Feynman path-integral representation of field operators and memory superoperators in a Liouville space

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The Feynman path-integral representation for memory superoperators is investigated. A physical interpretation of evolution superoperators in a Liouville space is given and shown to be closely related to the Feynman representation of quantum mechanics. From a general Huygens-like principle it is possible to obtain the Feynman path-integral representation for field operators. An application of the present formalism to a noninteracting many-boson system is provided.

I. INTRODUCTION

In recent years Feynman's path-integral formulation of quantum mechanics,[1,2] statistical mechanics,[3] and quantum field theory[4] has proven to be a surprisingly powerful method in a large variety of problems ranging through nuclear physics,[5] atomic and molecular physics,[6–13] solid-state physics,[14] polymer physics,[15–17] stochastic processes,[18] and quantum gravity.[19] However, no attempts have been made to extend the Feynman path-integral approach to the operator space in which useful information on the time development of field operators for both fermionic and bosonic systems could be extracted. This dynamical behavior was clearly analyzed (within the nonrelativistic theory) for the density matrix[20] by means of memory superoperators. An extension of this treatment to fermionic field operators allowed us to get a compact expression for the Green's function in a Liouville space within the regime of the interaction picture.[21] More recently a U-matrix theory in quantum mechanics has been developed into a workable approach based on Dyson's formulation[22] in which the wave function is written in terms of the U matrix within the regime of the interaction picture.[23] It was claimed that under the special case where the unperturbed Hamiltonian and the Lagrangian commute, the wave function can be reduced to the form identical to that obtained from the path-integral method and a comparison of the path-integral theory and this U-matrix theory was presented.[24]

The purpose of this article is to establish a link between memory superoperators and the quantum analog of classical action in operator space of a many-particle interacting system. This will allow us to generate a Feynman superpropagator in a Liouvillian space. Because one needs to specify essentially the action, this formalism will be potentially powerful in handling those physical situations where the Hamiltonian or Lagrangian cannot be written down explicitly. Such a situation arises, for example, in a reduced description of a many-particle system and is reflected through a memory term in the reduced action.

II. THEORETICAL BACKGROUND

We consider a many-particle interacting system described by a second-quantized Hamiltonian written as $H=H_0+H'$ where H_0 is a one-particle operator and H' contains the many-body effects, i.e.,

$$H_0=\int d\xi \int d\xi' \psi^+(\xi,t)\mathcal{H}_0(\xi|\xi')\psi(\xi',t) \ , \tag{1a}$$

$$H'=\frac{1}{2!}\int d\xi_1\int d\xi_2\int d\xi_1'\int d\xi_2'\psi^+(\xi_1,t)\psi^+(\xi_2,t)\mathcal{H}_1'(\xi_1\xi_2|\xi_1'\xi_2')\psi(\xi_2',t)\psi(\xi_1',t)+\cdots \ . \tag{1b}$$

Here ξ denotes the full set of coordinates (space and spin) of a particle; $\mathcal{H}_0(\xi|\xi'),\mathcal{H}_1'(\xi_1\xi_2|\xi_1'\xi_2'),\ldots$ are the kernels of the one-, two-, ... many-particle Schrödinger operators $H_0,H_1',\ldots,$ respectively, and $\psi^+(\xi,t)$ and $\psi(\xi,t)$ are the time-dependent field operators in the Heisenberg-picture representation satisfying the proper commutation relations, i.e.,

$$\begin{aligned}[\psi(\xi,t),\psi(\xi',t)]_\pm&=[\psi^+(\xi,t),\psi^+(\xi',t)]_\pm\\&=[\psi^+(\xi,t),\psi(\xi',t)]_\pm\\&\quad-\delta(\xi-\xi')=0 \ .\end{aligned} \tag{2}$$

Associated with these operators are the corresponding commutation superoperators $\hat{\mathcal{O}}_0=[H_0, \ ]_-$ and $\hat{\mathcal{O}}'$

$=[H', \ ]_-$, whose sum, $\hat{\mathcal{O}}=\hat{\mathcal{O}}_0+\hat{\mathcal{O}}'=[H, \ ]_-$, is the generator of motion for the field operators $\psi^+(\xi,t),\psi(\xi,t)$.[21] As seen, $\hat{\mathcal{O}}_0$, $\hat{\mathcal{O}}'$, and $\hat{\mathcal{O}}$ are Liouville-type superoperators, i.e., (linear Hermitian) operators that work in the Hilbert space of operators rather than the space of states. In this Liouvillian superoperators space the equation of motion of the interaction-picture field operator is given by[21] ($\hbar=1$)

$$i\frac{\partial}{\partial t}\psi_I(\xi,t)=-\hat{\mathcal{U}}(t)\psi_I(\xi,t) \ , \tag{3}$$

where $\hat{\mathcal{U}}(t)=[U(t), \ ]_-$ denotes the evolution superoperator associated with the interaction-picture Hamiltonian operator $U(t)=\exp(-i\hat{\mathcal{O}}_0 t)H'$.

The superoperator $\hat{\mathcal{U}}(t)$ is the generator of the motion group $\hat{\mathcal{G}}$ whose elements

$$\hat{\mathcal{G}}(t,t')=\mathcal{T}\exp\left[i\int_{t'}^{t}ds\,\hat{\mathcal{U}}(s)\right] \tag{4}$$

(defined in terms of the Dyson[22] time-ordering or chronological superoperator $\mathcal{T}$) exhibit well-known group properties and propagate the field operator $\psi(\xi,t)$ according to the prescription

$$\psi_I(\xi,t)=\hat{\mathcal{G}}(t,t')\psi_I(\xi,t') \ . \tag{5}$$

By introducing the mutually orthogonal projection superoperators $\mathcal{P}$ and $\mathcal{Q}=1-\mathcal{P}$ and defining the group of time-dependent superoperators $\hat{\mathcal{U}}_{\mathcal{R}\mathcal{S}}$, $\hat{\mathcal{G}}_{\mathcal{S}}(t,s)$, in the manner

$$\hat{\mathcal{U}}_{\mathcal{R}\mathcal{S}}(t)=\mathcal{R}\hat{\mathcal{U}}(t)\mathcal{S}, \quad \mathcal{R},\mathcal{S}=\mathcal{P},\mathcal{Q} \tag{6}$$

$$\hat{\mathcal{G}}_{\mathcal{S}}(t,s)=\mathcal{S}\mathcal{T}\exp\left[i\int_{s}^{t}ds_1\hat{\mathcal{U}}_{\mathcal{S}\mathcal{S}}(s_1)\right]\mathcal{S} \ , \tag{7}$$

it is found that the solution of the differential equation (3) for the two orthogonal complements of the field operator $\psi_{\mathcal{P}}(\xi,t)=\mathcal{P}\psi_I(\xi,t)$ and $\psi_{\mathcal{Q}}(\xi,t)=\mathcal{Q}\psi_I(\xi,t)$ can be expressed in the form

$$\psi_{\mathcal{P}}(\xi,t)=\hat{\mathcal{G}}_{\mathcal{P}}(t,t')[\hat{\mathcal{M}}_{\mathcal{P}\mathcal{Q}\mathcal{P}}(t,t')\psi_{\mathcal{P}}(\xi,t') +\hat{\mathcal{N}}_{\mathcal{P}\mathcal{Q}}(t,t')\psi_{\mathcal{Q}}(\xi,t')] \ , \tag{8}$$

where it can be verified that the memory superoperator $\hat{\mathcal{M}}_{\mathcal{P}\mathcal{Q}\mathcal{P}}(t,t')$ must satisfy the Volterra integral equation (here written for an arbitrary pair of complementary projections $\mathcal{R}$ and $\mathcal{S}$)

$$\hat{\mathcal{M}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(t,t')=\mathcal{S}-\int_{t'}^{t}ds\,\hat{\mathcal{W}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(t,s,t')\hat{\mathcal{M}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(s,t') \tag{9a}$$

and the $\hat{\mathcal{N}}_{\mathcal{P}\mathcal{Q}}(t,t')$ coefficient of $\psi_{\mathcal{Q}}(\xi,t')$ in Eq. (8) is connected to $\hat{\mathcal{M}}_{\mathcal{P}\mathcal{Q}\mathcal{P}}(t,t')$ through the expression

$$\hat{\mathcal{N}}_{\mathcal{P}\mathcal{Q}}(t,t')=\int_{t'}^{t}ds\,\hat{\mathcal{B}}_{\mathcal{P}\mathcal{Q}}(t',s)\hat{\mathcal{M}}_{\mathcal{Q}\mathcal{P}\mathcal{Q}}(s,t') \tag{9b}$$

whose kernels

$$\hat{\mathcal{W}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(t,s,t')=\left[\int_{s}^{t}ds'\hat{\mathcal{B}}_{\mathcal{S}\mathcal{R}}(t,s')\right]\hat{\mathcal{B}}_{\mathcal{R}\mathcal{S}}(t',s) \tag{10}$$

involve superoperators

$$\hat{\mathcal{B}}_{\mathcal{S}\mathcal{R}}(t,s)=\hat{\mathcal{G}}_{\mathcal{S}}(t,s)\hat{\mathcal{U}}_{\mathcal{S}\mathcal{R}}(s)\hat{\mathcal{G}}_{\mathcal{R}}(s,t) \tag{11}$$

which couple the $\mathcal{R}$ and $\mathcal{S}$ subspaces at the common instant of time t. This coupling proceeds by propagation through $\mathcal{R}$ space backward in time from t to s, a subsequent direct coupling between the two subspaces mediated by $\hat{\mathcal{U}}_{\mathcal{S}\mathcal{R}}(s)$ and a final forward propagation through $\mathcal{S}$ space from s to t. The kernel $\hat{\mathcal{W}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(t,s,t')$ defined by Eq. (10) couples the $\mathcal{S}$ and $\mathcal{R}$ spaces at time t' through motions forward to s and back again to t'. The integral of $\hat{\mathcal{B}}_{\mathcal{S}\mathcal{R}}(t',s')$ from $s'=s$ to $s'=t$ then recouples to $\mathcal{S}$ space the dynamic information that has been accumulated in $\mathcal{R}$ space during the interval.

The equations for the memory superoperators ($\hat{\mathcal{M}}_{\mathcal{S}\mathcal{R}\mathcal{S}}$ and $\hat{\mathcal{N}}_{\mathcal{S}\mathcal{R}}$) can be solved formally by iteration. Thus, using the Dyson U-matrix theory[23,24] and introducing a Dyson-like superoperator to order both the limit of integration and pairs of arguements of the integrand factors $\hat{\mathcal{W}}_{\mathcal{S}\mathcal{R}\mathcal{S}}$ appearing in Eq. (9a), it follows from this equation and after some rather involved analysis that[20,21]

$$\hat{\mathcal{M}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(t,t')=\mathcal{S}\hat{\mathcal{M}}^{C}(t,t')\mathcal{S} \tag{12}$$

and

$$\int_{t'}^{t}ds\,\hat{\mathcal{B}}_{\mathcal{R}\mathcal{S}}(t',s)\hat{\mathcal{M}}_{\mathcal{S}\mathcal{R}\mathcal{S}}(s,t')=\mathcal{R}\hat{\mathcal{M}}^{S}(t,t')\mathcal{S} \tag{13}$$

wherein

$$\hat{\mathcal{M}}^{C}(t,t')=\sum_{n\,(\geq 0)}(-1)^n\int_{t'}^{t}ds_1\int_{t'}^{s_1}ds_2\cdots\int_{t'}^{s_{2n-1}}ds_{2n}\hat{\mathcal{B}}(t',s_1)\hat{\mathcal{B}}(t',s_2)\cdots\hat{\mathcal{B}}(t',s_{2n})$$
$$=\mathcal{T}\cos\left[\int_{t'}^{t}ds\,\hat{\mathcal{B}}(t',s)\right] \tag{14}$$

and

$$\hat{\mathcal{M}}^{S}(t,t')=\sum_{n\,(\geq 0)}(-1)^n\int_{t'}^{t}ds_1\int_{t'}^{s_1}ds_2\cdots\int_{t'}^{s_{2n}}ds_{2n+1}\hat{\mathcal{B}}(t',s_1)\hat{\mathcal{B}}(t',s_2)\cdots\hat{\mathcal{B}}(t',s_{2n+1})$$
$$=\mathcal{T}\sin\left[\int_{t'}^{t}ds\,\hat{\mathcal{B}}(t',s)\right] \tag{15}$$

in which $\hat{\mathscr{B}}(t',s)$ stands for the Hermitian superoperator

$$\hat{\mathscr{B}}(t,s)\equiv\hat{\mathscr{B}}_{\mathcal{PQ}}(t,s)+\hat{\mathscr{B}}_{\mathcal{QP}}(t,s) . \quad (16)$$

Without going into mathematical details, the result, relevant to our discussion, is that in the Heisenberg-picture representations in a Liouville space, quantum field operators, as given by Eq. (8), evolve in time according to the prescription[21]

$$\psi(\xi,t)=\exp(i\hat{\mathcal{O}}_0 t)\hat{\mathcal{G}}_{\mathcal{D}}(t,t')\hat{\mathcal{G}}_{\mathcal{C}}(t,t')\exp(-i\hat{\mathcal{O}}_0 t')\psi(\xi,t') , \quad (17)$$

where $\hat{\mathcal{G}}_{\mathcal{D}}(t,t')$ and $\hat{\mathcal{G}}_{\mathcal{C}}(t,t')$ are the (operator subspaces) decoupling ($\mathcal{D}$) and coupling ($\mathcal{C}$) group superoperators[20,21] in the Zwanzig-Feshbach projection space technique[25,26] as follows:

$$\begin{aligned}\hat{\mathcal{G}}_{\mathcal{D}}(t,t')&=\hat{\mathcal{G}}_{\mathcal{P}}(t,t')+\hat{\mathcal{G}}_{\mathcal{Q}}(t,t')\\ &=\mathcal{T}\exp\left[i\int_{t'}^{t}ds[\hat{\mathcal{U}}_{\mathcal{PP}}(s)+\hat{\mathcal{U}}_{\mathcal{QQ}}(s)]\right] , \quad (18)\\ \hat{\mathcal{G}}_{\mathcal{C}}(t,t')&=\hat{\mathcal{M}}^{C}(t,t')+i\hat{\mathcal{M}}^{S}(t,t')\\ &=\mathcal{T}\exp\left[i\int_{t'}^{t}ds\,\hat{\mathscr{B}}(t',s)\right] . \quad (19)\end{aligned}$$

III. FEYNMAN REPRESENTATION OF MEMORY SUPEROPERATORS: PHYSICAL INTERPRETATION

The formal solution of the Heisenberg equation of motion in a Liouville space [see Eq. (3)]

$$i\frac{\partial}{\partial t}\psi(\xi,t)=-\hat{\mathcal{O}}\psi(\xi,t) \quad (20)$$

can be written in this operator space in the form

$$\psi(\xi,t)=\exp\left[2i\hat{\mathcal{O}}_0(t-t')-i\int_{t'}^{t}\hat{\mathcal{L}}(\xi,\dot{\xi};u)du\right]\psi(\xi,t') , \quad (21)$$

where $\hat{\mathcal{L}}$ is the super-Lagrangian operator in the Liouville space, i.e., $\hat{\mathcal{L}}=\hat{\mathcal{O}}_0-\hat{\mathcal{O}}'\equiv[L,\ ]_-$, such that

$$\hat{\mathscr{S}}=\int_{t'}^{t}\hat{\mathcal{L}}(\xi,\dot{\xi};u)du, \quad \hat{\mathscr{S}}\equiv\hat{\mathscr{S}}[\Gamma(t,t')] \quad (22)$$

where Γ stands for the paths in the operator space.

The exponential of the superoperator appearing in Eq. (21) can be decoupled as follows:

$$\exp\left[2i\hat{\mathcal{O}}_0(t-t')-i\int_{t'}^{t}\hat{\mathcal{L}}(\xi,\dot{\xi};u)du\right]=\exp[2i\hat{\mathcal{O}}_0(t-t')]\exp\left[-i\int_{t'}^{t}\hat{\mathcal{L}}(\xi,\dot{\xi};u)du\right]\hat{Z}(t,\tau) , \quad (23)$$

where $\hat{Z}(t,\tau)$ satisfies the operator equation[23,24]

$$\hat{Z}(t,\tau)=\exp\left[\int_0^{\tau}\hat{\zeta}(z)dz\right]+\sum_{n=2}^{\infty}\frac{1}{n!}\sum_{r=0}^{n-2}(r+1)!\left[\int_0^{\tau}\hat{\zeta}(z)dz\right]^{n-r-2}\hat{P}_r(\tau,0) , \quad (24)$$

with $\hat{\zeta}(z)$ given by

$$\hat{\zeta}(z)=-\sum_{\nu=1}^{\infty}\frac{z^{\nu}}{\nu!}\left[\int_{t'}^{t}2\hat{\mathcal{O}}_0 du,\hat{\mathscr{S}}[\Gamma(t,t')]\right]_{\nu} \quad (25)$$

and

$$\hat{P}_r(\tau,0)=\sum_{\alpha=0}^{r}\int_0^{\tau}dz_1\hat{\zeta}(z_1)\int_0^{z_1}dz_2\hat{\zeta}(z_2)\cdots\int_0^{z_\alpha}dz_{\alpha+1}\left[\int_0^{z_{\alpha+1}}dz[\hat{\zeta}(z),\hat{\zeta}(z_{\alpha+1})]\right]\hat{Y}_{r-\alpha}(z_{\alpha+1},0) . \quad (26)$$

$[A,B]_\nu$ stands for the iterated commutator,

$$[A,B]_\nu=[A,[A,[\ldots[A,B]]\ldots]] , \quad (27)$$

i.e., there are ν commutator terms in each term of the expansion (25). In Eq. (26) $\hat{Y}_n(\tau,0)$ satisfies the recurrence relation[24]

$$\begin{aligned}\hat{Y}_n(\tau,0)&=\int_0^{\tau}d\tau_1\hat{\zeta}(\tau_1)\int_0^{\tau_1}d\tau_2\hat{\zeta}(\tau_2)\cdots\int_0^{\tau_{n-1}}d\tau_n\hat{\zeta}(\tau_n)\\ &=\int_0^{\tau}dz\,\hat{\zeta}(z)\hat{Y}_{n-1}(z,0), \quad \hat{Y}_0(z,0)=\hat{\mathcal{I}} \quad (28)\end{aligned}$$

where $\hat{\mathcal{I}}$ is the identity superoperator.

It follows from Eqs. (21) and (23) that

$$\begin{aligned}\psi(\xi,t)=&\exp[2i\hat{\mathcal{O}}_0(t-t')]\exp\{-i\hat{\mathscr{S}}[\Gamma(t,t')]\}\\ &\times[\hat{\mathcal{I}}+\hat{\Xi}(t,\tau)]\psi(\xi,t') , \quad (29)\end{aligned}$$

where $\hat{Z}(t,\tau)$ in Eq. (23) has been written as $[\hat{\mathcal{I}}+\hat{\Xi}(t,\tau)]$. The two terms in Eq. (29) account for the time evolution of ψ at the phase-space points ξ. The first term is closely related to the usual path-integral wave functions[1,2] because it is seen to be modulated by the superphase $\exp(-i\hat{\mathscr{S}})$; the second term in Eq. (29) arises from the noncommutability property of the operators H_0 and L. A comparison of Eqs. (17) and (29) allows us to get a physical interpretation of the product of decoupling and coupling group superoperators

$$\begin{aligned}&\exp(i\hat{\mathcal{O}}_0 t)\hat{\mathcal{G}}_{\mathcal{D}}(t,t')\hat{\mathcal{G}}_{\mathcal{C}}(t,t')\exp(-i\hat{\mathcal{O}}_0 t')\\ &\quad=\exp[2i\hat{\mathcal{O}}_0(t-t')]\exp\{-i\hat{\mathscr{S}}[\Gamma(t,t')]\}[\hat{\mathcal{I}}+\hat{\Xi}(t,\tau)] . \quad (30)\end{aligned}$$

This expression may be considered the Feynman superphase representation of memory superoperators. It should be noted that Eq. (29) is not a standard Feynman path-integral representation. In fact, we are dealing with time evolution only, i.e., no space propagation is involved. Ths implies that the superaction $\hat{\mathcal{S}}$ as defined in Eq. (22) must not be interpreted as classical paths but rather as an action superoperator from which a Feynman path integral in operator space could be constructed. An action as a path in the operator space may be characterized as a way of implementing this kind of evolution scheme, and will be discussed elsewhere in a rigorous manner.

IV. FEYNMAN PATH-INTEGRAL REPRESENTATION OF FIELD OPERATORS

To get the counterpart of the Feynman path-integral formulation for quantum field operators as a quantum-mechanical analog of the state function, we write the field operator as a second-species inhomogeneous Fredholm integral equation

$$\psi(\xi,t)=-\int \psi(\xi',t)d^3\xi' , \tag{31}$$

where $\psi(\xi',t)$ may be considered the phase-space contribution for the field operator $\psi(\xi,t)$, namely, a Huygens-like principle.

The solution of Eq. (31) is arbitrary and hence, for the present purpose, is a useful way of expressing the field operator. Thus, introduction of the formal solution of Heisenberg's equation of motion (20) into Eq. (31) leads to

$$\psi(\xi,t)=-\int \exp\left[i\int_{t'}^{t}\hat{\mathcal{O}}du\right]\psi(\xi',t')d^3\xi' \tag{32}$$

and using the same procedure as that followed to express the time evolution of ψ in the operator space we get

$$\psi_\Gamma(\xi,t)=-\int \exp[2i\hat{\mathcal{O}}_0(t-t')]\exp\{-i\hat{\mathcal{S}}_\Gamma\}\times[\hat{\mathcal{J}}+\hat{\Xi}(t,\tau)]\psi(\xi',t')d^3\xi' . \tag{33}$$

As long as a particular choice of Γ is necessary for the evaluation of $\hat{\mathcal{S}}$, $\psi_\Gamma(\xi,t)$ can be considered as the field operator on the Γ path in operator space. In the time evolution equation for ψ [Eq. (21)] only one path is needed to describe it; in fact, no propagation in space coordinates is required. So ψ_Γ may be interpreted as a projection on the possible paths between t and t', and therefore the quantum field operator may be considered as sums over all those projections, i.e.,

$$\psi(\xi,t)=\sum_{\{\Gamma\}}\psi_\Gamma(\xi,t)=-\int \exp[2i\hat{\mathcal{O}}_0(t-t')]\sum_{\{\Gamma\}}\exp\{-i\hat{\mathcal{S}}_\Gamma\}[\hat{\mathcal{J}}+\hat{\Xi}(t,\tau)]\psi(\xi',t')d^3\xi' . \tag{34}$$

From this equation the Feynman path integral is recognized as

$$\sum_{\{\Gamma\}}\exp\{-i\hat{\mathcal{S}}_\Gamma\}[\hat{\mathcal{J}}+\hat{\Xi}(t,\tau)] . \tag{35}$$

Thus the field operator $\psi(\xi,t)$ satisfies the integral equation

$$\psi(\xi,t)=\int\hat{\mathcal{K}}(\xi t|\xi' t')\psi(\xi',t')d^3\xi' . \tag{36}$$

Equation (36) represents the Feynman path-integral formulation for quantum field operators, with $\hat{\mathcal{K}}$ being the Feynman superpropagator, defined as

$$\hat{\mathcal{K}}(\xi t\xi' t')=-\exp[2i\hat{\mathcal{O}}_0(t-t')]\times\sum_{\{\Gamma\}}\exp\{-i\hat{\mathcal{S}}_\Gamma\}[\hat{\mathcal{J}}+\hat{\Xi}(t,\tau)] . \tag{37}$$

This equation again takes into account the noncommutability of the operators H_0 and L by means of $\hat{\Xi}(t,\tau)$.

On introducing Eq. (36) into the Heisenberg equation of motion [Eq. (20)] it is easily verified that the differential equation for the superpropagator is

$$\left[i\frac{\partial}{\partial t}+\hat{\mathcal{O}}+\hat{\mathcal{J}}\right]\hat{\mathcal{K}}(\xi t|\xi' t')=\delta(\xi-\xi')\exp\left[-i\int_{t'}^{t}\hat{\mathcal{O}}\,du\right] . \tag{38}$$

V. APPLICATION TO A NONINTERACTING MANY-BOSON SYSTEM

In this section we shall develop functional integrals, within the formalism of superoperators for second-quantized Hamiltonians with Bose operators. For the purpose of developing our functional integrals we consider a many-boson system and use the discrete form of H_0. It follows from Eq. (1a) that it reads

$$H_0=\sum_k \varepsilon_k a_k^\dagger a_k , \tag{39}$$

where the ε_k's are the one-particle energies and $a_k^\dagger$ and a_k are the creation and annihilation bosonlike operators.

In order to proceed we need the superpropagator representation [Eq. (37)] in the original form as given by Eq. (32). Thus we write

$$\hat{\mathcal{K}}(\xi t|\xi' t')=\exp\left[i\int_{t'}^{t}\hat{\mathcal{O}}_0du\right] . \tag{40}$$

This equation may be written in terms of the short-time approach[27] as

$$\hat{\mathcal{K}}(\xi t|\xi' t')\equiv\hat{\mathcal{K}}=\exp(i\hat{\mathcal{O}}_0\delta t_1)\times\exp(i\hat{\mathcal{O}}_0\delta t_2)\cdots\exp(i\hat{\mathcal{O}}_0\delta t_N) , \tag{41}$$

where $\delta t_i\equiv\delta t=t/N$ $(i=1,\ldots,N)$ with N being the short-time partition.[27]

For obtaining expressions that could be handled in an easy way we introduce an operator space that will be the carrier space for the superoperator object. In the limit of no interaction this space is chosen to be

$$\mathcal{I}=\{a_k,a_l^+;\ k,l=1,\ldots,\infty\}\ . \tag{42}$$

This set spans the one-particle operator space and supports the closure relation

$$\hat{\mathcal{I}}=\sum_m |a_m)(a_m| \tag{43}$$

with the associated binary product operation

$$(\mathcal{X}|\mathcal{Y})=[\mathcal{Y},\mathcal{X}^\dagger]_-\ . \tag{44}$$

Thus the carrier space is metrized by

$$(a_i|a_k)=[a_i;a_k^\dagger]_-=\delta_{ik} \tag{45}$$

and the matrix elements for $\hat{\mathcal{K}}$ in the short-time approach becomes

$$(a_i|\hat{\mathcal{K}}a_k)=\sum_{i_1}\cdots\sum_{i_N}(a_i|\exp(i\hat{\mathcal{O}}_0\delta t)a_{i_1})(a_{i_1}|\exp(i\hat{\mathcal{O}}_0\delta t)a_{i_2})\cdots(a_{i_N}|\exp(i\hat{\mathcal{O}}_0\delta t)a_k)\ , \tag{46}$$

where the closure relation (43) has been introduced between each exponential function in Eq. (41). This procedure is analogous to that implicit in the summation over paths [Eq. (37)].

Realizing that each a_l is an eigenelement of $\hat{\mathcal{O}}_0$ with eigenvalues $n_l\varepsilon_l$ (where n_l is the corresponding occupation number associated with the lth energy level), and allowing N to tend toward infinity to get the exact propagator, yields[27]

$$(a_i|\hat{\mathcal{K}}a_k)=\exp(in_k\varepsilon_k t)\delta_{ik}\ . \tag{47}$$

Note that $\hat{\mathcal{K}}$ is diagonal in the operator basis set $\mathcal{I}$. Physically the matrix element (47) is the projection of the $\hat{\mathcal{K}}$ superoperator onto the one-particle operator space.

Performing the Wick rotation[28] (β is the inverse of the absolute temperature and t is imaginary time)

$$it=-\beta\ , \tag{48}$$

so as to realize the adequate analytic continuation into the Feynman superpropagator $\hat{\mathcal{K}}$ [Eq. (40)], straightforwardly yields the thermodynamic matrix elements

$$(a_i|\hat{\mathcal{K}}a_k)=\exp(-\beta n_k\varepsilon_k)\delta_{ik} \tag{49}$$

which are the unnormalized one-particle density-matrix elements. Thus, normalizing Eq. (49) and noting that $\hat{\mathcal{K}}=\hat{\rho}_1$ (one-particle density-matrix superoperator) leads to

$$\mathrm{Tr}(\hat{\rho}_1)=\sum_{\{R_i\}}(a_i|\hat{\rho}_1a_i)_{\{R_i\}}=\sum_{n_i=0}^{\infty}\exp(-\beta n_i\varepsilon_i)\ , \tag{50}$$

which is recognized to be the bosonic partition function for the one-state distribution. $\{R_i\}$ stands for the accessible one-particle configurations that are compatible with the ith energy level.[29]

We finally get, for the density superoperator matrix elements, the expression

$$(a_i|\hat{\rho}_1a_k)=\frac{\exp(-\beta n_k\varepsilon_k)\delta_{ik}}{\sum_{n_k=0}^{\infty}\exp(-\beta n_k\varepsilon_k)}\ . \tag{51}$$

Defining the number operator in the kth energy level in the usual form

$$\mathcal{N}_k=a_k^\dagger a_k \tag{52}$$

and its associated superoperator

$$\hat{\mathcal{N}}_k=[\mathcal{N}_k;\ ]_- \tag{53}$$

allows us to evaluate the mean occupation number for the kth level as

$$\langle\mathcal{N}_k\rangle\equiv\mathrm{Tr}(\hat{\rho}_1\hat{\mathcal{N}}_k)\equiv\bar{n}_k=\sum_{\{R_i\}}\sum_{\{R_l\}}(a_i|\hat{\rho}_1a_l)(a_l|\hat{\mathcal{N}}_ka_i) \tag{54a}$$

or equivalently

$$\bar{n}_k=\frac{\sum_{n_i=0}^{\infty}\sum_{\{R_l\}}\exp(-\beta n_i\varepsilon_i)(a_l|\hat{\mathcal{N}}_ka_i)\delta_{il}}{\sum_{n_i=0}^{\infty}\exp(-\beta n_i\varepsilon_i)}\ . \tag{54b}$$

Thus, using

$$(a_l|\hat{\mathcal{N}}_ka_i)=n_ka_i\delta_{ik}\delta_{il}\ , \tag{55}$$

we get immediately

$$\bar{n}_k=\frac{\sum_{n_k=0}^{\infty}n_k\exp(\beta n_k\varepsilon_k)}{\sum_{n_k=0}^{\infty}\exp(-\beta n_k\varepsilon_k)}\ , \tag{56}$$

which after a little algebra leads to

$$\bar{n}_k=\frac{1}{\exp(\beta\varepsilon_k)-1}\ , \tag{57}$$

which is the expected formula of the Planck distribution law.[29]

VI. FINAL REMARKS

Quantum time evolution for field operators is useful for describing not only equilibrium processes but also nonequilibrium evolutions. In fact, there exists a close relation with Green's functions expressed as propagators in operator space[21] and the reduced density functions that are the essential tools for evaluation of such properties. The Feynman path formulation for field operators leads us to state some interesting conclusions. As we have shown, the Feynman path representation of memory superoperators shows that the evolution depends on the whole history of the process, i.e., the evolution is noninstantaneous; in fact, the Feynman path-integral representation includes the history of the process by means of the action superoperator.

The superpropagator contains all the information about the system, it being a sum of contributions from all paths; thus the quantum superposition is already manifest in the present formulation. Finally, it should be stressed that since in an imaginary-time formalism the Feynman path intetral is mathematically equivalent to a partition function,[12,13] the present formalism should appear as a practical tool for the evaluation of magnetic and thermodynamic properties of many-body systems.

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[1]R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948); R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[2]L. S. Shulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[3]R. P. Feynman, *Statistical Mechanics* (Benjamin/Cummings, Reading, MA, 1972).

[4]D. J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).

[5]J. W. Negele, Rev. Mod. Phys. **54**, 913 (1982).

[6]I. H. Duru and H. Kleinert, Phys. Lett. **84B**, 185 (1979).

[7]R. Ho and A. Inomata, Phys. Rev. Lett. **48**, 231 (1982); A. Inomata, Phys. Lett. **101A**, 253 (1984).

[8]F. Steiner, Phys. Lett. **106A**, 356 (1984); **106A**, 363 (1984).

[9]G. C. Gerry, Phys. Lett. **142B**, 391 (1984).

[10]H. Grinberg, J. Marañón, and H. Vucetich, J. Chem. Phys. **78**, 839 (1983).

[11]H. Grinberg, J. Marañón, and H. Vucetich, Int. J. Quantum Chem. **23**, 329 (1983).

[12]H. Grinberg and J. Marañón, Phys. Lett. **94A**, 275 (1983).

[13]J. Marañón and H. Grinberg, J. Chem. Phys. **81**, 4537 (1984).

[14]K. K. Thornber, Phys. Rev. B **1**, 4099 (1970); K. K. Thornber, *ibid.* **3**, 1929 (1971).

[15]S. F. Edwards and K. F. Freed, J. Phys. C **3**, 739 (1970); K. F. Freed, J. Chem. Phys. **54**, 1453 (1971); S. F. Edwards and J. W. Grant, J. Phys. A **6**, 1169 (1973).

[16]F. W. Wiegel, *Introduction to Path-Integral Methods in Physics and Polymer Science* (World Scientific, Singapore, 1986).

[17]K. F. Freed, *Renormalization Group Theory of Macromolecules* (Wiley, New York, 1987).

[18]M. C. Gutzwiller, in *Path Integrals and Their Applications in Quantum, Statistical and Solid State Physics,* edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978).

[19]S. Hawking, in *General Relativity—and Einstein Centenary Survey,* edited by S. Hawking and W. Israel (Cambridge University Press, London, 1979).

[20]R. E. Turner and J. S. Dahler, J. Phys. B **13**, 161 (1980); R. E. Turner, J. S. Dahler, and R. F. Snider, Can. J. Phys. **60**, 1371 (1982).

[21]H. Grinberg, International Centre for Theoretical Physics Report No. IC/83/172.

[22]F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[23]C. C. Lam and P. C. W. Fung, Phys. Rev. A **27**, 1760 (1983).

[24]P. C. W. Fung and C. C. Lam, Phys. Rev. A **29**, 2364 (1984).

[25]R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960); Physica **30**, 1109 (1964).

[26]H. Feshbach, Ann. Phys. (N.Y.) **19**, 287 (1962).

[27]G. J. Papadopoulos, in Ref. 18.

[28]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1966); L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

[29]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill Kogakusha, Tokyo, 1965), Chap. 9.