Spin effects in highly relativistic systems

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Coupled integral equations are developed variationally and solved numerically for the two-particle bound-state problem in quantum electrodynamics. The ansatz incorporates explicit photon degrees of freedom, and the resulting equations, when perturbatively reduced, yield the correct perturbative α^4 coefficients. The results are of interest as they may be applied to the large spin-dependent effects in the highly relativistic bound systems of quantum chromodynamics.

I. INTRODUCTION

In a previous paper,[1] which we will refer to as I, integral equations were developed variationally[2,3] for the particle-antiparticle bound-state problem in quantum electrodynamics (QED). Our motivation for this work was to explore the highly relativistic nature of bound states at high α. This, we hope, in the first case, to be useful in the incorporation of relativistic effects in quark-model building and ultimately (perhaps) to the strong-coupling problem in quantum chromodynamics (QCD) itself.

In I, our ansatz was insensitive to explicit photon degrees of freedom and thus unable to give all the spin effects, which in perturbation theory appear at order α^4. We will remedy this now. For quark-model building this is important as the spin-spin and spin-orbit effects are found to be phenomenologically large in the hadronic spectrum.

In Sec. II we develop the coupled integral equations and demonstrate that a perturbative reduction yields the α^4 coefficients of perturbation theory. In Sec. III we numerically solve the (perturbatively decoupled) integral equations and present our results and discussion.

II. INTEGRAL EQUATIONS

For completeness we give the QED Hamiltonian. In radiation gauge it is

$$H=\int d^3x(\psi^\dagger\{\boldsymbol{\alpha}\cdot[(1/i)\nabla-e\,\mathbf{A}]+\beta m\}\psi+\tfrac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)) \ , \tag{1}$$

where

$$\mathbf{E}^2=\mathbf{E}_t^2+\mathbf{E}_l^2, \quad \mathbf{E}_t=-\dot{\mathbf{A}}, \quad \mathbf{B}=\nabla\times\mathbf{E} \ ,$$

$$\mathbf{E}_l^2=\frac{e^2}{4\pi}\int d^3y\frac{\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

Our expanded Fock-space variational ansatz, sensitive to all pieces of the Hamiltonian, is

$$\overline{|e^+e^-\rangle}=|e^+e^-\rangle+|e^+e^-\gamma\rangle+|e^+e^-e^+e^-\gamma\rangle+|\gamma\rangle \ . \tag{2}$$

In this extremely compact notation we have suppressed all integrals over internal momenta, sums over spins, and polarization vectors. For example,

$$\begin{aligned}|e^+e^-\rangle&=\sum_{ss'}\int d^3pF(\mathbf{p},s,s')b^\dagger(\mathbf{p},s)d^\dagger(-\mathbf{p},s')|0\rangle \ ,\\ |e^+e^-\gamma\rangle&=\sum_{ss'\lambda}\int d^3pd^3qG(\mathbf{p},\mathbf{q},s,s',\lambda)b^\dagger(\mathbf{p},s)\\ &\quad\times d^\dagger(\mathbf{q},s')a^\dagger(-\mathbf{p}-\mathbf{q},\lambda)|0\rangle \ ,\end{aligned} \tag{3}$$

where $F(\mathbf{p},s,s')=\bar{u}(\mathbf{p},s)\Gamma v(-\mathbf{p},s')f(p)$ and $\Gamma=1,\gamma_5,\boldsymbol{\gamma}$, $\gamma_5\boldsymbol{\gamma}$ depending on whether we wish to construct a scalar, pseudoscalar, vector, or pseudovector state, respectively. The function G is constructed in such a way as to generate three-particle states with identical quantum numbers generated by F [e.g., $G(\mathbf{p},\mathbf{q},s,s',\lambda)=\boldsymbol{\epsilon}(-\mathbf{p}-\mathbf{q},\lambda)\cdot\bar{u}(\mathbf{p},s)\boldsymbol{\gamma}v(\mathbf{q},s')$ for the scalar.] Angular-momentum considerations determine the dependence of $F(G)$ on all but one(three) variable(s). It is this remaining dependence which is (in principle) variationally optimized.

To summarize the procedure, the Hamiltonian is sandwiched between the ansatz of Eq. (2). The F,G-type functions for all four Fock-space components are then variationally optimized, resulting in coupled integral equations. The equations are rather complicated but fortunately a great deal of the complexity can be circumvented by physical considerations. We note that the coupling of the $|e^+e^-e^+e^-\gamma\rangle$ Fock-space component gives rise to three types of effects. These are the vacuum energy bubble, single-fermion (Z graph) self-energy contributions, and a single-photon (Z-graph) annihilation contribution. The latter, along with the $|\gamma\rangle$ Fock-space component, gives the entire annihilation contribution (only present in the 1^{--} state). Although we will present the annihilation contribution later, we would like to emphasize that for the eventual QCD application it is not relevant as quark-antiquark color singlets do not couple to single color-octet gluons.

Thus at the level of the Fock-space expansion of Eq. (2) the only interesting dynamical bound-state effects will come from mixing with $|e^+e^-\gamma\rangle$, resulting in the following coupled equations:

$$MF(\mathbf{p},\sigma,\delta)=2E_pF(\mathbf{p},\sigma,s)+\frac{\alpha}{2\pi^2}\sum_{\sigma',\delta'}\int d^3q\frac{m^2}{E_pE_q}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}F(\mathbf{q},\sigma',\delta')\bar{u}(p,\sigma)u(-q,\sigma')\bar{v}(-q,\delta')v(p,\delta)$$
$$+\frac{\sqrt{\alpha}}{2\pi}\sum_{\rho,\lambda}d^3q\frac{m}{(E_pE_q)^{1/2}}\frac{1}{\sqrt{|\mathbf{p}-\mathbf{q}|}}[G(\mathbf{p},-\mathbf{q},\sigma,\rho,\lambda)\bar{v}(-q,\rho)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(q-p,\lambda)v(-p,\delta)$$
$$-G(\mathbf{q},-\mathbf{p},p,\delta,\lambda)\bar{u}(p,\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(p-q,\lambda)u(q,\rho)]\ , \tag{4}$$

$$MG(\mathbf{p},-\mathbf{q},\sigma,\delta,\lambda)=(E_p+E_q+|\mathbf{p}-\mathbf{q}|)G(\mathbf{p},-\mathbf{q},\sigma,\delta,\lambda)$$
$$+\frac{\alpha}{2\pi^2}\sum_{\sigma',\delta'}\int\int d^3kd^3k'\frac{m^2}{(E_pE_qE_kE_{k'})^{1/2}}G(\mathbf{k},-\mathbf{k}',\sigma',\delta',\lambda)\delta(\mathbf{p}-\mathbf{q}-\mathbf{k}+\mathbf{k}')$$
$$\times\left[\frac{\bar{u}(p,\sigma)u(-k,\sigma')\bar{v}(-k',\delta')v(q,\delta)}{|\mathbf{p}-\mathbf{k}|^2}-\frac{\bar{u}(p,\sigma)v(q,\delta)\bar{v}(-k',\delta')u(-k,\sigma')}{|\mathbf{p}-\mathbf{q}|^2}\right]$$
$$+\frac{\sqrt{\alpha}}{2\pi}\sum_{\rho}\frac{m}{(E_pE_q)^{1/2}}\frac{1}{\sqrt{|\mathbf{p}-\mathbf{q}|}}[F(\mathbf{p},\sigma,\rho)\bar{v}(-p,\rho)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(q-p,\lambda)v(-q,\delta)$$
$$-F(\mathbf{q},\rho,\delta)\bar{u}(p,\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(q-p,\lambda)u(q,\rho)]\ . \tag{5}$$

Note that by setting $G=0$ we recover the integral equations presented and solved in I.

A solution of this coupled system of Eqs. (4) and (5) would give a true variational upper bound to the two-particle bound-state mass. The problem is difficult, however, as it is effectively a coupled system of a one-dimensional and three-dimensional integral equation and there is no way to decouple the equations easily. We can, however, obtain an approximate equation[4] by setting the second term of Eq. (5) to zero, then solving for G in terms of F and obtaining a one-dimensional integral equation solely in terms of F. Certainly this procedure is justified perturbatively as the neglected term gives contributions of higher order in α. Exactly the same procedure is followed to obtain the annihilation contribution.

We now present our new improved kernels for the same states considered in I:

$$Mf(p)=2E_pf(p)-\frac{\alpha}{4\pi}\int\frac{q}{p}K(p,q)f(q)dq\ ; \tag{6}$$

pseudoscalar,

$$K(p,q)=\frac{E_p^2+E_q^2+6E_pE_q-4m^2}{E_pE_q}\ln\left|\frac{p+q}{p-q}\right|-\frac{2pq}{E_pE_q}\quad\text{(Ref. 5)}\ ; \tag{7a}$$

scalar,

$$K(p,q)=\left[\frac{(E_p^2+E_q^2)(E_pE_q-m^2)+6E_pE_q(E_pE_q+m^2)-4m^2(E_p+E_q)^2+4m^4}{pqE_pE_q}\ln\left|\frac{p+q}{p-q}\right|-2-\frac{2m^2}{E_pE_q}\right]\ ; \tag{7b}$$

pseudovector,

$$K(p,q)=\left[\frac{4(E_p^2+E_q^2)[(E_p+E_q)^2-4m^2]-16m^2(E_pE_q-m^2)}{4pqE_pE_q}\ln\left|\frac{p+q}{p-q}\right|-4-\frac{2E_p}{E_q}-\frac{2E_q}{E_p}+\frac{4m^2}{E_pE_q}\right]\ ; \tag{7c}$$

vector

$$K(p,q)=\frac{2pq}{[(2E_p^2+m^2)(2E_q^2+m^2)]^{1/2}}$$
$$\times\begin{cases}\left[\dfrac{4(E_p^2+E_q^2)[(E_p+E_q)^2+2m^2]-2m^2(E_p+E_q)^2-8m^2(E_pE_q+m^2)}{4pqE_pE_q}\ln\left|\dfrac{p+q}{p-q}\right|\right.\\ \left.\quad-4-\dfrac{2E_p}{E_q}-\dfrac{2E_q}{E_p}-\dfrac{3m^2}{E_pE_q}\right]\quad\text{(no annihilation)}\\ \left[-\dfrac{2}{3m^2}\dfrac{(2E_p^2+m^2)(2E_q^2+m^2)}{E_pE_q}\right]\quad\text{(plus annihilation)}\ .\end{cases} \tag{7d}$$

The kernels can be expanded in p/m. Retaining terms up to $(p/m)^4$ one can show that this yields the Breit Hamiltonian in the momentum representation. Then performing a nonrelativistic reduction of our functions F yields correspondingly the Clebsch-Gordan coefficients of the $(^{2s+1}L_J, J^{PC}) = (^1S_0, 0^{-+}), (^3S_1, 1^{--}), (^3P_0, 0^{++}), (^3P_1, 1^{++})$ states. Sandwiching the reduced Hamiltonian between the nonrelativistic positronium wave functions gives the following: Pseudoscalar,

$$\frac{E}{m} = 2 - \frac{\alpha^2}{4} - \frac{63\alpha^4}{192} + O(\alpha^5) ;$$

scalar ,

$$\frac{E}{m} = 2 - \frac{\alpha^2}{16} - \frac{95\alpha^4}{3072} + O(\alpha^5) ; \tag{8}$$

vector ,

$$\frac{E}{m} = 2 - \frac{\alpha^2}{4} + \begin{Bmatrix} \alpha^4/192 \text{ (no annihilation)} \\ 49\alpha^4/192 \text{ (plus annihilation)} \end{Bmatrix} + O(\alpha^5) ;$$

pseudovector ,

$$\frac{E}{m} = 2 - \frac{\alpha^2}{16} - \frac{47\alpha^4}{3072} + O(\alpha^5) .$$

These are in fact the α^4 coefficients of standard perturbation theory.[6]

We can claim that the kernels which were obtained from an approximate variational calculation (and therefore capable of being pushed to higher α) are perturbatively correct to order α^4. They will therefore yield all the correct spin-dependent physics encountered up to that level.

III. NUMERICAL RESULTS AND DISCUSSION

Equations (6) and (7) are solved numerically as in I. Again we used an optimized discretization procedure to solve the integral equations which reduced effectively to a matrix diagonalization problem. Convergence was obtained typically to five figures with 100 points for most of the curve. At the critical point α_c (i.e., where the bound-state mass vanishes) 300 points were required to obtain close to three-figure accuracy.

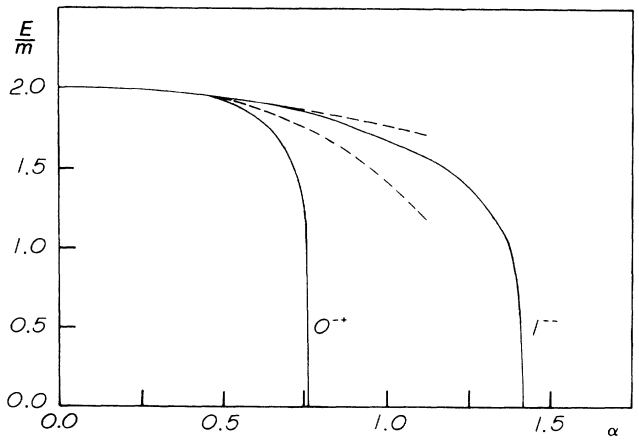

FIG. 1. The pseudoscalar-vector bound-state mass splitting. The solid curves are the solutions of the integral equations (7). The dashed curves are the result of perturbation theory, Eq. (8). The annihilation contribution is not included.

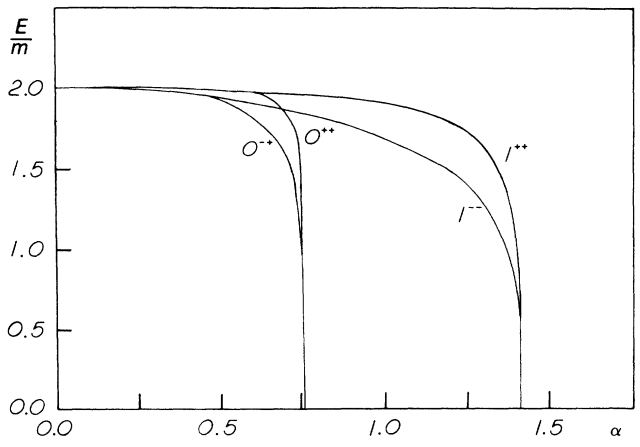

FIG. 2. Bound-state mass vs α. The quantum numbers of the curves are indicated $0^{-+}, 1^{--}, 0^{++}, 1^{++}$

In Fig. 1 we plot the bound-state mass versus α for the pseudoscalar and vector states. We compare our results to α^4 perturbation theory and note that up to $\alpha = 0.5$ the curves are indistinguishable. Keeping in mind an eventual QCD application, we note that at higher α a large pseudoscalar-vector mass splitting arises naturally. In Fig. 2 we plot the bound-state mass curves for all four states. As in I there is dramatic turnaround behavior near the critical point as well as persistence of the degeneracy of the scalar, pseudoscalar and vector, pseudovector states at the critical point. It appears that the degeneracy observed in I was not an artifact of the limited Fock-space expansion used but it survives an enhanced and thus improved calculation.

The degeneracy can be understood in the following way. Examination of the kernels reveals that in the $m = 0$ limit the corresponding kernels are identical. This is expected as $m = 0$ QED is chirally invariant and therefore could result in parity doubling in the spectrum.[3] We can explicitly solve the massless equations and obtain for α_c the values

$$\alpha_c(0^{-+}, 0^{++}) = \frac{4\sqrt{3}}{\sqrt{3}\pi/2 + 2/\pi\sqrt{3}} \approx 0.748 \text{ (Ref. 7)} ,$$
$$\alpha_c(1^{--}, 1^{++}) = \frac{\sqrt{8}}{\pi/\sqrt{8} + \sqrt{8}/\pi} \approx 1.406 . \tag{9}$$

These are very close to our best numerical values for the massive theory

$$\alpha_c(0^{++}) = 0.756 ,$$
$$\alpha_c(0^{-+}) = 0.755 ,$$
$$\alpha_c(1^{--}) = 1.414 \text{ (no annihilation)} , \tag{10}$$
$$\alpha_c(1^{++}) = 1.415 .$$

The lack of rapid convergence near the critical value is the result of large cancelations between the kinetic and potential energies which just precisely balance when the wave function goes over to the massless form $f(p)=p^{-2}$. This is a state which is extremely localized in configuration space sitting deep in the potential well at the origin. The energies involved are so large that a small fermion mass becomes irrelevant.

To summarize and conclude, within the present level of Fock-space expansion, the solution of the integral equations yield, at low α, physics that corresponds to perturbation theory up to order α^4. At higher α the equations reveal the kinematics and some dynamics of the ultrarelativistic region. Renormalization effects at this level can all be reabsorbed into a mass reparametrization. (There are no effects due to "off-shell" electron propagation in the bound state which would appear at higher order.) We feel the technique presented which is relativistic and nonperturbative and which incorporates all the short-range physics of one-gluon exchange will be useful in the study of the bound systems in quark models in QCD.

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