\right] . \quad (4.54)$$

The singularity in the spectrum can be understood in the following intuitive sense. A vanishing correlation function in the long time limit means that the initial conditions for the noise process are immaterial, which is clearly not the case for the telegraph noise. Here, for each realization of the stochastic process, even after infinitely long time, the phase is either equal to the initial value or minus this value. Thus, the complex field amplitude has a constant component which is symmetric in β, resulting in a singular (monochromatic) contribution to the spectrum.

Incomplete anticorrelation results in jump size $|\beta|$ that fluctuates in time in each realization of field, which changes the situation qualitatively: the field phase is now not confined to two values, and can diffuse. As a result of this, now $\lim_{t\to\infty}k(t)=0$ and the field spectrum becomes nonsingular. This situation is discussed in Sec. IV E 2.

2. Incomplete anticorrelation ($0<1+\gamma\ll1$)

Incomplete anticorrelation means $\gamma\approx-1$ and $h(\beta',\beta)$ is only approximately a δ function. In this case, Eq. (2.10) can be reduced to a form reminiscent of the

Fokker-Planck equation. We do that by a method that is an extension of that used in Ref. 48 to derive the Fokker-Planck equation. We rewrite $h(\beta',\beta)$ as $h(\beta';\alpha)$ where $\alpha=\beta+\beta'$, and consider $h(\beta';\alpha)$ to be a sharply peaked function of α, but a slowly varying function of β'. We also assume that $\tau_0(\beta)$ and $r(\beta,t)$ are slowly varying functions of β, and expand the integrand in Eq. (2.10) in a Taylor series around $\beta'=-\beta$. Neglecting third- and higher-order terms yields the two coupled equations

$$\dot{r}_+=-\frac{r_+-e^{-i\beta}r_-}{\tau_0(\beta)}+e^{-i\beta}L_-r_- , \tag{4.55a}$$

$$\dot{r}_-=-\frac{r_--e^{-i\beta}r_+}{\tau_0(\beta)}+e^{i\beta}L_+r_+ \tag{4.55b}$$

with $r_\pm=r(\pm\beta,t)$,

$$L_\pm r_\pm=\pm\frac{\partial}{\partial\beta}[a_1(\pm\beta)r_\pm]+\frac{\partial^2}{\partial\beta^2}[a_2(\pm\beta)r_\pm] , \tag{4.56}$$

$$a_n(\beta)=\{1/[n!\tau_0(\beta)]\}\int d\alpha\,\alpha^n h(\beta;\alpha) . \tag{4.57}$$

For $\gamma=-1$, $L_\pm=0$ and Eq. (4.55a) reduces to Eq. (4.47). If γ is close enough to -1 (the exact condition is defined below), the last terms on the right-hand sides of Eqs. (4.55) are much less than other terms and can be neglected in the interval $0<t\lesssim\tau_{av}$. In this interval the solution (4.49) is recovered.

Consider the long-time asymptotics $t\gg\tau_{av}$. One can rewrite (4.55b) as $r_+=e^{-i\beta}r_-+\tau_0(\beta)e^{-i\beta}\dot{r}_-+\tau_0(\beta)L_+r_+$. The last two terms are of order $(1+\gamma)B^2/4$, and can be neglected for γ sufficiently close to -1. Under these conditions $r_+\approx e^{-i\beta}r_-$. Multiplying both sides of Eq. (4.55b) by $e^{-i\beta}$ and summing it with Eq. (4.55a) yields

$$\dot{r}_+=\tfrac{1}{2}(e^{-i\beta}L_+e^{i\beta}r_++L_+r_+) . \tag{4.58}$$

Here we assumed that $h(\beta',\beta)=h(-\beta',-\beta)$ yielding the relation $L_-=L_+$. The matching condition between the short- and long-time asymptotes yield the following initial condition:

$$r(\beta,0)=\frac{f(\beta)(1+e^{-i\beta})}{2} . \tag{4.59}$$

Introducing a new variable $R(\beta,t)=e^{i\beta/2}r(\beta,t)$ transforms Eq. (4.58) into the equation

$$\dot{R}=[-a_2(\beta)/4+L_+]R \tag{4.60}$$

with the initial condition $R(\beta,0)=f(\beta)\cos(\beta/2)$.

For the KSM $a_1(\beta)=\nu'_\beta\beta$ and $a_2(\beta)=\nu'_\beta B^2$ where $\nu'_\beta=(1+\gamma)/\tau_0$. The solution of Eq. (4.60) is

$$R(\beta,t)=e^{-\nu_a t}\int d\beta_0 G(\beta_0,\beta,t)R(\beta_0,0) \tag{4.61}$$

where $\nu_a=\nu'_\beta B^2/4$ and the Green's function $G(\beta_0,\beta,t)$ is the solution of the equation $\dot{G}=L_+G$:

$$G(\beta_0,\beta,t)=\frac{1}{B[2\pi(1-e^{-2\nu'_\beta t})]^{1/2}}\times\exp\left[-\frac{(\beta-\beta_0e^{-\nu'_\beta t})^2}{2B^2(1-e^{-2\nu'_\beta t})}\right] . \tag{4.62}$$

The final expression for $k(t)$ in the long-time limit $(t\gg\tau_0)$ is

$$k(t)=e^{-(B^2/4)-\nu_a t}\cosh(\tfrac{1}{4}B^2e^{-\nu'_\beta t}) . \tag{4.63}$$

An interpolation formula that correctly describes both the short- and long-time behavior is

$$k(t)=e^{-B^2/4}[e^{-2t/\tau_0}\sinh(\tfrac{1}{4}B^2)+e^{-\nu_a t}\cosh(\tfrac{1}{4}B^2e^{-\nu'_\beta t})] . \tag{4.64}$$

Equation (4.64), the main result of this section, shows that the correlation function of the field for $\gamma\approx-1$ consists of two components: fast decaying [the first term in Eq. (4.64)] and slowly decaying (the second term). For $B^2\ll4$, this expression reduces to the Born-approximation result found above.

If $B\gg1$ each component accounts for half of the decay. In this case Eq. (4.64) reduces for the interval $t\ll\nu_\beta'^{-1}$, where most of the decay occurs, to a sum of two exponentials:

$$k(t)=\tfrac{1}{2}(e^{-2t/\tau_0}+e^{-2\nu_a t}) \quad (0\le t\ll\nu_\beta'^{-1}) . \tag{4.65}$$

The spectrum is a sum of two Lorentzians of the same integral intensity and centered at the same frequency ω_L but of significantly different widths $2/\tau_0$ and $2\nu_a$.

For the case of intermediate jumps $B\sim1$ the second term in Eq. (4.64) is responsible for the major portion of decay. It is now nonexponential and produces a narrow non-Lorentzian component superimposed on a board Lorentzian pedestal of width $2/\tau_0$.

The above results are valid when many jumps are required to randomize the phase. For the KSM [see Eq. (3.5)], this requires $(1-\gamma^2)B^2\ll1$, or since γ is close to -1, $1+\gamma\ll1/2B^2$. For any fixed γ, for large enough jumps such that $B^2\gg1/(1+\gamma)$, the large-jump limit applies (Sec. IV C).

An alternative expression valid in the limit $B\gg1$ can be derived[60] yielding the result

$$k(t)=e^{-t/\tau_0}\cosh[(t/\tau_0)e^{-2\nu_a\tau_0}] \quad (t\ll1/\nu'_\beta) . \tag{4.66}$$

This result coincides with the results of this section and of Sec. IV C, when appropriate, and correctly describes the transition between regions E and C.

V. RESULTS AND DISCUSSION

A very important part of this work is the ability to simulate the phase jumps numerically, calculate the correlation functions [Eq. (2.7)], and obtain the predicted spectrum [Eq. (2.8)]. In all cases the analytical results have been compared to the simulations. While the theoretical derivation is general, the simulation was performed for the specific case of the KSM. Extensions are of course possible. The purpose of the simulation is to generate, on the computer, a field with random phase jumps as described by the different limits, and calculate the spectrum and correlation function.

The simulation is performed as follows. At $t=0$ a random phase and initial jump $\beta(0)$ [from $f(\beta)$] are assigned to the field. Given the mean time between jumps τ_0, the time of the next jump is selected from a Poissonian distribution by a random number generator. At each jump time, the phase jumps by an amount β chosen at random from $h(\beta',\beta)$. Time is divided into N equally spaced segments and the field is sampled at the end of each segment. Each such calculation constitutes a single realization of the stochastic process. The correlation function is averaged over many (typically 10 000) such realizations. The spectrum of the field is obtained from $k(t)$ by the numerical algorithm of complex fast Fourier transform.

It should be noted that although $k(t)$ is a real function, in practice the simulation always gives a small, nonzero imaginary part, so that the calculated spectrum is not necessarily symmetric. The choice of the length of time over which averages are performed is of crucial importance. There are several requirements. (a) The time has to be long enough so that (according to the sampling theorem), it provides enough resolution in the frequency domain. (b) It has to fulfill the need that the correlation function $k(t)$ practically vanishes outside the range of calculation to get the correct Fourier transform. (c) The time cannot be too long, or else the residual noise in $k(t)$ at long times will result in a "rough" spectrum. Thus, for each case a suitable time range was chosen. The general rule was that the length of the time was approximately five times the decay time of the correlation function. In most cases this choice proved to be adequate. The simulation procedure as described here is straightforward, and does not consume significant amounts of computer time. It is completely independent of the approximations used in this paper, and can provide a useful test of the approximations.

In the previous sections we derived detailed mathematical expressions for the field correlation function in each of the regions of the B, γ phase space. The approximations involved in deriving the results in each region were detailed separately. In this section the spectral line shapes and correlation functions are shown for the different regions, and wherever appropriate, the transitions between the regions are discussed. In all the figures presented in this section, both simulation and analytical results are displayed.

Figure 2 shows schematic plots of the generic form of the correlation functions that are obtained in each region. The quantity that is plotted is the negative logarithm of the correlation function against the time delay t. The different regions are marked by the letters corresponding to the subsection of Sec. IV where they are discussed. The reader is referred to that discussion for detailed mathematical expressions, but the results for the KSM are summarized in Table II. Unless otherwise stated, these are the expressions actually used as the "theory" in the following figures.

On the line $\gamma=0$ (region A) the function is linear with a slope that starts quadratically for small B, and saturates to a constant. A linear logarithm of the correlation function means a Lorentzian line shape, and this is the predicted shape, in accordance with the Burshtein model. In the Born-approximation region (region B), the initial slope for times smaller than $\tau_\beta=\tau_0/(1-\gamma)$ is $\nu_1[=B^2/(2\tau_0)]$ independent of γ. For longer times, the correlation takes an effect, causing the slope to become γ dependent: $[\nu=\nu_1(1+\gamma)/(1-\gamma)]$, larger than ν_1 for positive γ and smaller for negative γ. Note that the intersection between the two linear asymptotes occurs at $\tau_0/(1-\gamma)=(1/\nu_\beta)$.

In the large-jump region (region C) the function is linear for all values of γ (except γ very close to -1). The slope in this region is $1/\tau_0$, independent of B and γ, and there is no signature of non-Markovian behavior in this region. The region of $\gamma\approx1$ is divided into two subregions D_1 and D_2, depending on the size of B. For moderately small B $(1-\gamma\ll B\ll1)$ the logarithm of the correlation function approximates a parabola and for larger B it turns into a straight line regaining the results of the large-jump region.

In the anticorrelation region (region E), the spectrum consists of a Lorentzian of width of $2/\tau_0$ and a much narrower central component. The correlation function is thus the sum of a fast decaying component, and a slowly decaying long-lived asymptote. The solution in this region does not merge with that obtained for large jumps, and there is always a transition region between the two, described by Eq. (4.66).

The following set of figures depicts the correlation function, presenting both analytical results and numerical simulations. For each figure the values of B and γ are given, and general trends are observed. We have chosen to present the correlation function rather than the spectrum, as the differences are much more pronounced, since all spectra (on a linear scale) look alike.

Figure 3 shows the Born-approximation (BA) region, where the change of slope is clearly seen for negative and positive γ. In Figs. 3(a)–3(d), the theory is the BA expression from Table II.

Figure 4 presents the intermediate case ($B=1$). For anticorrelation ($\gamma=-0.95$), the slowly decaying component is visible in Fig. 4(a). For Burshtein's uncorrelated case ($\gamma=0$), a straight line is observed in Fig. 4(b), as expected. For the slowly varying jump ($\gamma=0.95$), the slope is increasing [Fig. 4(c)]. Here, Eqs. (4.36) and (4.37) were used for the theory.

As can be observed from the phase-space diagram (Fig. 2), for $B\approx1$, $\gamma\neq0$, and γ not too close to ±1 none of the approximations are strictly valid, and in fact, we do not have a closed-form expression to describe the correlation function. The simulation, however, can be performed everywhere, and the limits of applicability of the various approximations may be tested against it. An example of such cases is shown in Fig. 5, where the cases of $B=1$ and $\gamma=\pm0.5$ are depicted. The various theoretical expressions relevant for each case are plotted with the letters designating each region and compared to the simulation results. For $\gamma=-0.5$ $B^2/(1-\gamma)^2=\frac{4}{9}$, while for $\gamma=0.5$ it is 4. Thus, one expects the BA approximation to be much better for $\gamma=-0.5$, which is, indeed, seen in Fig. 5(a). In both cases, the simulation results are between the limiting cases [A and E in Fig. 5(a) and A and D_1 in Fig. 5(b)]. The strength of the present analysis is in the ability to compare the theoretical results to simulated ones, even in regions where one has no *a priori* justification to expect the theory to be right. As can be seen, even far from the regions defining the approximations the theoretical expressions are similar to the "real" data, and in some cases the agreement is far better than expected.

For $B=3$ the large-jump (LJ) approximation is applicable except for γ close to -1. Figure 6 shows this situation. For $\gamma=-0.95$ [Fig. 6(a)] we have used the highly anticorrelation jump expression, and for $\gamma=-0.5$ [Fig. 6(c)] the LJ expression (Table II). For $\gamma>-0.5$ both simulation and theory look identical to Fig. 6(c) and there is no point in showing additional graphs. Between these two regions, an intermediate region always exists. The transition is demonstrated in Fig. 7(b) for the $\gamma=-0.85$ value. The expression for the interim region is the C-E transition region expression in Table II.

As discussed in Sec. IV E, the spectrum of the highly anticorrelated region consists of a narrow sharp feature riding on a broader Lorentzian. In Fig. 7 the spectrum for $\gamma=-0.99$ and $B=3$ is shown on a semilogarithmic scale. The solid curve is the highly anticorrelated jump expression, which includes both the narrow feature and the broad Lorentzian. The dashed curve depicts only the narrow Lorentzian sharp feature. The total intensity is divided equally between the two Lorentzians.

In the lower right-hand corner of the phase-space dia-

TABLE II. Results for the main regions in the (B,γ) phase space, specialized to the KSM case. The conditions on B and γ, the formulas for the correlation function, and the spectrum are given. Here $\Delta\omega_L=\omega-\omega_L$ and $\nu_1=[1-\exp(-B^2/2)]/\tau_0$.

Limit	(B,γ)	Region	$k(t)$	$J(\Delta\omega_L)$
Short-time asymptotes (far wings of the spectrum)	$-1\le\gamma\le 1$	any	$1-\nu_1 t$	$\dfrac{\nu_1}{\pi\Delta\omega_L^2}$
Uncorrelated jumps	$\gamma=0$	A	$e^{-\nu_1 t}$	$\dfrac{\nu_1}{\pi(\Delta\omega_L^2+\nu_1^2)}$
Born approximation (quasidiffusion)	$B^2\ll(1-\gamma)^2$	B	$\left[1+\dfrac{\gamma B^2}{(1-\gamma)^2}\right]e^{-\nu t} - \left[\dfrac{\gamma B^2}{(1-\gamma)^2}e^{-\nu_\beta t}\right]$	$\dfrac{1}{\pi}\dfrac{\nu_1+(\nu-\nu_1)(\Delta\omega_L^2/\nu_\beta^2+1)^{-1}}{\Delta\omega_L^2+\nu^2}$
Frequency fluctuations (slowly varying jump)	$(1-\gamma)^2\ll B^2$, $B^2\ll 1$	$D_1\cap D_2$	$\exp\left[-\dfrac{B^2t^2}{2\tau_0^2}\right]$	$\dfrac{\tau_0}{(2\pi)^{1/2}B}\exp\left[-\dfrac{\Delta\omega_L^2\tau_0^2}{2B^2}\right]$
Frequency fluctuations (general case)	$\gamma\approx 1$, $B^2\ll 1$	D_2	$\exp\left[-\dfrac{B^2}{(1-\gamma)^2}\times(\nu_\beta t-1+e^{-\nu_\beta t})\right]$	
Large jumps	$(1+\gamma)^{-1}\ll B^2$	C	e^{-t/τ_0}	$\dfrac{\tau_0}{\pi(1+\Delta\omega_L^2\tau_0^2)}$
Highly anticorrelated jumps	$\gamma\approx -1$	E	$e^{-B^2/4}\left[e^{-2t/\tau_0}\sinh\dfrac{B^2}{4} + e^{-\nu_a t}\cosh\dfrac{B^2e^{-\nu'_\beta t}}{4}\right]$	
Highly anticorrelated jumps (small-jump limit)	$1+\gamma\ll 1$, $B^2\ll 4$	$E\cap B$	$(1-B^2/4)e^{-\nu_a t} +(B^2/4)e^{-2t/\tau_0}$	$\dfrac{\nu_1}{\pi}\dfrac{\Delta\omega_L^2+2(1+\gamma)/\tau_0^2}{(\Delta\omega_L^2+4/\tau_0^2)(\Delta\omega_L^2+\nu_a^2)}$
Highly anticorrelated jumps (large-jump limit)	$(1+\gamma)^{-1}\gg B^2$, $B^2\gg 1$	E	$(e^{-2t/\tau_0}+e^{-2\nu_a t})/2$	$\dfrac{1}{\pi}\left[\dfrac{1/\tau_0}{\Delta\omega_L^2+4/\tau_0^2}+\dfrac{\nu_a}{\Delta\omega_L^2+4\nu_a^2}\right]$
C-E transition region	$1+\gamma\ll 1$, $B\gg 1$	C-E	$e^{-t/\tau_0}\cosh[(t/\tau_0)e^{-2\nu_a\tau_0}]$	$\dfrac{1}{\pi}\sum_{j=\pm 1}\dfrac{(1+je^{-2\nu_a\tau_0})/\tau_0}{\Delta\omega_L^2+(1+je^{-2\nu_a\tau_0})^2/\tau_0^2}$

gram the frequency fluctuation (FF) approximation applies. This region overlaps the BA for small B and the slowly varying jump region when γ is close to 1. The situation is shown in Fig. 8. In Fig. 8(a) ($\gamma=0.995$, $B=0.05$), both the general FF expression and the slowly varying jump limit of the FF agree with the simulation. In Fig. 8(c) ($\gamma=0.95$, $B=0.005$), both BA and the general FF expressions yield the same curve. In Fig. 8(b) ($\gamma=0.95$, $B=0.05$), the simulation is compared to three theoretical expressions —the BA, the FF, and the slowly varying jump limit of FF. As is clearly seen, only the general FF is valid. Note that the actual decay in Fig. 8(c) is much smaller than in Figs. 8(a) and 8(b) due to the smaller value of B.

In the FF region a transition from a Lorentzian (BA) to a Gaussian spectrum is predicted and is depicted in Fig. 9 for the same cases shown in Fig. 9. The BA is the motionally narrowed limit of the Kubo oscillator, giving a Lorentzian line shape. The slowly varying jump limit, on the other hand, corresponds to the quasistatic limit, known to give a Gaussian spectrum. In order to compare the spectral shapes, we have normalized each spectrum to its half-width, scaling the intensity so that the area remains unchanged.

If the spectral line shape of the real laser deviates from a Lorentzian towards a Gaussian, one may compare the observed line shape to a figure like the present one, and estimate two ratios connecting B, γ, and τ_0: the ratio $B/(1-\gamma)$, which is analogous to the $\Delta\tau_c$ parameter of the Kubo oscillator theory,[52] and $\nu_\beta=(1-\gamma)/\tau_0$. Note that a transition region between Gaussian and Lorentzian line shapes exists not only in the FF region, but also within the slowly varying jump region and the input field analysis may not be sufficient to distinguish between them. As will be shown in paper II, a nonlinear optical experiment readily gives the distinction.

VI. EXPERIMENTAL RAMIFICATIONS OF THE THEORY

The results derived in Secs. IV and V predict the correlation function and the spectral line shape for the GJM model for phase fluctuations. When actual lasers are involved, the experimentalist may measure some parameters of the laser field, but has no *a priori* knowledge of the statistical nature of the noise in the laser. Thus, traditionally lasers have been described by their "linewidth," which is an averaged quantity. As is clear from the discussion above, a single parameter like a linewidth cannot describe in full the statistical properties of a stochastic

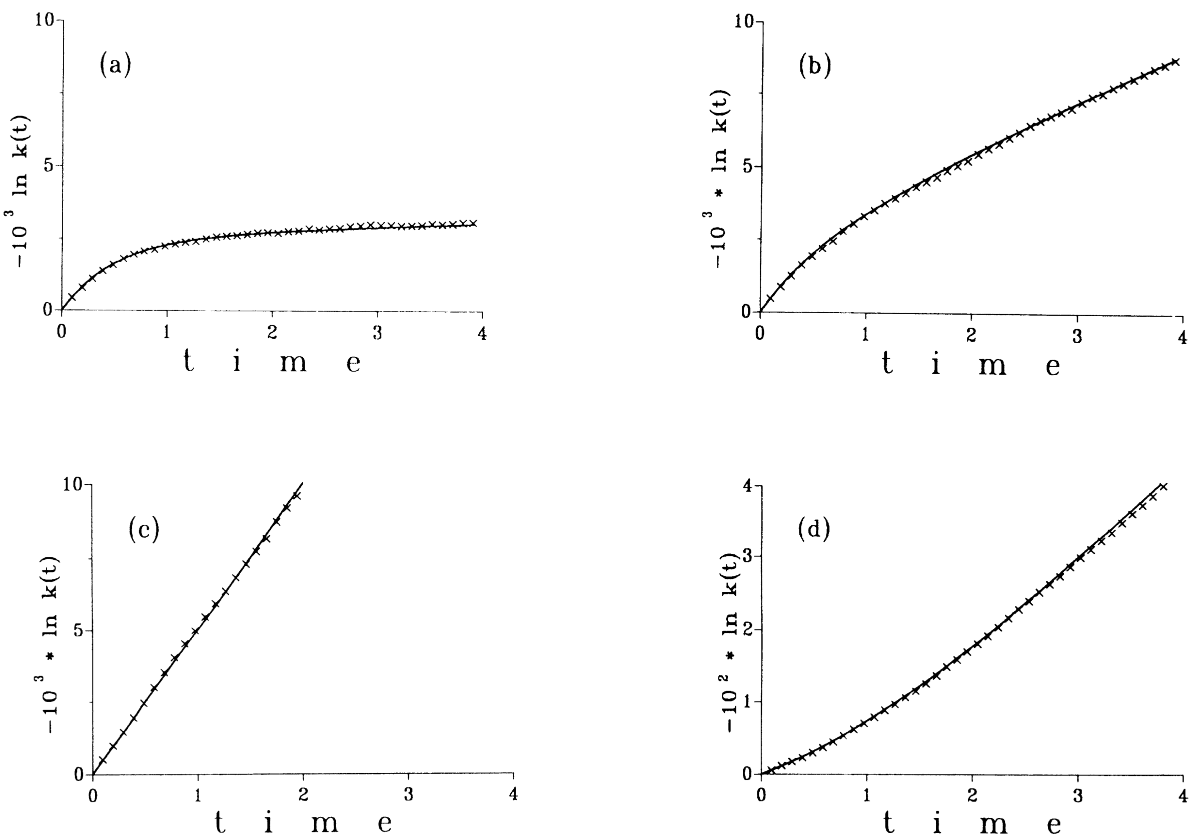

FIG. 3. Negative logarithm of the correlation function against time delay for $\tau_0=1$, $B=0.1$, and $\gamma=$ (a) -0.95, (b) -0.5, (c) 0, (d) 0.5. The solid line is the Born-approximation (BA) expression. The dots are simulation results.

field, and as was recently demonstrated by us, two laser fields of identical linewidths may give rise to totally different observable line shapes in experiments like resonance fluorescence.[10]

The inverse problem of determining the stochastic character of a field from measured quantities is extremely complex, and is certainly not fully resolved in this article. However, based on the derivation presented here, and within the assumptions of the GJM, several conclusions may be reached regarding the non-Markovian character of the field. Without specifically assuming a particular functional dependence of the jump distribution function, one may determine the type of correlations in the field, and estimate the average jump size.

In particular, we propose a procedure that characterizes the stochastic nature of the field by a simple measurement of its autocorrelation function. The procedure involves the following steps.

(i) Measure the field amplitude autocorrelation function $k(t)$. (One way to do this is to split the laser beam into two parts, delay one with respect to the other, and measure the cross term in the total intensity of the two fields on the same square law detector.)

(ii) Plot $-\ln k(t)$ versus delay time t for the above measured function.

(iii) Compare the observed shape of this plot to the shapes predicted for the different regions in this paper (see Fig. 2).

(iv) If two (asymptotically) straight lines, or a parabola, were measured, the region is identified unambiguously, and the applicable section above describes the field. Read that section.

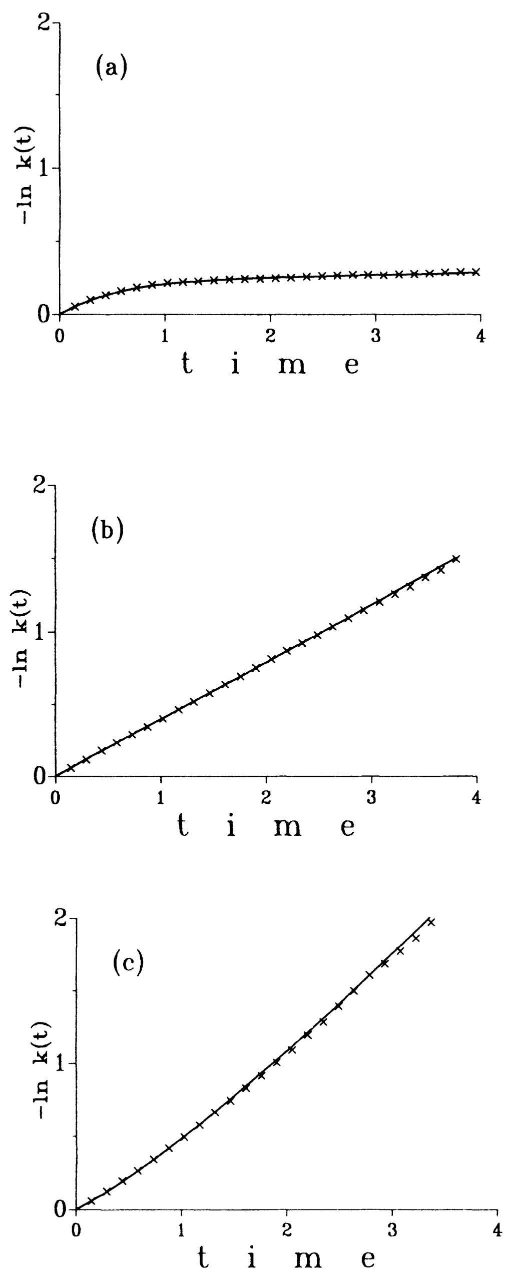

FIG. 4. Negative logarithm of the correlation function against time delay for $\tau_0=1$, for $B=1$ and $\gamma=$ (a) -0.95, (b) 0, (c) 0.95. The theory is taken from Table II for (a) and (b) and from (4.36) and (4.37) for (c).

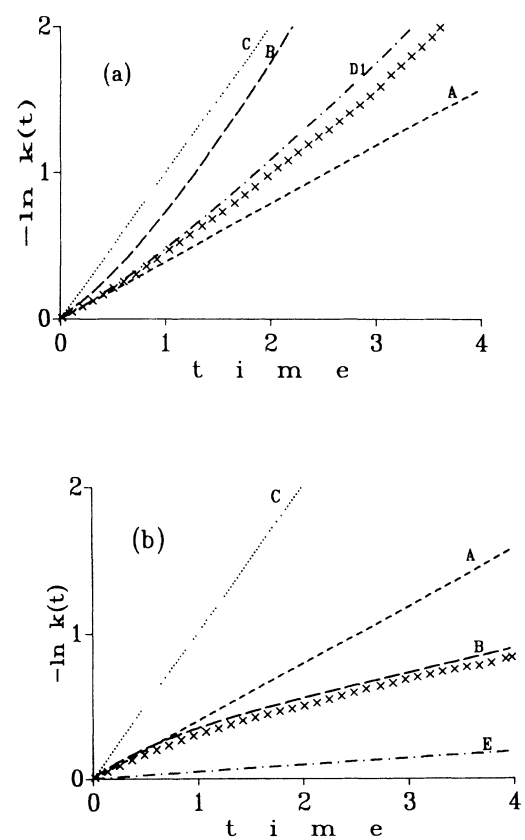

FIG. 5. Negative logarithm of the correlation function against time delay for $\tau_0=1$, $B=1$, and $\gamma=$ (a) -0.5, (b) 0.5. The different line types are the approximations for different regions marked by the corresponding letters.

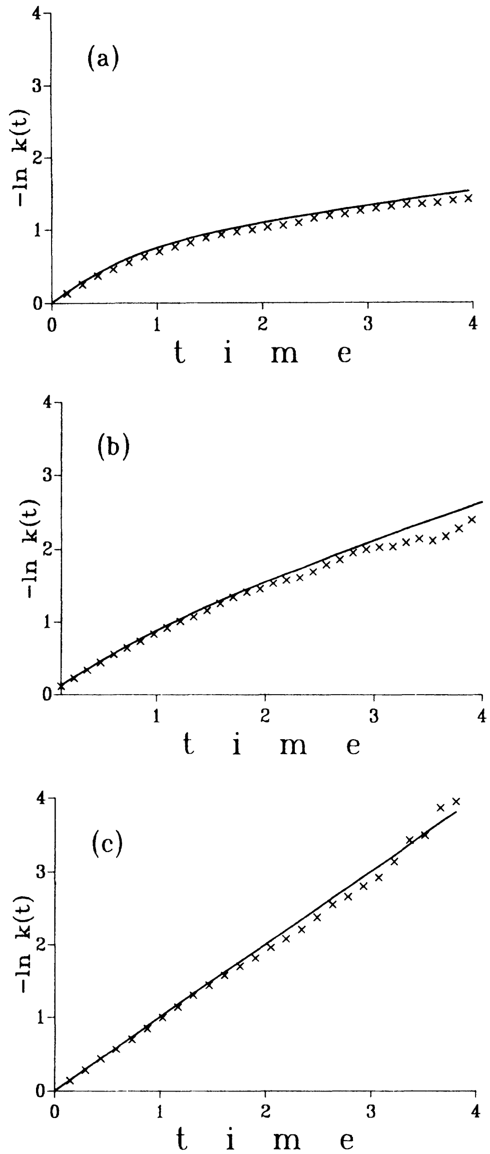

FIG. 6 Negative logarithm of the correlation function against time delay for $\tau_0=1$, $B=3$, and $\gamma=$(a) -0.95, (b) -0.85, (c) -0.5. The solid lines are taken from Table II (see text).

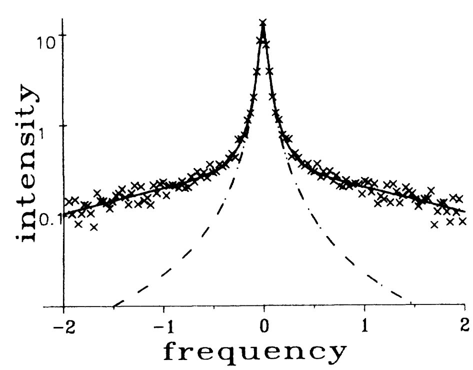

FIG. 7. Spectrum for $\tau_0=1$, $B=3$, and $\gamma=-0.99$. The solid line is the highly anticorrelated solution, the dash-dotted line is a Lorentzian of width $2\nu_a$.

(v) If a single straight line is measured, the situation is less fortunate. The underlying stochastic process may be Markovian ($\gamma=0$), it may mean a large B limit for $\gamma\neq 0$ or it may mean that the correlation function was not measured accurately enough to resolve the (very) small B limit. In this case, the procedures proposed in this article may not be sufficient, and the method of paper II should

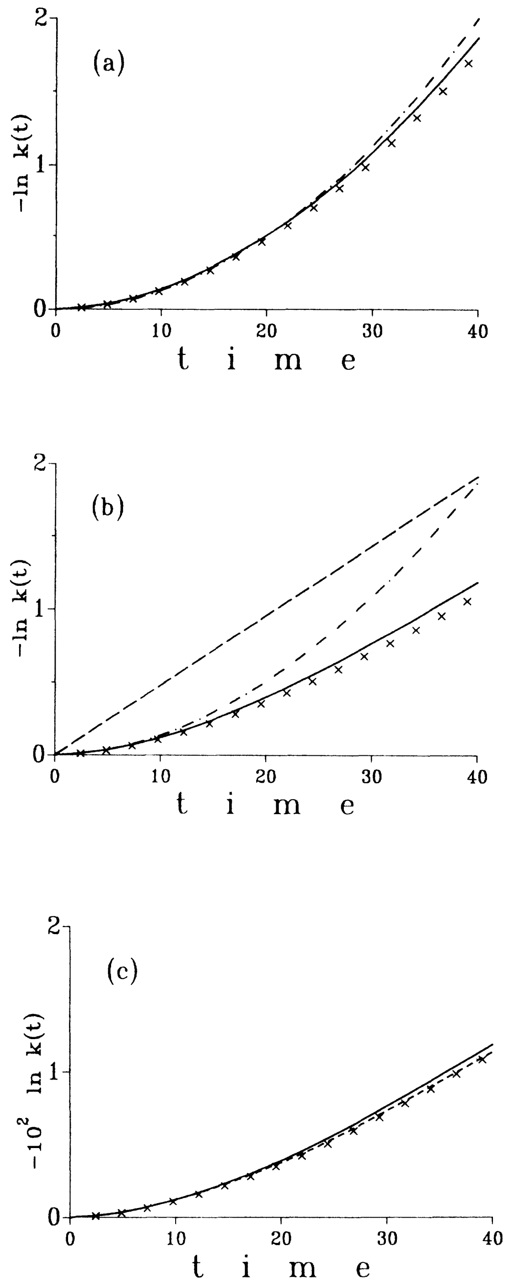

FIG. 8. Negative logarithm of the correlation function against time delay in the FF region for $\tau_0=1$. (a) $B=0.05$ and $\gamma=0.995$, (b) $B=0.05$ and $\gamma=0.95$, and (c) $B=0.005$ and $\gamma=0.95$. The solid line is the FF solution everywhere, the dashed lines in (a) and (b) are the slowly varying jump limit of the FF solution and the dash-dotted lines in (b) and (c) are BA.

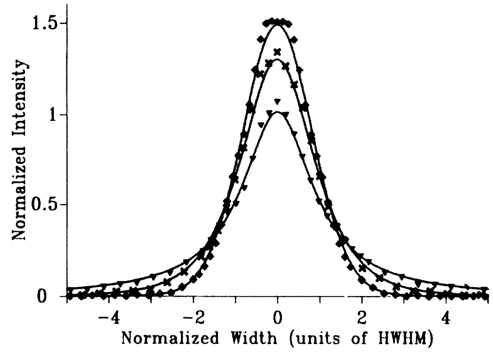

FIG. 9. Normalized spectra in the FF region. The diamonds correspond to 8(a), the X's to 8(b), and the triangles to 8(c). The solid lines are FF expressions for each.

be adopted.

(vi) If the observed function is neither one of the above, a more complicated statistical process is involved. For example, a jump process involving a nonconstant $\tau_0(\beta)$ with large jumps, or a nonjump process.

Experiments of this kind are under way in our laboratory.

VII. CONCLUSIONS

In the present paper a generalized jump model was developed for phase fluctuations in an electromagnetic field. The model is very general, and encompasses all standard models as limiting cases. In particular, we show that the well-known results of the phase-diffusion model, the telegraph-noise model, the Burshtein uncorrelated jump model, and the Kubo oscillator treatments are all regained from the present work.

The analytical work presented in this paper is fully supported by detailed numerical simulations. The analytical results and the simulations are compared, and the agreement is excellent where it should be. The computer simulations enable us to derive results even between the regions treated analytically, and to test the quality of the various approximations used for the different analytical derivations. The mathematical methods developed here are generally applicable to other problems of stochastic effects in physical systems, like pressure broadening, line shapes in condensed phase, electron scattering in random media, and many more.

Since this is a very long and detailed paper, no attempt will be made here to summarize the results. If a short summary is needed, Fig. 2 and the KSM results in Table II are the best such overview. Here, we will address three important questions.

(i) Is it justified to use a discontinuous jump model to describe the phase of a laser? The answer is not clear, but it is not *a priori* unreasonable to assume that the environment affects the laser by changing its phase suddenly. The test of this assumption, of course, is in the predictions of the model, but from the experimental point of view, any measurement of the phase is performed over a finite time, so any information available on the phase will always be for discrete times. Thus, in this respect the assumption is reasonable. From the mathematical point of view, the jumps are handled rigorously, and there is no problem with this assumption.

(ii) How realistic is the assumption of correlated jumps? The sources for the noise discussed in this paper are not specified, but may safely be assumed to originate from the environment in which the laser is operating. A free running laser interacts with its surroundings, and is influenced by such factors as temperature changes, pressure changes, etc. The time scale and range of such changes may be different than what is required to jump the phase of the laser, and it is plausible that a gradual temperature rise will cause several jumps in the same direction. In the other extreme, any stabilized laser operates on the principle of anticorrelation, namely, the parameter to be stabilized is monitored, and when a change is sensed, there is feed back into the laser to undo the change. Admittedly, most lasers are frequency rather than phase stabilized, but the connection is obvious. Thus, the range of B, γ phase space covered by this paper is a reasonable one. *Indeed, it is unreasonable to assume that fluctuations in nature are uncorrelated.*

(iii) Even if mathematically sound and physically reasonable, is the model relevant to work of current interest or is it merely a curiosity? The answer to this question, as we tried to show, is that it is relevant, and that the characterization of a laser by a single parameter is not enough. Only one example is given here: In applications to optical communication, the bit error rate in an optical system is usually calculated directly from the laser linewidth where a Lorentzian shape is assumed. A laser that is stabilized[61,62] (highly anticorrelated in the language of this paper) may have a spectrum that contains a narrow sharp central component, and a broad pedestal. Such a laser will not perform well in communication systems, even though its half width is very narrow. Clearly, for such a laser more information about the line shape is needed, and our model provides the tools to handle such situations.

The very interesting question of the light-matter nonlinear interaction of a stochastic field of the type discussed here is the topic of a separate paper.

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APPENDIX A: GENERALIZED CUMULANT EXPANSIONS

In this appendix the theory of the generalized cumulant expansions is summarized, and a new version of this theory suitable for the study of the GJM is developed.

1. The general equation

Consider a linear equation

$$\dot{x}=[A+B(t)]x \tag{A1}$$

where $x(t)$ is a vector and A and $B(t)$ are operators. Suppose that the quantity of interest is not $x(t)$, but rather the projection $Px(t)$ of $x(t)$ into the "relevant" subspace of the functional space where $x(t)$ is defined. If the initial condition for Eq. (A1) $x(0)$ belongs to the relevant subspace

$$Px(0)=x(0) \tag{A2}$$

and A commutes with P,

$$PA=AP\ , \tag{A3}$$

then the following equation can be written[63] for $Px(t)$:

$$P\dot{x}=[A+PB(t)]Px+\int_0^{+}K(t,t')Px(t')dt'\ . \tag{A4}$$

Here the kernel $K(t,t')$ is the following perturbation expansion[64,65] in orders of $B(t)$:

$$K(t,t')=\theta_2(t,t')+\sum_{n=3}^{\infty}\int_{t'}^{t}dt_{n-2}\cdots\int_{t'}^{t_2}dt_1\theta_n(t,t_{n-2},\ldots,t_1,t') \tag{A5}$$

where θ_n's are totally ordered cumulants,[64–66]

$$\theta_n(t_n,\ldots,t_1)=PB(t_n)e^{A(t_n-t_{n-1})}(1-P)B(t_{n-1})\cdots e^{A(t_2-t_1)}(1-P)B(t_1)\quad(t_n\geq\cdots\geq t_1)\ . \tag{A6}$$

Cumulants θ_n are combinations[64,67,68] of generalized moments M_k $(1\leq k\leq n)$ of $B(t)$,

$$M_k(t_k,\ldots,t_1)=PB(t_k)e^{A(t_k-t_{k-1})}B(t_{k-1})\cdots e^{A(t_2-t_1)}B(t_1)\quad(t_k\geq\cdots\geq t_1)\ . \tag{A7}$$

In particular

$$\theta_2(t,t')=M_2(t,t')-M_1(t)e^{A(t-t')}M_1(t')\ . \tag{A8}$$

By a rearrangement of the cumulant expansion Eq. (A4) can be transformed[64] to the differential equation[69–71]

$$P\dot{x}=[A+PB(t)+R(t)]Px\ . \tag{A9}$$

Here

$$R(t)=\sum_{n=2}^{\infty}R_n(t)$$

$$=\sum_{n=2}^{\infty}\int_0^{t}dt_{n-1}\cdots\int_0^{t_2}dt_1C_n(t,t_{n-1},\ldots,t_1) \tag{A10}$$

where C_n's are partially ordered cumulants. In particular,

$$C_2(t,t')=\theta_2(t,t')e^{-A(t-t')}\ . \tag{A11}$$

The cumulant C_n is the nth order in B, and can be expressed[71,72] either in terms of moments M_n or of totally ordered[65] cumulants θ_n. Assume that the moments M_n have the product property,

$$M_n(t_n,\ldots,t_1)\to M_{n-i}(t_n,\ldots,t_{i+1})e^{A(t_{i+1}-t_i)}\times M_i(t_i,\ldots,t_1)\quad\text{for } t_{i+1}-t_i\to\infty\quad(1\leq i\leq n-1)\ . \tag{A12}$$

Then the cumulants possess the cluster property,[64,72]

$$\theta_n(t_n,\ldots,t_1),\ C_n(t_n,\ldots,t_1)\to 0\quad\text{for } t_{i+1}-t_i\to\infty\quad(1\leq i\leq n-1)\ . \tag{A13}$$

Equation (A13) provides the convergence of the integrals in the cumulant expansions (A5) and (A10) for $t\to\infty$. Both Eqs. (A4) and (A9) are exact, and thus have the same solution but their forms are significantly different, so that for a specific problem one may be more suitable than the other.

2. The Born approximation

Assume that the nth moment of $B(t)$ is of order b^n where b is a small parameter. We will need approximate equations valid up to the second order in the perturbation parameter. In this Born approximation Eqs. (A4) and (A9) reduce to

$$P\dot{x}=[A+PB(t)]Px+\int_0^{t}\theta_2(t,t')Px(t')dt'\ , \tag{A14}$$

and

$$P\dot{x}=[A+PB(t)+R_2(t)]Px\ . \tag{A15}$$

Because of the approximation, Eqs. (A14) and (A15) yield the *same* solutions[67,68] with an accuracy of b^2. The present generalized cumulant expansion technique validates (for *any* time $t\geq 0$) the decoupling method used in early works to obtain equations of the type (A14) in special cases.[73,74]

The exact equations (A4) and (A9) provide higher-order corrections to Eqs. (A14) and (A15). Assume that the cumulants decay exponentially with a characteristic time τ_c, and that the nth-order cumulant is of order b^n, then it can be shown that for long times $[t\gg(n-1)\tau_c]$ $\|R_n(t)\|\sim b^n\tau_c^{n-1}$. Hence, in Eq. (A10) one can neglect $R_n(t)$ for $n\geq 3$ yielding Eq. (A15) if

$$b\tau_c\ll 1\ . \tag{A16}$$

Sometimes the functions $R_{2k+1}(t)$ $(k\geq 1)$ vanish or are anomalously small (e.g., $\|R_{2k+1}\|\sim\|R_{2k+2}\|$, as in Appendix B). Then the validity condition becomes somewhat less restrictive,

$$(b\tau_c)^2 \ll 1 \ . \tag{A17}$$

3. Elimination of irrelevant variables

Consider an equation

$$\dot{x}(V,t) = (A + L_V + B_V)x(V,t) \ . \tag{A18}$$

Here A and B_V are matrices in the vector space X, where the vector x is defined, B_V is an operator acting on function of V, and L_V is a stochastic operator as in Eq. (2.1), being a scalar in the vector space X. We assume that the initial condition for Eq. (A24) is

$$x(V,0) = x(0)f(V) \ . \tag{A19}$$

We also assume for convenience, that A, B_V, and L_V are time independent. Equation (A18) reduces to a Burshtein-like equation in the particular case when B_V is a function of V rather than an operator.

We are interested in the quantity

$$\bar{x}(t) = \int x(V,t)dV \tag{A20}$$

rather than in $x(V,t)$ itself. We can eliminate the irrelevant variables V by defining a projection operator P as

$$P \cdots = f(V)\int dV \cdots \ . \tag{A21}$$

For the L_V operator possessing the properties (2.2) $PL_V = L_V P = 0$ and since A is not a function of V, $PA = AP$. In addition $Px(V,0) = x(V,0)$. Therefore, the results of the generalized cumulant expansion discussed above may be applied to the calculation of $Px = f(V)\bar{x}$ by making the substitution $A \to A + L_V$ and $B(t) \to B_V$. Integrating Eq. (A4) over V yields the result

$$\dot{\bar{x}} = (A + \bar{B})\bar{x} + \int_0^t K(t-t')\bar{x}(t')dt' \tag{A22}$$

where

$$\bar{B} = \int B_V f(V)dV \ . \tag{A23}$$

The kernel $K(t-t')$ can be calculated from Eq. (A5) using the totally ordered cumulants θ_n obtained from Eq. (A6) by operating on $f(V)$ and integrating over V. It is helpful to define an averaging operator Q by

$$Q \cdots = \int \cdots f(V)dV \ . \tag{A24}$$

Then the totally ordered cumulants are given by Eq. (A6) with the substitutions $P \to Q$, $A \to A + L_V$, and $B(t) \to B_V$. Defining the generalized moments M_k by Eq. (A7) with these substitutions produces the same connection between cumulants and moments as in Sec. I of this appendix. The calculation is simplified by observing that the conditional probability $f(V';V,t-t')$ is given by

$$f(V';V,t-t') = e^{L_V(t-t')}\delta(V-V') \ . \tag{A25}$$

For example, the second moment is

$$M_2(t-t') = QB_V e^{(A+L_V)(t-t')}B_V \ , \tag{A26}$$

which may be rewritten using Eq. (A25) as

$$M_2(t-t') = \int dV \int dV' B_V e^{At} f(V';V,t-t')B_{V'}f(V') \ . \tag{A27}$$

Truncating the cumulant expansion (A22) to second order yields

$$\dot{\bar{x}} = (A + \bar{B})\bar{x} + \int_0^t \theta(t-t')\bar{x}(t')dt' \tag{A28}$$

where, from Eq. (A8),

$$\theta(t) \equiv \theta_2(t) = M_2(t) - \bar{B}e^{At}\bar{B} \ . \tag{A29}$$

An analogous procedure can be used on Eq. (A9) to obtain the alternative form:

$$\dot{\bar{x}} = [A + \bar{B} + \int_0^t \theta(t')e^{-At'}dt']\bar{x} \ . \tag{A30}$$

In this paper, we are interested in solving Eq. (2.11). This is in the form of Eq. (A18) with $x = r$, $V = \beta$, $A = 0$, and $B_V = B_\beta$. The correlation function $k(t)$ is then $\bar{x}(t)$ [cf. Eq. (2.9)] and $\bar{B} = \nu_1$ [see Eq. (4.2)], as follows from Eqs. (A23), (2.12), and (2.5b). In this way, we obtain Eqs. (4.10), (4.11), and (4.24) used in the main text. In particular, combining Eqs. (A29), (A27), (2.12), and (2.5b) yields Eq. (4.11).

APPENDIX B: VALIDITY CONDITIONS FOR THE BORN APPROXIMATION

To investigate the validity of the approximation made in obtaining Eqs. (4.10) and (4.24) we need estimates of the moments of the operator B_β,

$$M_n(t,\ldots,t) \equiv \tilde{M}_n = \int B_\beta^n f(\beta)d\beta \ . \tag{B1}$$

The upper bound will be the fully correlated case. The estimate becomes

$$\|\tilde{M}_n\| \sim \begin{cases} (B/\tau_{\rm av})^{2k}, & n = 2k \quad (k = 1,2,\ldots) \\ B^{2k}/\tau_{\rm av}^{2k-1}, & n = 2k-1 \quad (k = 1,2,\ldots) \ . \end{cases} \tag{B2}$$

Since $\tau_c = \nu_\beta^{-1} = \tau_{\rm av}/(1-\gamma)$, one obtains for $R_n \sim \|\tilde{M}_n\|\tau_c^{n-1}$ the estimates

$$\|R_{2k}\| \sim \|R_{2k-1}\|/(1-\gamma) \sim B^{2k}/[(1-\gamma)^{2k-1}\tau_{\rm av}] \ . \tag{B3}$$

Hence, from Eq. (A17), the Born approximation is justified for $B^2 \ll (1-\gamma)^2$.

The estimation is not necessarily valid for γ close to -1 when the random process is characterized by two time scales $1/\nu_\beta$ and $1/\nu'_\beta$. Another approach is to compare an explicit solution for $k(t)$ to the Born approximation calculated for $\gamma = -1$. In that case, Eq. (2.4) has the solution

$$f(\beta_0;\beta,t)=\tfrac{1}{2}\{\delta(\beta-\beta_0)+\delta(\beta+\beta_0)+[\delta(\beta-\beta_0)-\delta(\beta+\beta_0)]e^{-2t/\tau_0(\beta)}\} . \quad \text{(B4)}$$

Using (B4) in Eq. (4.17), one can calculate $\theta_0(t)$ to lowest order in B:

$$\theta_0(t)=k_\epsilon(t)=\int d\beta f(\beta)[\beta/\tau_0(\beta)]^2 e^{-2t/\tau_0(\beta)} . \quad \text{(B5)}$$

The resulting expression for $k(t)$ [Eq. (4.23)] can be shown to be identical with the expansion up to second order in B of the exact expression Eq. (4.49). In this way, we have proven that the validity condition is $B^2 \ll (1-\gamma)^2$ for all values of γ.

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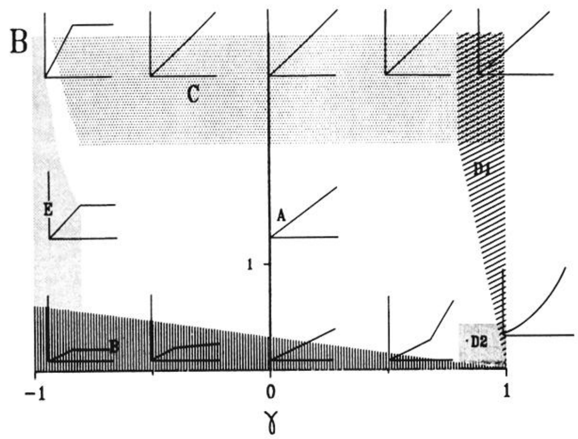

FIG. 2. (B,γ) phase space, the regions where specific solutions are obtained, are shaded and marked by the letter corresponding to that region. The plots describe schematically the negative logarithm of the correlation functions against time delay.