Linear canonical transformations of coherent and squeezed states in the Wigner phase space. III. Two-mode states

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It is shown that the basic symmetry of two-mode squeezed states is governed by the group SP(4) in the Wigner phase space which is locally isomorphic to the (3+2)-dimensional Lorentz group. This symmetry, in the Schrödinger picture, appears as Dirac's two-oscillator representation of O(3,2). It is shown that the SU(2) and SU(1,1) interferometers exhibit the symmetry of this higher-dimensional Lorentz group. The mathematics of two-mode squeezed states is shown to be applicable to other branches of physics including thermally excited states in statistical mechanics and relativistic extended hadrons in the quark model.

I. INTRODUCTION

In our previous papers,[1,2] we noted that the Wigner phase-space picture of quantum mechanics[3,4] is the natural language for squeezed states of light. The basic symmetry of the Wigner distribution function in phase space consisting of one pair of canonical variables is governed by the inhomogeneous symplectic group ISp(2). Its homogeneous subgroup Sp(2) is locally isomorphic to the (2+1)-dimensional Lorentz group. It was thus possible to study some aspects of special relativity using squeezed states of light.[1,5]

Since most of the squeezed states observed in laboratories are two-mode states,[6,7] we study in this paper canonical transformations of two-mode squeezed states within the framework of the phase-space picture of quantum mechanics. In this case, the basic symmetry of linear canonical transformations is that of Sp(4) which is locally isomorphic to the (3+2)-dimensional Lorentz group.[8,9]

The groups SU(2) and SU(1,1) play important roles in quantum optics.[10–22] They are locally isomorphic to the three-dimensional rotation group and the (2+1)-dimensional Lorentz group, respectively. Both of these groups have their respective generators of rotations. However, the interferometers of Yurke, McCall, and Klander[10] prove that the rotation generator of the SU(1,1) group is not one of the three generators of the SU(2) group. Thus there are four generators of rotations. This is an indication that those SU(2) and SU(1,1) groups cannot be isomorphic to subgroups of the familiar (3+1)-dimensional Lorentz group which has only three rotation generators.

We shall show in this paper that the above-mentioned SU(2) and SU(1,1) groups are isomorphic to the O(3) and O(1,2) subgroup of the (3+2)-dimensional Lorentz group.[13] This group, together with the (4+1)-dimensional Lorentz group, is often called the de Sitter group.[14] The de Sitter group has been extensively discussed in the literature as one of the post-Minkowskian space-time symmetries in general relativity and elementary particle physics.[15] While the de Sitter group is first introduced in physics for describing a curved space in these disciplines,[16] it contains many interesting subgroups.[17] Starting from this group, it is possible to construct representations of the Poincaré group for relativistic particles.[18,19]

As in the case of one-mode squeezed states, the local isomorphism between Sp(4) and O (3,2) allows us to study space-time symmetries of the relativistic world in terms of canonical transformations of the Wigner function for two-mode squeezed states. The correspondence between these two groups is well known among group theoreticians.[20] However, in quantum optics, proving theorems is not enough. In this paper, we shall construct explicit representations suitable for studying the two-photon system starting from canonical transformations in classical mechanics.[21]

The squeezed state of light is relatively new in physics.[22] However, its mathematical language is based on the Lorentz group and the harmonic oscillator,[1,2,23] which form the backbone for many other theories. For this reason, in addition to the O(3,2) symmetry mentioned above, the two-mode squeezed state of light shares many interesting mathematical properties with other branches of physics. Of particular interest is the reduction of a two-mode squeezed state into a one-mode state by integrating the Wigner function over one pair of canonical variables. We shall study this reduction of phase space in thermal excitations of the one-dimensional harmonic oscillator and relativistic extended particles in the quark model.

In Sec. II, we construct a representation of the group of homogeneous linear canonical transformations using the Pauli spin matrices. It is shown in Sec. III that transformations of Sp(4) consist of two pairs of rotations and squeezes of the Wigner function in four-dimensional phase space. Section IV is devoted to the discussion of the O(3,2) de Sitter group and its correspondence to the group of homogeneous linear canonical transformations in four-dimensional phase space.

In Sec. V, Dirac's two-oscillator representation of the O(3,2) de Sitter group is constructed from Sp(4) which is the group of linear canonical transformations in four-dimensional phase space. In Sec. VI, we study symmetries of the O(3,2) de Sitter group which may be observed in optics laboratories. We discuss also how to extract measurable numbers from the Wigner function in four-dimensional phase space.

In Sec. VII, we discuss the connection between the density matrix and the Wigner distribution function. The density matrix is the standard language for incoherent thermal excitations. In Sec. VIII, it is shown that the mathematics of Lorentz-boosted relativistic hadrons is the same as the harmonic oscillator in a thermally excited state. The conclusions of Secs. VII and VIII are that there are physical processes which lead to a radial expansion in phase space. Thus, in Sec. IX, we study group theoretical implications of this noncanonical transformation in relation to canonical transformations.

II. CANONICAL TRANSFORMATIONS IN CLASSICAL MECHANICS

For a dynamical system consisting of two pairs of canonical variables x_1,p_1 and x_2,p_2, we can introduce the four-dimensional coordinate system

$$(\eta_1,\eta_2,\eta_3,\eta_4)=(x_1,x_2,p_1,p_2) . \tag{2.1}$$

Then the transformation of the variables from η_i to ξ_i is canonical if

$$MJ\tilde{M}=J , \tag{2.2}$$

where

$$M_{ij}=\frac{\partial}{\partial\eta_j}\xi_i$$

and

$$J=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} . \tag{2.3}$$

For linear canonical transformations, we can work with the group of 4×4 real matrices satisfying the condition of Eq. (2.2). This group is called the four-dimensional symplectic group or Sp(4). While there are many physical applications of this group,[8,9] we are interested here in constructing the representations relevant to the study of two-mode squeezed states.

It is more convenient to discuss this group in terms of its generators G, defined as

$$M=e^{-i\alpha G} , \tag{2.4}$$

where G represents a set of purely imaginary 4×4 matrices. The symplectic condition of Eq. (2.2) dictates that G be symmetric and anticommute with J or be antisymmetric and commute with J.

In terms of the Pauli spin matrices and the 2×2 identity matrix, we can construct the following four antisymmetric matrices which commute with J of Eq. (2.3):

$$\begin{gathered} J_1=\frac{i}{2}\begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}, \quad J_2=\frac{1}{2}\begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} , \\ J_3=\frac{i}{2}\begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}, \quad J_0=\frac{i}{2}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} . \end{gathered} \tag{2.5}$$

The following six symmetric generators anticommute with J:

$$\begin{gathered} K_1=\frac{i}{2}\begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}, \quad K_2=\frac{i}{2}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad K_3=\frac{i}{2}\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} , \\ Q_1=\frac{i}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \quad Q_2=-\frac{i}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad Q_3=\frac{i}{2}\begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} . \end{gathered} \tag{2.6}$$

These generators satisfy the commutation relations

$$\begin{gathered} [J_i,J_j]=i\epsilon_{ijk}J_k, \quad [J_i,J_0]=0 , \\ [J_i,K_j]=-i\epsilon_{ijk}K_k, \quad [J_i,Q_j]=-i\epsilon_{ijk}Q_k , \\ [K_i,K_j]=[Q_i,Q_j]=-i\epsilon_{ijk}J_k , \\ [K_i,Q_j]=-i\delta_{ij}J_0 , \\ [K_i,J_0]=iQ_i, \quad [Q_i,J_0]=iK_i . \end{gathered} \tag{2.7}$$

The group of homogeneous linear transformations with this closed set of generators is called the symplectic group Sp(4).

The above generators will be useful in studying two-mode squeezed states. This set of generators is not the only solution of the commutation relations. As we shall see in Sec. IV, the generators of the O(3,2) de Sitter group also satisfy the same set of commutation relations. This allows to study the space-time symmetry in terms of modern optics and vice versa.

III. LINEAR CANONICAL TRANSFORMATIONS OF THE WIGNER FUNCTION

If $\psi(x)$ is a wave function in the Schrödinger picture of quantum mechanics, then the Wigner distribution function is[1-4]

$$W(x,p)=\frac{1}{\pi}\int e^{2ipy}\psi^*(x+y)\psi(x-y)dy \ . \tag{3.1}$$

The parameters x and p are c numbers. Therefore this form is a distribution function defined over two-dimensional phase space of x and p.

The basic properties of the Wigner function have been exhaustively discussed in the literature.[4] In quantum optics, every coherent and squeezed state can be obtained from the vacuum state through a canonical transformation of the vacuum state, whose Wigner function is[1,2]

$$W(x,p)=\left[\frac{1}{\pi}\right]\exp[-(x^2+p^2)] \ . \tag{3.2}$$

The group of linear canonical transformations consists of translations, rotations, and squeezes.

We can obtain a coherent state by translating this function, whose generators are

$$N_1=-i\frac{\partial}{\partial x}, \quad N_2=-i\frac{\partial}{\partial p} \ . \tag{3.3}$$

The role of translations is not essential in two-mode squeezed states. Furthermore, it is very easy to deal with translational symmetry in phase space. Thus, we shall return to these operators in Sec. IX where the thermal excitation of one-mode states is discussed.

The squeezes along the x axis and along the direction which makes an angle of 45° with the x axis are generated by

$$B_1=\frac{i}{2}\left[x\frac{\partial}{\partial x}-p\frac{\partial}{\partial p}\right], \quad B_2=\frac{i}{2}\left[x\frac{\partial}{\partial p}+p\frac{\partial}{\partial x}\right], \tag{3.4}$$

respectively, while rotations around the origin are generated by

$$L=\frac{i}{2}\left[p\frac{\partial}{\partial x}-x\frac{\partial}{\partial p}\right] \ . \tag{3.5}$$

These operators satisfy the commutation relations for the generators of the $(2+1)$-dimensional Lorentz group:

$$[B_1,B_2]=-iL, \quad [L,B_2]=-iB_1, \quad [L,B_1]=iB_2 \ . \tag{3.6}$$

We can change the sign of two of the three generators without affecting the commutation relations. The group generated by B_1, B_2, and L is called the symplectic group Sp(2) which is unitarily equivalent to SU(1,1) and is locally isomorphic to the $(2+1)$-dimensional Lorentz group.

If the wave function depends on two coordinate variables x_1 and x_2, the Wigner function will be a function of two pairs of canonical variables:

$$W(x_1,x_2,p_1,p_2)=\frac{1}{\pi}\int \exp[2i(p_1y_1+p_2y_2)] \times\psi^*(x_1+y_1,x_2+y_2) \times\psi(x_1+y_1,x_2+y_2)dy_1dy_2 \ . \tag{3.7}$$

Let us then consider the ground-state wave function for this two-oscillator system:

$$\psi(x_1,x_2)=\left[\frac{1}{\pi}\right]^{1/2}\exp\left[-\left[\frac{1}{2}\right](x_1^2+x_2^2)\right] \ . \tag{3.8}$$

Then the Wigner function will be

$$W(x_1,x_2,p_1,p_2)=\left[\frac{1}{\pi}\right]^2\exp[-(x_1^2+x_2^2+p_1^2+p_2^2)] \ . \tag{3.9}$$

We are now interested in performing rotations and squeezes with respect to two pairs of variables. There are three possible ways of choosing two pairs among the four variables, and the above Wigner function can be written in three different ways:

$$\begin{aligned} W(x_1,x_2,p_1,p_2) &= \left[\frac{1}{\pi}\right]^2\exp[-(x_1^2+p_1^2)]\exp[-(x_2^2+p_2^2)] \\ &= \left[\frac{1}{\pi}\right]^2\exp[-(x_1^2+p_2^2)]\exp[-(x_2^2+p_1^2)] \\ &= \left[\frac{1}{\pi}\right]^2\exp[-(x_1^2+x_2^2)]\exp[-(p_1^2+p_2^2)] \ . \end{aligned} \tag{3.10}$$

For the Wigner function, the generators of Sp(4) given in Eqs. (2.5) and (2.6) can be written in differential forms. They can be written in terms of rotation or squeeze generators. There are four generators of rotations:

$$\begin{aligned} J_1 &= +\left[\frac{i}{2}\right]\left[\left[x_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_1}\right]+\left[x_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_2}\right]\right], \\ J_2 &= -\left[\frac{i}{2}\right]\left[\left[x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right]+\left[p_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial p_1}\right]\right], \\ J_3 &= +\left[\frac{i}{2}\right]\left[\left[x_1\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_1}\right]-\left[x_2\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_2}\right]\right], \\ J_0 &= +\left[\frac{i}{2}\right]\left[\left[x_1\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_1}\right]+\left[x_2\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_2}\right]\right], \end{aligned} \tag{3.11}$$

and there are six squeeze generators:

$$
\begin{aligned}
K_1 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_1}\right)\right. \\
&\qquad \left. -\left(x_2\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_2}\right)\right\}, \\
K_2 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial x_1}-p_1\frac{\partial}{\partial p_1}\right)\right. \\
&\qquad \left. +\left(x_2\frac{\partial}{\partial x_2}-p_2\frac{\partial}{\partial p_2}\right)\right\}, \\
K_3 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_1}\right)\right. \\
&\qquad \left. +\left(x_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_2}\right)\right\}, \\
Q_1 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial x_2}-p_1\frac{\partial}{\partial p_1}\right)\right. \\
&\qquad \left. -\left(x_2\frac{\partial}{\partial x_2}-p_2\frac{\partial}{\partial p_2}\right)\right\}, \\
Q_2 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_1}\right)\right. \\
&\qquad \left. +\left(x_2\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_2}\right)\right\}, \\
Q_3 &= +\left(\frac{i}{2}\right)\left\{\left(x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}\right)\right. \\
&\qquad \left. -\left(p_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial p_2}\right)\right\}.
\end{aligned}
\tag{3.12}
$$

According to the above expressions, we can separate the four-dimensional phase space into a pair of two-dimensional spaces, and perform canonical transformations in each space. If all the transformations are done within a given pair, the symmetry group is the same as the one in the two-dimensional phase space. If, on the other hand, we perform a transformation generated by K_1 followed by a K_3 or Q_3 transformation, the pairs do not remain separated. The group Sp(4) is much more complicated than a direct product of two Sp(2) groups.

IV. (3+2)-DIMENSIONAL DE SITTER GROUP

In the usual $(3+1)$-dimensional Lorentz group, transformations on the vector space (x,y,z,t) leave $x^2+y^2+z^2-t^2$ invariant. In the $(3+2)$-dimensional de Sitter group, transformations on the vector space (x,y,z,s,t) leave the quantity

$$x^2+y^2+z^2-s^2-t^2 \tag{4.1}$$

invariant. In this space, there are two timelike variables.

There are three generators of rotations in the space of x, y, and z. They are

$$
J_1=\begin{pmatrix} 0&0&0&0&0\\ 0&0&-i&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},\quad
J_2=\begin{pmatrix} 0&0&i&0&0\\ 0&0&0&0&0\\ -i&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},
$$
$$
J_3=\begin{pmatrix} 0&-i&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}. \tag{4.2}
$$

These generators satisfy the commutation relations for the three-dimensional rotation group:

$$[J_i,J_j]=i\epsilon_{ijk}J_k\ . \tag{4.3}$$

In addition, it is possible to perform rotations in the plane of t and s, generated by

$$
J_0=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&i&0 \end{pmatrix}. \tag{4.4}
$$

This matrix commutes with the generators of the three-dimensional rotation group given in Eq. (4.2):

$$[J_0,J_i]=0\ . \tag{4.5}$$

There are three generators of boosts with respect to three spacelike directions and the first time variable t. They are

$$
K_1=\begin{pmatrix} 0&0&0&i&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},\quad
K_2=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&i&0\\ 0&0&0&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0 \end{pmatrix},
$$
$$
K_3=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&i&0\\ 0&0&i&0&0\\ 0&0&0&0&0 \end{pmatrix}. \tag{4.6}
$$

These generators satisfy the commutation relations for the $(3+1)$-dimensional Lorentz group:

$$[K_i,K_j]=-i\epsilon_{ijk}J_k,\quad [K_i,J_j]=i\epsilon_{ijk}K_k\ . \tag{4.7}$$

In addition, there are three boosts with respect to the second time variable s. They are generated by

$$Q_1=\begin{bmatrix}0&0&0&0&i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{bmatrix},\quad Q_2=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0\end{bmatrix},$$
$$Q_3=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&i\\0&0&0&0&0&0\\0&0&0&i&0&0\end{bmatrix}. \tag{4.8}$$

These generators also satisfy the commutation relations for the $(3+1)$-dimensional Lorentz group:

$$[Q_i,Q_j]=-i\epsilon_{ijk}J_k,\quad [Q_i,J_j]=i\epsilon_{ijk}Q_k\ . \tag{4.9}$$

These Q matrices satisfy the following three sets of commutation relations, with K_i and J_0.

$$\begin{aligned}&[K_i,J_0]=Q_i,\quad [Q_i,J_0]=-K_i\ ,\\&[K_i,Q_j]=-i\delta_{ij}J_0\ ,\end{aligned} \tag{4.10}$$

which are like those for the $(2+1)$-dimensional Lorentz group. Indeed, they generate such a group. However, they are not usual transformations with two spacelike coordinates and one timelike coordinate. Instead, they represent transformations with two timelike coordinates and one spacelike direction. This is the characteristic of the O(3,2) group which has two timelike directions. As we shall see in Sec. VI, this $(1+2)$-dimensional Lorentz group is the most important subgroup of O(3,2) in the physics of two-mode squeezed states.

V. DIRAC'S TWO-OSCILLATOR FORMALISM

We are now interested in corresponding transformation on the Schrödinger wave function. For this purpose, let us write the transformation on $W(x,p)$ in terms of the generators

$$e^{-i\epsilon G(x,p)}W(x,p)\ , \tag{5.1}$$

where G is the generator of transformations applicable to the Wigner function, and ϵ is the transformation parameter. We then write the corresponding transformation in the Schrödinger picture as

$$e^{-i\epsilon\hat{G}(x)}\psi(x)\ . \tag{5.2}$$

The operator $\hat{G}(x)$ depends only on x. Then for small ϵ,

$$\begin{aligned}G(x,p)W(x,p)=\frac{1}{\pi}\int e^{2ipy}\{&\psi^*(x+y)[\hat{G}(x-y)\psi(x-y)]\\&-[\hat{G}(x+y)\psi(x+y)]^*\\&\times\psi(x-y)\}dy\ .\end{aligned} \tag{5.3}$$

This relation can be generalized to $\psi(x_1,x_2)$ and $W(x_1,x_2,p_1,p_2)$. We can see from the above expression that the application of $\hat{G}=x_ix_j$ on $\psi(x_1,x_2)$ leads to

$$G=i\left[x_i\frac{\partial}{\partial p_j}+x\frac{\partial}{\partial p_i}\right] \tag{5.4}$$

on the Wigner function $W(x_1,x_2,p_1,p_2)$. This is valid also for $i=j$. The operator

$$\hat{G}=\left[-i\frac{\partial}{\partial x_i}\right]\left[-i\frac{\partial}{\partial x_j}\right], \tag{5.5}$$

applicable to $\psi(x_1,x_2)$, corresponds to

$$G=-i\left[p_i\frac{\partial}{\partial x_j}+p_j\frac{\partial}{\partial x_i}\right] \tag{5.6}$$

in phase space. The operation of

$$\hat{G}=-\left[\frac{i}{2}\right]\left[x_i\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x_j}x_i\right] \tag{5.7}$$

in the Schrödinger picture leads to

$$G=-i\left[x_i\frac{\partial}{\partial x_j}-p_j\frac{\partial}{\partial p_i}\right] \tag{5.8}$$

applicable to the Wigner function.

It is now possible to translate the operators in the phase-space picture given in Eqs. (3.11) and (3.12) into those applicable to the Schrödinger wave function. They take the form

$$\begin{aligned}&\hat{J}_1=\tfrac{1}{2}(x_1x_2+\hat{p}_2\hat{p}_1),\quad \hat{J}_2=\tfrac{1}{2}(x_1\hat{p}_2-x_2\hat{p}_1)\ ,\\&\hat{J}_3=\tfrac{1}{4}[(x_1x_1+\hat{p}_1\hat{p}_1)-(x_2x_2+\hat{p}_2\hat{p}_2)]\ ,\\&\hat{J}_0=\tfrac{1}{4}[(x_1x_1+\hat{p}_1\hat{p}_1)+(x_2x_2+\hat{p}_2\hat{p}_2)]\ ,\\&\hat{K}_1=-\tfrac{1}{4}[(x_1x_1-\hat{p}_1\hat{p}_1)-(x_2x_2-\hat{p}_2\hat{p}_2)]\ ,\\&\hat{K}_2=\tfrac{1}{4}(x_1\hat{p}_1+\hat{p}_1x_1+x_2\hat{p}_2+\hat{p}_2x_2)\ ,\\&\hat{K}_3=-\tfrac{1}{2}(x_1x_2-\hat{p}_1\hat{p}_2)\ ,\\&\hat{Q}_1=\tfrac{1}{4}(x_1\hat{p}_1+\hat{p}_1x_1-x_2\hat{p}_2-\hat{p}_2x_2)\ ,\\&\hat{Q}_2=\tfrac{1}{4}[(x_1x_1-\hat{p}_1\hat{p}_1)+(x_2x_2-\hat{p}_2\hat{p}_2)]\ ,\\&\hat{Q}_3=-\tfrac{1}{2}(x_1\hat{p}_2+x_2\hat{p}_1)\ ,\end{aligned} \tag{5.9}$$

where

$$\hat{p}_1=-i\frac{\partial}{\partial x_1}\ \text{and}\ \hat{p}_2=-i\frac{\partial}{\partial x_2}\ .$$

Except K_2 and Q_1, the above set of generators is the same as that given in Dirac's 1963 paper.[13] We are using here the Hermitian form for K_2 and Q_1.

All of the above operators are Hermitian. Indeed, also in the case of two-mode squeezed states, linear canonical transformations in the phase-space picture correspond to unitary transformations in the Schrödinger picture of quantum mechanics. In quantum optics, it is more convenient to express these generators in terms of the annihilation and creation operators: $a=(x+i\hat{p})/\sqrt{2}$ and $a^\dagger=(x-i\hat{p})/\sqrt{2}$. They are[19]

$$\begin{aligned}
&\hat{J}_1=\tfrac{1}{2}(a_1^\dagger a_2+a_2^\dagger a_1), \quad \hat{J}_2=\frac{1}{2i}(a_1^\dagger a_2-a_2^\dagger a_1)\ ,\\
&\hat{J}_3=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2), \quad \hat{J}_0=\tfrac{1}{2}(a_1^\dagger a_1+a_2 a_2^\dagger)\ ,\\
&\hat{K}_1=-\tfrac{1}{4}(a_1^\dagger a_1^\dagger+a_1a_1-a_2^\dagger a_2^\dagger-a_2a_2)\ ,\\
&\hat{K}_2=-\frac{i}{4}(a_1^\dagger a_1^\dagger-a_1a_1+a_2^\dagger a_2^\dagger-a_2a_2)\ ,\\
&\hat{K}_3=\tfrac{1}{2}(a_1^\dagger a_2^\dagger+a_1a_2)\ ,\\
&\hat{Q}_1=-\frac{i}{4}(a_1^\dagger a_1^\dagger-a_1a_1-a_2^\dagger a_2^\dagger+a_2a_2)\ ,\\
&\hat{Q}_2=\tfrac{1}{4}(a_1^\dagger a_1^\dagger+a_1a_1+a_2^\dagger a_2^\dagger+a_2a_2)\ ,\\
&\hat{Q}_3=-\frac{i}{2}(a_1^\dagger a_2^\dagger-a_1a_2)\ .
\end{aligned} \tag{5.10}$$

VI. OBSERVABLE SYMMETRIES OF O(3,2)

It is possible to study squeezed states in terms of special relativity and vice versa by exploiting the correspondence between SU(1,1) and O(2,1).[1,5,10] In the single-mode case, the Wigner rotation and the Thomas precession were the key items in special relativity which correspond to repeated noncollinear squeezes in quantum optics.[1,5,24]

There are many O(2,1)-like subgroups of O(3,2). It is thus possible to propose many experiments to test the Thomas or Wigner rotation. This is not what we intend to discuss in this section. The key question is whether there are experiments to test the symmetry of the O(3,2) group. It would be ideal if we could design an experiment where all of the ten generators of Dirac's two-oscillator representation, but we are not able to suggest such an experiment at this time. On the other hand, it is still possible to detect the O(3,2) characteristics with its subgroups.

The O(3,2) de Sitter group has many interesting subgroups. One important subgroup is the "ordinary" $(3+1)$-dimensional Lorentz group, which in turn has as subgroups one three-dimensional rotation group and three $(2+1)$-dimensional Lorentz groups. We could therefore be misled to believe that the symmetries of SU(2) [or O(3)] and SU(1,1) [or O(2,1)] are those from the ordinary Lorentz group. If this were the case, there would not be any need for the O(3,2) symmetry. Since there are only three generators of rotations in O(3,1), in order that the O(3,1) group be the fundamental group, it is necessary for the rotation generator of the O(2,1) subgroup be one of the three generators of rotations forming the rotation subgroup.

On the other hand, let us look at the interferometers of Yurke, McCall, and Klauder. Their SU(2) interferometer is based on $\hat{J}_1$, $\hat{J}_2$, and $\hat{J}_3$, with

$$[\hat{J}_i,\hat{J}_j]=i\epsilon_{ijk}\hat{J}_k\ , \tag{6.1}$$

where the forms of these operators are given in Eq. (5.11), while their SU(1,1) interferometer is based on $\hat{J}_0$, $\hat{K}_3$, and $\hat{Q}_3$ satisfying the commutation relations

$$[\hat{K}_3,\hat{Q}_3]=-i\hat{J}_0, \quad [\hat{J}_0,\hat{K}_3]=i\hat{Q}_3, \quad [\hat{J}_0,\hat{Q}_3]=-i\hat{K}_3\ . \tag{6.2}$$

It is important to note that $\hat{J}_0$, while being a generator of rotation, is not one of the generators of the SU(2) group given in Eq. (6.1). The SU(1,1) group of Yurke, McCall, and Klauder is clearly one of the three O(1,2) subgroups of the (3,2) de Sitter group. It is not a subgroup of the $(3+1)$-dimensional Lorentz group, while the rotation group is.

In the system of interferometers of Yurke, McCall, and Klauder, there are three generators for SU(2) and three for SU(1,1). There are altogether six generators. They do not form a closed system of commutation relations. They need four more generators. If we add them together, the result is the set of ten generators of the two-oscillator representation of the O(3,2) group.

One of the characteristics of O(3,2) is that $\hat{K}_3$ in the SU(1,1) interferometer is capable of forming another SU(1,1) group. It would be very helpful if we could design experiments to test the set of the O(2,1) commutation relations involving $\hat{K}_3$ and the generators not contained in the interferometer of Yurke, McCall, and Klauder.[10] K_3 can form the following sets of closed commutation relations:

$$[\hat{J}_1,\hat{K}_3]=-i\hat{K}_2, \quad [\hat{J}_1,\hat{K}_2]=i\hat{K}_3, \quad [\hat{K}_2,\hat{K}_3]=-i\hat{J}_1\ , \tag{6.3}$$

or

$$[\hat{J}_2,\hat{K}_3]=i\hat{K}_1, \quad [\hat{J}_2,\hat{K}_1]=-i\hat{K}_3, \quad [\hat{K}_1,\hat{K}_3]=i\hat{J}_2\ . \tag{6.4}$$

The same reasoning is applicable to the expressions where $\hat{K}_1$ and $\hat{K}_2$ are replaced by $\hat{Q}_1$ and $\hat{Q}_2$, respectively. The experiment based on one or more of the above four sets of commutation relations will prove the existence of both the O(2,1) and O(1,2) symmetries. This will reinforce the evidence of the O(3,2) symmetry in the two-mode system.

Perhaps, the most ambitious experiment on the O(3,2) symmetry in two-mode optics can be stated in the following way. The original purpose of introducing the O(3,2) de Sitter group was to study the curvature of the universe.[14,16] The correspondence between this group and Sp(4) may allow us to design an optical analog computer for studying the curvature of the universe. This is a future possibility.

In the meantime, let us study how we can extract measurable numbers from the Wigner function. In quantum mechanics, we calculate those numbers from the overlap of distribution functions and the expectation value of operators[2,4] for two given wave functions $\psi(x)$ and $\phi(x)$, and their corresponding Wigner function $W_\psi(x,p)$ and $W_\phi(x,p)$. Then the transition probability takes the form[2-4]

$$|(\phi(x),\psi(x))|^2=2\pi\int W_\psi(x,p)W_\phi(x,p)dx\,dp\ . \tag{6.5}$$

This expression is useful when we calculate the probability of a certain state being in a particular eigenstate. For two pairs of canonical variables, the overlap integral may be written as[25]

$$|(\phi(x_1,x_2),\psi(x_1,x_2))|^2 = (2\pi)^2 \int W_\psi(x_1,x_2;p_1,p_2) \times W_\phi(x_1,x_2;p_1,p_2)dx_1dx_2dp_1dp_2 \; . \tag{6.6}$$

Let us consider next the expectation value of an operator applicable to $\psi(x)$ or the momentum wave function $f(p)$. If the operator Q is a function only of x or p, then the expectation value is[2,26]

$$\langle Q \rangle = (\psi(x), Q(x)\psi(x)) = \int Q(x)W(x,p)dx\,dp \; , \tag{6.7}$$

with a similar expression for $Q(p)$. If Q is a function of both x and p, we are not aware of any simple expression. On the other hand, it is possible to prove that[2,4]

$$\int W(x,p)(x^n p^m)dx\,dp = (-i)^m(\tfrac{1}{2})^n \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \left[\psi(x), x^{n-r} \left[\frac{\partial}{\partial x} \right]^m x^r \psi(x) \right] . \tag{6.8}$$

Among the many operators in quantum mechanics, the photon number and the (photon number)^2 operators are the two most important operators. In the Schrödinger representation, the number operator takes the form

$$N = \frac{1}{2} \left[x^2 - \left[\frac{\partial}{\partial x} \right]^2 - 1 \right] . \tag{6.9}$$

This means that in the Wigner phase-space picture, the expectation value of this operator is

$$\langle N \rangle = \frac{1}{2} \int (x^2+p^2-1)W(x,p)dx\,dp \; , \tag{6.10}$$

which is a straightforward application of Eq. (6.8). The formula N^2 is more complicated because there is a term proportional to x^2p^2. The application of Eq. (6.8) leads to

$$\langle N^2 \rangle = \frac{1}{4} \int [(x^2+p^2-1)^2-1]W(x,p)dx\,dp \; . \tag{6.11}$$

In the single-mode case, it was observed that every squeezed state can be represented by the Wigner function of the Gaussian form localized within an elliptic region in phase space. If the region of localization is a circle, the Wigner function corresponds to an unsqueezed coherent state. The circle centered around the origin is the vacuum state. It is possible to obtain a coherent or squeezed state by canonically transforming the circle centered around the origin. If we translate the circle, the result is a coherent state. Without loss of generality, we can obtain every squeezed state by squeezing the vacuum followed by translation.

It is quite clear from Eq. (5.11) that $\hat{J}_0$ measures the total number of photons. It is also clear that $\hat{J}_3$ measures the difference between the photon numbers of the first and second kinds.

$$\langle N_1 \rangle = \langle (\hat{J}_0 + \hat{J}_3) \rangle = \frac{1}{2} \int (x_1^2+p_1^2-1) \times W(x_1,x_2,p_1,p_2)dx_1dx_2dp_1dp_2 \; , \tag{6.12}$$

$$\langle N_2 \rangle = \langle (\hat{J}_0 - \hat{J}_3 + \tfrac{1}{2}) \rangle = \frac{1}{2} \int (x_2^2+p_2^2-1) \times W(x_1,x_2,p_1,p_2)dx_1dx_2dp_1dp_2 \; .$$

Likewise,

$$\langle N_1^2 \rangle = \frac{1}{4} \int [(x_1^2+p_1^2-1)^2-1] \times W(x_1,x_2,p_1,p_2)dx_1dx_2dp_1dp_2 \; ,$$

$$\langle N_2^2 \rangle = \frac{1}{4} \int [(x_2^2+p_2^2-1)^2-1] \times W(x_1,x_2,p_1,p_2)dx_1dx_2dp_1dp_2 \; , \tag{6.13}$$

$$\langle N_1 N_2 \rangle = \frac{1}{4} \int (x_2^2+p_2^2-1)(x_1^2+p_1^2-1) \times W(x_1,x_2,p_1,p_2)dx_1dx_2dp_1dp_2 \; .$$

These quantities are needed in calculating the photon number variations $\langle (\Delta N)^2 \rangle$, $\langle (\Delta N_1)^2 \rangle$, and $\langle (\Delta N_2)^2 \rangle$.

VII. DENSITY MATRICES IN THE PHASE-SPACE PICTURE OF QUANTUM MECHANICS

In quantum mechanics and quantum optics, we often have to deal with nonpure mixed states. For instance, in the case of the one-dimensional harmonic oscillator, the most general form of normalized solution is

$$\psi(x,t) = e^{-i\omega t/2} \sum_n C_n e^{-in\omega t} \psi_n(x) \; , \tag{7.1}$$

where $\psi_n(x)$ is the solution of the time-independent oscillator equation with the energy level $\omega(n+\frac{1}{2})$. The wave function $\psi(x,t)$ is normalized:

$$(\psi(x,t),\psi(x,t)) = \sum_n |C_n|^2 = 1 \; . \tag{7.2}$$

The expectation value $\langle A \rangle = (\psi(x,t), A\psi(x,t))$ of an operator $A(x)$ can be written as

$$\langle A \rangle = \sum_n |C_n|^2 (\psi_n(x), A(x)\psi_n(x)) + \sum_{\substack{n,m \\ n\neq m}} C_m^* C_n e^{i\omega(m-n)t} (\psi_m(x), A(x)\psi_n(x)) \; . \tag{7.3}$$

If we take the ensemble average for many oscillators prepared independently with different initial times, the net effect is the same as that of taking the time average, and the second term in the above expression vanishes. As a consequence, the ensemble average is

$$\overline{\langle A \rangle} = \sum_n |C_n|^2 (\psi_n(x), A(x)\psi_n(x)) \; . \tag{7.4}$$

We use the word "mixed" or "nonpure" in order to describe this ensemble average.

It is very convenient to treat this problem if we introduce the density matrix defined as[26]

$$\rho(x,x')=\sum_n |C_n|^2\psi_n(x)\psi_n^*(x') \tag{7.5}$$

and

$$\overline{\langle A\rangle}=\int dx'\int A(x',x)\rho(x,x')dx , \tag{7.6}$$

with

$$A(x',x)=\delta(x'-x)A(x) .$$

The above expression is then the trace of the matrix $A(x',x)\rho(x,x')$ often written as

$$\overline{\langle A\rangle}=\mathrm{Tr}(\rho A) . \tag{7.7}$$

If $C_n=\delta_{nm}$ for a given value of m, we say that the system is in a pure state. Otherwise, the system is in a mixed state.

It is possible to derive this result without taking the ensemble average, if we introduce an auxiliary Hilbert space consisting of $\psi_n(\bar{x})$ and attach it to C_n.[26–29] We can consider the wave function of the form

$$\psi(x,x')=\sum_n [C_n\psi_n(\bar{x})]\psi_n(x) . \tag{7.8}$$

The auxiliary coordinate $\bar{x}$ is called the "shadow" coordinate in the literature. It is possible to derive the result of Eq. (7.4) by treating $\psi(x,\bar{x})$ as a pure-state wave function defined in the total Hilbert space consisting both of $\psi_n(x)$ and $\psi_n(\bar{x})$. The expectation value of $A(x)$

$$\langle A\rangle=\sum_{n,m} C_m^*C_n(\psi_m(\bar{x}),\psi_n(\bar{x}))(\psi_m(x),A(x)\psi_n(x)) \tag{7.9}$$

is the same as the ensemble average $\langle\overline{A}\rangle$ given in Eq. (7.4). It is possible to obtain the density matrix by integrating $\psi(x,\bar{x})\psi^*(x',\bar{x})$ over the $\bar{x}$ variable:

$$\rho(x,x')=\int \psi(x,\bar{x})\psi^*(x',\bar{x})d\bar{x} . \tag{7.10}$$

The evaluation of this integral leads to the expression for $\rho(x,x')$ given in Eq. (7.5).

The best known example of the above procedure is to derive the thermally excited oscillator state with the density matrix:[26]

$$\rho_T(x,x')=(1-e^{-\omega/kT})\sum_n e^{-n\omega/kT}\psi_n(x)\psi_n^*(x') . \tag{7.11}$$

It is possible to obtain this form by taking an ensemble average. However, we are interested in deriving this expression by introducing a shadow coordinate. Let us start with the ground-state harmonic oscillator wave function

$$\psi_0(x,\bar{x})=\psi_0(x)\psi_0(\bar{x})=\left[\frac{1}{\pi}\right]^{1/2}\exp[-(\tfrac{1}{2})(x^2+\bar{x}^2)] , \tag{7.12}$$

where x is measured in units of $1/\sqrt{m\omega}$. Let us now make a coordinate transformation:

$$\begin{bmatrix} x' \\ \bar{x}' \end{bmatrix}=\begin{bmatrix} 1/(1-e^{-\omega/kT})^{1/2} & e^{-\omega/2kT}/(1-e^{-\omega/kT})^{1/2} \\ e^{-\omega/2kT}/(1-e^{-\omega/kT})^{1/2} & 1/(1-e^{-\omega/kT})^{1/2} \end{bmatrix}\begin{bmatrix} x \\ \bar{x} \end{bmatrix} . \tag{7.13}$$

This leads to the squeezed wave function of the form

$$\psi_T(x,\bar{x})=\left[\frac{1}{\pi}\right]^{1/2}\exp\left\{-\left[\frac{1}{4}\right]\left[\left[\tanh\frac{\omega}{4kT}\right](x+\bar{x})^2+\left[\coth\frac{\omega}{4kT}\right](x-\bar{x})^2\right]\right\} . \tag{7.14}$$

This transformation is generated by the squeeze generator $\hat{Q}_3$ of Eq. (5.9) where x_1 and x_2 are replaced by x and $\bar{x}$, respectively. Indeed, the transformation from $\psi_0(x,\bar{x})$ of Eq. (7.12) to $\psi_T(x,\bar{x})$ of Eq. (7.14) is unitary. The above expression can be expanded as[30]

$$\psi_T(x,\bar{x})=[1-\exp(-\omega/kT)]^{1/2}\sum_n [\exp(-\omega/2kT)]^n\psi_n(x)\psi_n(\bar{x}) . \tag{7.15}$$

The transformation from the ground state to the above series through the coordinate transformation of Eq. (7.13) is often called the Bogoliubov transformation.[27] The evaluation of the integral

$$\rho_T(x,x')=\int \psi_T(x,\bar{x})\psi_T^*(x_1',\bar{x})d\bar{x} \tag{7.16}$$

leads to the density matrix of Eq. (7.11). The sum of the series will lead to

$$\rho_T(x,x')=\left[\left[\frac{1}{\pi}\right]\tanh\left[\frac{\omega}{2kT}\right]\right]^{1/2}\exp\left\{-\left[\frac{1}{4}\right]\left[(x+x')^2\tanh\left[\frac{\omega}{2kT}\right]+(x-x')^2\coth\left[\frac{\omega}{4kT}\right]\right]\right\} . \tag{7.17}$$

Next, we would like to show that the above procedure can be carried out in the phase-space picture of quantum mechanics. For one pair of canonical variables, the transformation of $\rho(x,x')$ into the Wigner function is achieved through[3,4]

$$W(x,p)=\frac{1}{\pi}\int \rho(x+y,x-y)e^{2ipy}dy \ . \tag{7.18}$$

A similar expression can be given for two pairs of canonical variables. The density matrix can be recovered from $W(x,p)$ through the inverse transformation:

$$\rho(x,x')=\int W\left[\frac{x+x'}{2},p\right]e^{-ip(x-x')}dp \ . \tag{7.19}$$

Let us start with the Wigner function for the ground-state harmonic oscillator:

$$W_0(x,\tilde{x},p,\tilde{p})=\left[\frac{1}{\pi}\right]^2 \exp[-(x^2+\tilde{x}^2+p^2+\tilde{p}^2)] \ . \tag{7.20}$$

Then the coordinate transformation of Eq. (7.13) leads to[30]

$$W_T(x,\tilde{x},p,\tilde{p})=\left[\frac{1}{\pi}\right]^2 \exp\left\{-\left[\frac{1}{2}\right]\left[(x+\tilde{x})^2\tanh\left[\frac{\omega}{4kT}\right]+(x-\tilde{x})^2\coth\left[\frac{\omega}{4kT}\right]\right.\right.$$
$$\left.\left.+(p-\tilde{p})^2\tanh\left[\frac{\omega}{4kT}\right]+(p+\tilde{p})^2\coth(\omega 4kT)\right]\right\} \ . \tag{7.21}$$

This transformation is generated by Q_3 of Eq. (3.13) applicable to the Wigner function.

We can now construct the Wigner function $W_T(x,p)$ by integrating the above expression over $\tilde{x}$ and $\tilde{p}$:

$$W_T(x,p)=\int W_T(x,\tilde{x},p,\tilde{p})d\tilde{x}\,d\tilde{p} \ . \tag{7.22}$$

The result of this integration is

$$W_T(x,p)=\left[\frac{\tanh(\omega/2kT)}{\pi}\right] \times \exp\left[-(x^2+p^2)\tanh\left[\frac{\omega}{2kT}\right]\right] \ . \tag{7.23}$$

It is possible to obtain this form also from the definition of the Wigner function given in Eq. (7.18) and from $\rho_T(x,x')$ of Eq. (7.17). This is a clear indication that the Wigner function can also be effective in dealing with mixed states.

When $T=0$, $W_T(x,p)$ becomes that of the ground state. As the temperature rises, the distribution in phase space becomes widespread. Thus, the transformation is not a canonical transformation. This corresponds to the lack of unitarity during the process of taking an ensemble average, or the integration over the shadow coordinate in the Schrödinger picture. We shall discuss the group theoretical implication of the thermal expansion in Sec. IX.

It is important to realize that the shadow coordinate $\tilde{x}$ does not describe any physical world. It was introduced purely for mathematical convenience of producing the result same as the ensemble average. At this point, we become eager to find an example in which the shadow coordinate has its own physics. We shall study such an example in Sec. VIII.

VIII. TWO-MODE SQUEEZED STATES IN RELATIVISTIC QUANTUM MECHANICS

It was noted in Sec. VII that the mathematics of the Bogoliubov transformation is that of Lorentz transformations in special relativity. Thus we are led to the question of whether there is a physical example of this mathematics in special relativity. The answer to this question is yes.

The relativistic quark model within the framework of the covariant harmonic oscillator formalism is a case in point. Since this model has been widely discussed in the literature,[30-33] we shall study here only the aspect of the covariant oscillator formalism which exhibits a parallelism with the thermal excitation of one-mode squeezed states through a squeezed two-mode state.

Let us consider a hadron consisting of two quarks. If the space-time position of two quarks are specified by x_a and x_b, respectively, the system can be described by the variables[31]

$$X=(x_a+x_b)/2, \quad x=(x_a-x_b)/2\sqrt{2} \ . \tag{8.1}$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. As for the four-momenta of the quarks p_a and p_b, we can combine them into the total hadronic four-momentum and momentum-energy separation between the quarks:[31]

$$P=p_a+p_b, \quad p=\sqrt{2}(p_a-p_b) \ , \tag{8.2}$$

where P is the hadronic four-momentum conjugate to X. The internal momentum-energy separation is conjugate to x.

In the convention of Feynman, Kislinger, and Ravndal,[31] the internal motion of the quarks can be described by the Lorentz-invariant oscillator equation

$$\frac{1}{2}\left[x_\mu^2-\left(\frac{\partial}{\partial x_\mu}\right)^2\right]\psi(x)=\lambda\psi(x)\ , \tag{8.3}$$

where we use the space-favored metric $x^\mu=(x,y,z,t)$. The four-dimensional covariant oscillator wave functions are Hermite polynomials multiplied by a Gaussian factor, which dictates the localization property of the wave function. As Dirac suggested,[34] the Gaussian factor takes the form

$$\left(\frac{1}{\pi}\right)\exp\left[-\left(\frac{1}{2}\right)(x^2+y^2+z^2+t^2)\right]\ . \tag{8.4}$$

Since the x and y components are invariant under Lorentz boosts along the z direction, and since the oscillator wave functions are separable in the Cartesian coordinate system, we can drop the x and y variables from the above expression, and restore them whenever necessary.

The Lorentz boost along the z direction takes a simple form in the light-cone coordinate system, in which the variables $(z+t)/\sqrt{2}$ and $(z-t)/\sqrt{2}$ are transformed to $e^{\lambda}(z+t)/\sqrt{2}$ and $e^{-\lambda(z-t)}/\sqrt{2}$, respectively, where λ is the boost parameter and is $\tanh^{-1}(v/c)$. Then the ground-state wave function will be Lorentz squeezed as[25,30,33]

$$\psi_\lambda(z,t)=\left(\frac{1}{\pi}\right)^{1/2}\exp\left[-\left(\frac{1}{4}\right)[e^{-2\lambda}(z+t)^2+e^{2\lambda}(z-t)^2]\right]\ . \tag{8.5}$$

Indeed, this expression is the same as that for the two-mode squeezed state given in Eq. (7.14), if $e^{-2\lambda}$ is identified as $\tanh(\omega/4kT)$. The variables z and t correspond to x and $\bar{x}$, respectively:

$$\phi_\lambda(p_z,p_0)=\frac{1}{\pi}\int\psi_\lambda(z,t)\exp[i(p_z z-p_0 t)]dz\,dt\ . \tag{8.6}$$

Thus p and $\bar{p}$ in Sec. VII will become p_z and $-p_0$, respectively. Thus the Wigner function will be

$$W_\lambda(z,t,p_z,p_0)=\left(\frac{1}{\pi}\right)^2\exp\left[-\left(\frac{1}{2}\right)[e^{-2\lambda}(z+t)^2+e^{2\lambda}(p_z-p_0)^2]\right]\exp\left[-\left(\frac{1}{2}\right)[e^{2\lambda}(z-t)^2+e^{-2\lambda}(p_z+p_0)^2]\right]\ . \tag{8.7}$$

If the t variable is not measured, the Wigner function will become

$$W_\lambda(z,p_z)=\left(\frac{1}{\pi}\right)\left(\frac{1}{\cosh(2\lambda)}\right)\times\exp[-(z^2+p_z^2)/\cosh(2\lambda)]\ . \tag{8.8}$$

As the hadronic speed v increases, the distribution becomes widespread in phase space. This is derivable from the two-mode picture of the covariant harmonic oscillator formalism.

In one of our recent papers,[35] we discussed the possibility of deriving the concept of hadronic temperature from the parallelism between the Lorentz and Bogoliubov transformations. This correspondence can be made in a simpler way in terms of the expansion of the Wigner distribution function in phase space. By comparing W_T of Eq. (7.23) with W_λ of Eq. (8.8), we can make the correspondence

$$\tanh\left(\frac{\omega}{2kT}\right)=\frac{1}{\cosh(2\lambda)}\ . \tag{8.9}$$

This leads to the conclusion that $(v/c)^2=\exp(-\omega/kT)$.[35]

IX. EXPANSION IN PHASE SPACE

We discussed in our earlier papers[1,2] the group of linear canonical transformations in two-dimensional phase space consisting of one pair of canonical variables and their generators which are given in Eqs. (3.4), (3.5), and (3.6) of the present paper. These generators can be represented by 3×3 matrices applicable to the three-dimensional space of $(x,p,1)$.[1,2] The generators of translations are

$$N_1=\begin{pmatrix}0&0&i\\0&0&0\\0&0&0\end{pmatrix},\quad N_2=\begin{pmatrix}0&0&0\\0&0&i\\0&0&0\end{pmatrix}\ . \tag{9.1}$$

The squeeze and rotation generators are

$$B_1=\begin{pmatrix}i/2&0&0\\0&-i/2&0\\0&0&0\end{pmatrix},\quad B_2=\begin{pmatrix}0&i/2&0\\i/2&0&0\\0&0&0\end{pmatrix},\quad L=\begin{pmatrix}0&-i/2&0\\i/2&0&0\\0&0&0\end{pmatrix}\ . \tag{9.2}$$

We can achieve the thermal expansion of the Wigner distribution function from that of the ground state:

$$W_0(x,p)=\left(\frac{1}{\pi}\right)\exp[-(x^2+p^2)]\ , \tag{9.3}$$

to $W_T(x,p)$ of Eq. (7.23) by applying the operator

$$F(\lambda)=\exp\left\{-\left[\ln\left[\tanh\frac{1}{2kT}\right]\right]\left[x\frac{\partial}{\partial x}+p\frac{\partial}{\partial p}+1\right]\right\} \tag{9.4}$$

to $W_0(x,p)$. This transformation is unitary in phase space in the sense that the Wigner function of Eq. (7.23) remains normalized. However, this is not an area-preserving canonical transformation. We have observed in Sec. V that the thermal excitation is not a unitary transformation in the Schrödinger picture of quantum mechanics.

The generator of this expansion in phase space is[29]

$$E=i\left[x\frac{\partial}{\partial x}+p\frac{\partial}{\partial p}+1\right]. \tag{9.5}$$

In matrix form, this generator is

$$E=\begin{bmatrix} i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{9.6}$$

This generator satisfies the following commutation relations with the generators of canonical transformations.

$$\begin{gathered}[E,L]=0,\quad [E,B_1]=0,\quad [E,B_2]=0\,,\\ [E,N_1]=iN_1,\quad [E,N_2]=iN_2\,.\end{gathered} \tag{9.7}$$

The expansion commutes with the squeeze and rotation generators. However, the translation does not commute with the expansion. The expansion changes the scale of translations.

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