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Sub-Poissonian photon statistics in time-dependent collective resonance fluorescence

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We have discussed the photon statistics of the spectral components of N-atom time-dependent resonance fluorescence. It is shown that in contrast to the stationary limit, sub-Poissonian photon statistics in the sidebands occur for any number N of atoms including the case $N \gg 1$. Reduction in Mandel's parameters Q_{+1} is found with increasing numbers of atoms. The typical time for the presence of sub-Poissonian statistics is proportional to $1/N$.

I. INTRODUCTION

In recent years the quantum properties of light such as squeezing (see Refs. 1–3 and references quoted therein) and sub-Poissonian statistics[4–14] have attracted a great deal of attention. The first experimental observation of sub-Poissonian photon statistics was reported by Short and Mandel.[7] Applications of sub-Poissonian light for quantum-noise-limited measurements have been discussed in various papers.[4–6]

The photon statistics in the stationary resonance fluorescence from a single atom as well as from an ensemble of atoms have been studied theoretically.[8–12,15–17] Experimental studies of the photon statistics of single-atom fluorescence have been reported.[7,18,19] The photon statistics of the spectral components and the cross correlation between them have been investigated in previous papers.[15,16,18] In particular, it has been shown that in the steady-state limit, sub-Poissonian statistics disappear if the number of atoms is large.

In the present paper we investigate sub-Poissonian photon statistics of the spectral components of N-atom time-dependent resonance fluorescence. In contrast with the stationary limit, sub-Poissonian statistics in the sidebands occur even for large numbers of atoms. The sidebands become more sub-Poissonian with an increasing in number of atoms. It is shown that the typical time for observing sub-Poissonian statistics is proportional to $1/N$.

II. THE MASTER EQUATION

We consider N identical two-level atoms (the Dicke model) interacting with an external classical field at frequency ω_L and with the vacuum of the radiation field. For simplicity the external field is assumed to be in exact resonance with the atomic transition frequency ω. By using the rotating-wave and Born-Markov approximations for describing the coupling between the system of atoms and the vacuum reservoir, one finds a master equation (in the interaction picture) for the reduced atomic system in the following form:[20,21]

$$\frac{\partial\rho}{\partial t} = -i[\Omega(J_{12}+J_{21}),\rho] - \frac{\gamma}{2}(J_{21}J_{12}\rho - 2J_{12}\rho J_{21} + \rho J_{21}J_{12}) = L\rho \,, \tag{2.1}$$

where γ is the atomic spontaneous transition rate, Ω is the resonant Rabi frequency, and the operators J_{ij} $(i,j=1,2)$ are the usual collective atomic angular operators defined by

$$J_{ij} = \sum_{k=1}^{N} |i\rangle_k\,{}_k\langle j| \,.$$

These operators satisfy the commutation relation

$$[J_{ij},J_{lm}] = J_{im}\delta_{jl} - J_{lj}\delta_{lm} \,.$$

Following Ref. 22, we introduce the Schwinger representation for these operators:

$$J_{ij} = a_i^\dagger a_j \quad (i,j=1,2) \,,$$

where the operators a_i and a_j obey the boson commutation relations

$$[a_i,a_j^\dagger] = \delta_{ij} \,.$$

After performing the canonical (dressing) transformation

$$a_1 = \frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{2}}c_2 \,, \qquad a_2 = -\frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{2}}c_2 \,, \tag{2.2}$$

one finds that the Liouville operator L appearing in Eq. (2.1) splits into two components, L_0 and L_1. The component L_0 slowly varies in time, whereas L_1 contains rapidly oscillating terms at frequencies 2Ω and 4Ω. In the case of the intense external field, such that the condition

$$\Omega \gg \gamma \tag{2.3}$$

is fulfilled, one can use the secular approximation,[16,17] i.e., to retain only the slowly varying part, and the master equation (2.1) can be reduced to the form

$$\frac{\partial\tilde{\rho}}{\partial t}=-i\Omega[R_3,\tilde{\rho}]+\frac{\gamma}{8}(2R_3\tilde{\rho}R_3-R_3^2\tilde{\rho}-\tilde{\rho}R_3^2)$$
$$+\frac{\gamma}{8}(2R_{12}\tilde{\rho}R_{21}-R_{21}R_{12}\tilde{\rho}-\tilde{\rho}R_{21}R_{12})$$
$$+\frac{\gamma}{8}(2R_{21}\tilde{\rho}R_{12}-R_{12}R_{21}\tilde{\rho}-\tilde{\rho}R_{12}R_{21})\ , \qquad (2.4)$$

where $\tilde{\rho}=U\rho U^\dagger$ and U is the unitary operator of the canonical transformation (2.2); $R_3=R_{22}-R_{11}$ and $R_{ij}=c_i^\dagger c_j$ $(i,j=1,2)$ are the collective angular operators of the dressed atoms. The operators c_i and $c_j^\dagger$ satisfy the boson commutation relation

$$[c_i,c_j^\dagger]=\delta_{ij}\ ,$$

so that

$$[R_{ij},R_{lm}]=R_{im}\delta_{jl}-R_{lj}\delta_{im}\ . \qquad (2.5)$$

The equation of motion for the mean value of an atomic operator B follows from the master equation (2.4):

$$\frac{d}{dt}\langle B\rangle_t=i\Omega\langle[B,R_3]\rangle_t$$
$$+\frac{\gamma}{8}(\langle[R_3,B]R_3\rangle_t+\langle R_3[B,R_3]\rangle_t)$$
$$+\frac{\gamma}{8}(\langle[R_{21},B]R_{12}\rangle_t+\langle R_{21}[B,R_{12}]\rangle_t)$$
$$+\frac{\gamma}{8}(\langle[R_{12},B]R_{21}\rangle_t+\langle R_{12}[B,R_{21}]\rangle_t)\ , \qquad (2.6)$$

where

$$\langle B\rangle_t=\langle B(t)\rangle=\mathrm{Sp}[\rho(t)B]\ .$$

In the following, Eq. (2.4) will be used in an investigation of the dynamical evolution of the photon statistics of the photon spectral components.

III. SUB-POISSONIAN PHOTON STATISTICS OF THE SPECTRUM COMPONENTS

The dynamical evolution of the resonance fluorescence spectrum has been considered previously for a single atom.[23-27] Squeezing in time-dependent collective resonance fluorescence has been considered in Ref. 28. In this section we will investigate the dynamical evolution of the spectral components. From the canonical transformation (2.2) it follows that

$$J_{21}=\tfrac{1}{2}R_{21}-\tfrac{1}{2}R_{12}+\tfrac{1}{2}R_3\ ,$$
$$J_{12}=\tfrac{1}{2}R_{12}-\tfrac{1}{2}R_{21}+\tfrac{1}{2}R_3\ . \qquad (3.1)$$

It is easy to show that in the case of a large Rabi frequency Ω, such that the condition (2.3) is fulfilled (the secular approximation), the operators $\frac{1}{2}R_{21}$ $(\frac{1}{2}R_{12})$, $\frac{1}{2}R_3$ $(\frac{1}{2}R_3)$, and $-\frac{1}{2}R_{12}$ $(-\frac{1}{2}R_{21})$, which will be denoted as $S_1^\dagger$ (S_1), $S_0^\dagger$ (S_0), and $S_{-1}^\dagger$ (S_{-1}), can be considered as the operator sources of the spectral components at frequencies $\omega_L+2\Omega$, ω_L, and $\omega_L-2\Omega$. Further we shall consider the relative intensity fluctuations of the spectral components S_i:

$$Q_i=\frac{\langle S_i^\dagger S_i^\dagger S_i S_i\rangle_t-(\langle S_i^\dagger S_i\rangle_t)^2}{\langle S_i^\dagger S_i\rangle_t}\quad (i=0,\pm 1)\ . \qquad (3.2)$$

The factors Q_i for the atomic operators are equal (up to constant terms) to Mandel's parameters[9] and describe the photon statistics of the spectral component S_i. The condition for sub-Poissonian photon statistics of the spectral component S_i $(i=0,\pm 1)$ is taken in the form[9-11]

$$Q_i<0\ . \qquad (3.3)$$

The equations of motion for the atomic operators $\langle S_i^\dagger S_i^\dagger S_i S_i\rangle_t$ and $\langle S_i^\dagger S_i\rangle_t$ follow from Eq. (2.6). In particular, for the case when the atoms are initially in the symmetric Dicke state

$$\rho_0=|n_1,n_2\rangle\langle n_1,n_2|\ , \qquad (3.4)$$

we have

$$\frac{d}{dt}\langle R_3^2\rangle_t=-\frac{3\gamma}{2}\langle R_3^2\rangle_t+\frac{\gamma}{2}(N^2+2N)\ , \qquad (3.5)$$

$$\frac{d}{dt}\langle R_{21}R_{12}\rangle_t=-\frac{3\gamma}{2}\langle R_{21}R_{12}\rangle_t+\frac{\gamma}{4}(N^2+2N)$$
$$=\frac{d}{dt}\langle R_{12}R_{21}\rangle_t\ , \qquad (3.6)$$

$$\frac{d}{dt}\langle R_3^4\rangle_t=-5\gamma\langle R_3^4\rangle_t+\frac{\gamma}{2}(6N^2+12N-29)\langle R_3^2\rangle_t$$
$$+\frac{7\gamma}{2}(N^2+2N)\ , \qquad (3.7)$$

$$\frac{d}{dt}\langle R_{21}^2R_{12}^2\rangle_t=-5\gamma\langle R_{21}^2R_{12}^2\rangle_t+2\gamma\langle R_3^2\rangle_t$$
$$+\gamma(N^2+2N-7)\langle R_{21}R_{12}\rangle_t$$
$$=\frac{d}{dt}\langle R_{12}^2R_{21}^2\rangle_t\ . \qquad (3.8)$$

Equations (3.5)–(3.8) are exactly solvable and their solutions are

$$\langle R_3^2\rangle_t=\tfrac{1}{3}(N^2+2N)$$
$$+\exp(-3\gamma t/2)[\langle R_3^2\rangle_0-\tfrac{1}{3}(N^2+2N)]\ , \qquad (3.9)$$

$$\langle R_{21}R_{12}\rangle_t=\tfrac{1}{6}(N^2+2N)+\exp\left[-\frac{3\gamma}{2}t\right]$$
$$\times[\langle R_{21}R_{12}\rangle_0-\tfrac{1}{6}(N^2+2N)]$$
$$=\langle R_{12}R_{21}\rangle_t\ , \qquad (3.10)$$

$$\langle R_3^4\rangle_t=\langle R_3^4\rangle_s+\exp(-5\gamma t)(\langle R_3^4\rangle_0-\langle R_3^4\rangle_s-\tfrac{1}{7}X_1)$$
$$+\tfrac{2}{7}X_1\exp\left[-\frac{3\gamma t}{2}\right]\ , \qquad (3.11)$$

$$\langle R_{21}^2 R_{12}^2 \rangle_t = \langle R_{21}^2 R_{12}^2 \rangle_s + \exp(-5\gamma t)(\langle R_{21}^2 R_{12}^2 \rangle_0 - \langle R_{21}^2 R_{12}^2 \rangle_s - \tfrac{2}{7}X_2) + \tfrac{2}{7}X_2 \exp\left[-\frac{3\gamma}{2}t\right] = \langle R_{12}^2 R_{21}^2 \rangle , \quad (3.12)$$

$$\langle R_3^2 \rangle_0 = \tfrac{1}{2}(N^2+2N-D^2) , \quad (3.13)$$

$$\langle R_3^2 \rangle_s = \tfrac{1}{3}(N^2+2N) , \quad (3.14)$$

$$\langle R_{21} R_{12} \rangle_0 = \tfrac{1}{8}(N^2+2N+D^2) , \quad (3.15)$$

$$\langle R_{21} R_{12} \rangle_s = \tfrac{1}{6}(N^2+2N) , \quad (3.16)$$

$$X_1 = \tfrac{1}{2}(6N^2+12N-29)(\langle R_3^2 \rangle_0 - \langle R_3^2 \rangle_s) , \quad (3.17)$$

$$X_2 = 2(\langle R_3^2 \rangle_0 - \langle R_3^2 \rangle_s) + (N^2+2N-7)(\langle R_{21} R_{12} \rangle_0 - \langle R_{21} R_{12} \rangle_s) , \quad (3.18)$$

$$\langle R_3^4 \rangle_s = \tfrac{1}{30}[6(N^2+2N)^2 - 8(N^2+2N)] , \quad (3.19)$$

$$\langle R_{21}^2 R_{12}^2 \rangle_s = \tfrac{1}{30}(N^2+2N)(N^2+2N-3) , \quad (3.20)$$

$$\langle R_3^4 \rangle_0 = \tfrac{1}{8}[3(N^2+2N-D^2)^2 - 8(N^2+2N-D^2) + 12D^2] , \quad (3.21)$$

$$\langle R_{21}^2 R_{12}^2 \rangle_0 = \tfrac{1}{64}[(N^2+2N+D^2)^2 + \tfrac{1}{2}(N^2+2N-D^2)^2 + 4N^2+8N-30D^2] . \quad (3.22)$$

The initial population inversion of the atoms is denoted by $D = n_2 - n_1$. It is clear from Eqs. (3.10) and (3.12) that the sidebands $S_{\pm 1}$ have the same photon statistics ($Q_1 = Q_{-1}$). The factors Q_i defined by Eq. (3.2) now take the form

$$Q_0 = \frac{1}{4} \frac{\langle R_3^4 \rangle_t - (\langle R_3^2 \rangle_t)^2}{\langle R_3^2 \rangle_t} , \quad (3.23)$$

$$Q_{\pm 1} = \frac{1}{4} \frac{\langle R_{21}^2 R_{12}^2 \rangle_t - (\langle R_{21} R_{12} \rangle_t)^2}{\langle R_{21} R_{12} \rangle_t} , \quad (3.24)$$

where the quantities $\langle R_3^2 \rangle_t$, $\langle R_{21} R_{12} \rangle_t$, $\langle R_3^4 \rangle_t$, and $\langle R_{21}^2 R_{12}^2 \rangle_t$ are given by the relations (3.9)–(3.12).

In the steady-state limit relations (3.23) and (3.24) become

$$Q_0 = \tfrac{1}{15}(N^2+2N-3) , \quad (3.25)$$

$$Q_{\pm 1} = \tfrac{1}{120}(N^2+2N-18) . \quad (3.26)$$

From Eqs. (3.25) and (3.26) it is easy to see that in the steady-state limit the center component of the spectrum S_0 has no sub-Poissonian statistics ($Q_0 \geq 0$) for any number of the atoms, although the sidebands $S_{\pm 1}$ have sub-Poissonian statistics ($Q_{\pm 1} < 0$) for the cases $N = 1,2,3$. When the number of the atoms is large, all three spectral components have super-Poissonian photon statistics.

In the short-time limit $N\gamma t \ll 1$, Eqs. (3.23) and (3.24) take the following form:

$$Q_0 \cong \frac{1}{8} \frac{(N^2+2N-D^2)^2 - 8(N^2+2N-D^2) + 12D^2}{N^2+2N-D^2} , \quad (3.27)$$

$$Q_{\pm 1} \cong \frac{1}{64} \frac{(N^2+2N-D^2)^2 + 8N^2 + 16N - 60D^2}{(N^2+2N+D^2)} . \quad (3.28)$$

It is seen from the relation (3.27) that $Q_0 \geq 0$ for any value of D and N. This means that sub-Poissonian photon statistics are absent in the central spectral component S_0 for the short times. Moreover, one can show that the super-Poissonian statistics of the central spectral component are preserved during the whole time.

On the other hand, it is seen from Eq. (3.28) that $Q_{\pm 1} < 0$ for the case $|D| \cong N$. In particular, when $D = \pm N$ we find that

$$Q_{\pm 1} \cong -\frac{1}{8} \frac{3N^2 - N}{N^2 + N} , \quad (3.29)$$

which means that during the first moments of the evolution the sidebands $S_{\pm 1}$ have sub-Poissonian photon statistics ($Q_{\pm 1} < 0$) for any number of atoms N (including the case $N \gg 1$).

The dynamical evolution of the factors $Q_{\pm 1}$ as a function of the scaled time $N\gamma t$ for various numbers of the atoms is shown in Fig. 1. This figure shows reduction in Mandel's parameters $Q_{\pm 1}$ (i.e., the sidebands $S_{\pm 1}$ become more sub-Poissonian) with increasing the number of the atoms during the initial period of time. It is also seen that the characteristic time during which sub-Poissonian photon statistics are present is equal to $1/N\gamma$. From Eqs. (3.9)–(3.12) one can see that the stationary state can be reached for times proportional to $1/\gamma$.

Finally, we should like to note that the intensity of the sidebands, i.e., the quantity $\langle R_{21} R_{12} \rangle_t$, is proportional to N^2. Thus it follows that an increase in N enhances the intensity of the sidebands, which then have sub-Poissonian photon statistics over a shorter time.

IV. CONCLUSIONS

We have considered the photon statistics of the spectral components of collective time-dependent resonance

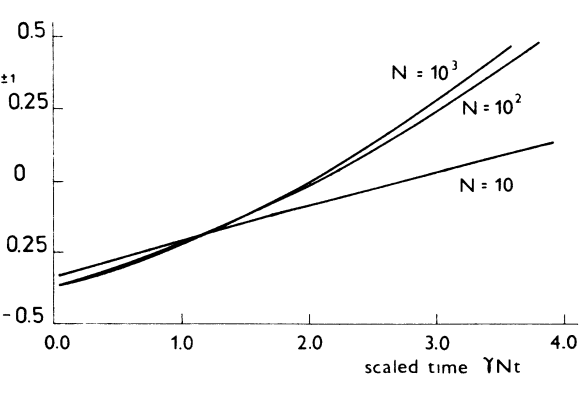

FIG. 1. Factor $Q_{\pm 1}$ as a function of scaled time $N\gamma t$ for various values of N and with $D = \pm N$.

fluorescence. It has been shown that in contrast to the steady-state case, the sidebands can exhibit sub-Poissonian statistics even for large numbers of atoms. We have shown that during the initial period of time, the sidebands $S_{\pm 1}$ become more sub-Poissonian with increasing numbers of atoms. The characteristic time during which sub-Poissonian statistics can be observed is equal to $1/N\gamma$. The occurrence of sub-Poissonian photon statistics in the sidebands of the time-dependent resonance, together with squeezing and the violation of the Cauchy-Schwarz inequality in the steady-state limit,[29-31] clearly demonstrates the quantum nature of the sidebands of the Mollow's triplet.

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[1] D. F. Walls, Nature (London) **306**, 141 (1983).

[2] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[3] J. Opt. Soc. Am. B **4**, No. 10 (1987); J. Mod. Opt. **34**, No. 6/7 (1987).

[4] H. P. Yuen, Phys. Rev. Lett. **56**, 2176 (1986).

[5] Y. Yamamoto and H. A. Haus, Rev. Mod. Phys. **58**, 1001 (1986).

[6] Y. Yamamoto, S. Machida, and O. Nilsson, Phys. Rev. A **34**, 4025 (1986).

[7] R. Short and L. Mandel, Phys. Rev. Lett. **51**, 384 (1983).

[8] L. Mandel, Phys. Rev. Lett. **49**, 136 (1982).

[9] L. Mandel, Opt. Lett. **4**, 205 (1979).

[10] L. Mandel, Phys. Scr. **T12**, 34 (1986).

[11] B. J. Dalton, Phys. Scr. **T12**, 43 (1986).

[12] A. S. Shumovsky and Tran Quang, Phys. Lett. **132A**, 164 (1988).

[13] M. A. Dupertuis and S. Stenholm, Phys. Rev. A **37**, 1226 (1988).

[14] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988).

[15] P. A. Apanasevich and J. Ja. Kilin, J. Phys. B **12**, L83 (1979).

[16] N. N. Bogolubov, Jr., F. I. Aliskenderov, A. S. Shumovsky, and Tran Quang, J. Phys. B **20**, 1885 (1987).

[17] G. S. Agarwal, L. M. Narducci, Da Hsuan Feng, and R. Gilmore, Phys. Rev. Lett. **42**, 1250 (1978).

[18] A. Aspect, G. Roger, S. Reynaud, J. Dalibard, and C. Cohen-Tannoudji, Phys. Rev. Lett. **45**, 617 (1980).

[19] F. Diedrich and H. Walther, Phys. Rev. Lett. **203**, 58 (1987).

[20] R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A **4**, 302 (1971); **4**, 854 (1971).

[21] G. S. Agarwal, *Quantum Statistical Theories of Spontaneous Emission and Their Relation to Other Approaches,* Vol. 70 of *Springer Tracts in Modern Physics* (Springer-Verlag, Berlin, 1974).

[22] J. V. Schwinger, in *Quantum Theory of Angular Momentum,* edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965).

[23] J. H. Eberly, C. Kunasz, and K. Wodkiewicz, J. Phys. B **13**, 217 (1980).

[24] X. Y. Huang, R. Tanas, and J. H. Eberly, Phys. Rev. A **26**, 892 (1982).

[25] Tak San Ho and H. Rabitz, Phys. Rev. A **37**, 1576 (1988).

[26] B. Renaud, R. M. Whitley, and C. R. Stroud, Jr., J. Phys. B **10**, 19 (1977).

[27] E. Courtens and A. Szoke, Phys. Rev. A **15**, 1588 (1977).

[28] Z. Ficek, R. Tanas, and S. Kielich, J. Phys. (Paris) **48**, 1697 (1987).

[29] N. N. Bogolubov, Jr., A. S. Shumovsky, and Tran Quang, Opt. Commun. **64**, 351 (1987).

[30] Q. V. Lawande and S. V. Lawande, Phys. Rev. A **38**, 800 (1988).

[31] N. N. Bogolubov, Jr., A. S. Shumovsky, and Tran Quang, Phys. Lett. **123A**, 71 (1987).