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Quantum logistic map

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By coupling a kicked quantum system to a bath of harmonic oscillators we derive a logistic map with quantum corrections. We find a period-doubling route to the classical behavior as a dissipation parameter is increased, and other interesting features at intermediate values of this parameter.

A subject of much current interest is the effect of external noise on nonlinear dynamical systems. It has been found, for instance, that the addition of external random noise destroys localization[1] and increases irreversibility[2] in the quantum standard map. On the other hand, the addition of quantum fluctuations to the Lorenz equations[3] suppresses the chaos in that system.[4]

A number of authors have introduced noise into quantum systems by including dissipation.[5-11] The general approach has been to solve a master equation for the density matrix of the system of interest coupled to a bath of oscillators, and then to calculate the quantities of interest from the density matrix. Graham and co-workers have shown that, in certain limits, maps of dissipative quantum systems can be written as classical maps with random-noise terms.[5] In this paper we describe a dissipative map in which quantum corrections effectively add noise, the dynamics become more classical as the dissipation is increased, and there is a period doubling in the transition to the classical behavior. We suggest that this behavior is generic.

Our approach is to derive equations of motion for a kicked quantum system coupled to a bath of oscillators, introduce a quasicontinuum model to describe the dissipation from the bath, and then take an expectation value and study the resulting expectation-value map. In order to study the effects of quantum correlations we write $\hat{a}=\langle\hat{a}\rangle+\delta\hat{a}$, where $\delta\hat{a}$ represents a quantum fluctuation about $\langle\hat{a}\rangle$ and obviously has the property $\langle\delta\hat{a}\rangle=0$. In this way we study what effects correlations of the form $\langle\delta\hat{a}\delta\hat{a}\rangle$, $\langle\delta\hat{a}^\dagger\delta\hat{a}\rangle$, etc., have as the coupling to the bath is varied.

We start with the Hamiltonian

$$\hat{H}=\hbar\sum_k(\omega_k-\omega_0)\hat{b}_k^\dagger\hat{b}_k+\hbar C\sum_k(\hat{a}^\dagger b_k+\hat{b}_k^\dagger\hat{a})+\hbar V(\hat{a},\hat{a}^\dagger)\sum_n\delta(t-nT)\,, \quad (1)$$

where $\hat{a}$ ($\hat{a}^\dagger$) is the boson annihilation (creation) operator of the system of interest with frequency ω_0, $\hat{b}_k$ ($\hat{b}_k^\dagger$) are the boson annihilation (creation) operators of the bath, C is the coupling constant of the bath to the system, and $V(\hat{a},\hat{a}^\dagger)$ will be specified later. The boson operators obey the usual equal-time commutation relations: $[\hat{a},\hat{a}^\dagger]=1$, $[\hat{b}_{k_1},\hat{b}_{k_2}^\dagger]=\delta_{k_1k_2}$, $[\hat{b}_k,\hat{a}]=0$, etc. The Heisenberg equations of motion are then

$$\dot{\hat{a}}(t)=-iC\sum_k\hat{b}_k(t)-i[\hat{a},V(\hat{a},\hat{a}^\dagger)]\sum_n\delta(t-nT)\,, \quad (2a)$$

$$\dot{\hat{b}}_k(t)=-i\Delta_k\hat{b}_k(t)-iC\hat{a}(t)\,, \quad (2b)$$

where $\Delta_k=\omega_k-\omega_0$. Substitution of the formal solution of (2b) into (2a) yields a single equation for $\hat{a}(t)$,

$$\dot{\hat{a}}(t)=-iC\sum_k\hat{b}_k(0)\exp(-i\Delta_kt)-C^2\int_0^t dt'\,\hat{a}(t')\sum_k\exp[i\Delta_k(t-t')]-i[\hat{a},V(\hat{a},\hat{a}^\dagger)]\sum_n\delta(t-nT)\,. \quad (3)$$

By introducing a quasicontinuum model[12] with $\Delta_k=k\Delta$, $k=0,\pm1,\pm2,\ldots$, such that $2\pi C^2/\Delta\to2\Gamma$ as $\Delta,C\to0$, the second term on the right-hand side of (3) becomes $-\Gamma\hat{a}(t)$ and we are left with

$$\dot{\hat{a}}(t)=-iC\sum_k\hat{b}_k(0)\exp(-i\Delta_kt)-\Gamma\hat{a}(t)+f(\hat{a},\hat{a}^\dagger)\sum_n\delta(t-nT)\,, \quad (4)$$

where $f(\hat{a},\hat{a}^\dagger)=-i[\hat{a},V(\hat{a},\hat{a}^\dagger)]$. It is worth mentioning that if we ignore the potential term in Eq. (4) and integrate the resulting oscillator equation, the commutation relation $[\hat{a}(t),\hat{a}(t)^\dagger]=1$ is preserved. The quasicontinuum model gives us, then, a consistent description of dissipation. In other words, the quantum Langevin noise term defined by the first term on the right-hand side of (4) is consistent with the dissipative second term. This is simply a reflection of the general fluctuation-dissipation theorem.

Integration of (4) from $t=nT-\epsilon$ to $t=(n+1)T-\epsilon$ gives us the following operator map:

$$\hat{a}_{n+1}=\hat{a}_ne^{-\beta}+\hat{G}_n+f(\hat{a}_n,\hat{a}_n^\dagger)e^{-\beta}\,, \quad (5)$$

where

$$\hat{G}_n\equiv-iC\sum_k\hat{b}_k(0)\exp[-i\Delta_k(n+1)T]\times[1-\exp(i\Delta_kT-\beta)]/(\Gamma-i\Delta_k)$$

and $\beta = \Gamma T$. We now take the expectation value of (5), assuming $\langle \hat{b}_k(0) \rangle = 0$ for all k, resulting in

$$\langle \hat{a}_{n+1} \rangle = [\langle \hat{a}_n \rangle + \langle f(\hat{a}_n, \hat{a}_n^\dagger) \rangle] e^{-\beta} . \qquad (6)$$

Because we want to compare our quantum results to the familiar logistic map[13] we choose the "force"

$$f(\hat{a}_n, \hat{a}_n^\dagger) = -\hat{a}_n + e^{\beta} r (\hat{a}_n - \hat{a}_n^\dagger \hat{a}_n) , \qquad (7)$$

where r is an adjustable parameter. Equation (6) then becomes

$$\langle \hat{a}_{n+1} \rangle = r(\langle \hat{a}_n \rangle - \langle \hat{a}_n^\dagger \hat{a}_n \rangle) , \qquad (8)$$

which is similar to the classical logistic map.

In order to study the effect of the quantum correlations inherent in the last term of (8) we now write $\hat{a} \equiv \langle \hat{a} \rangle + \delta \hat{a}$ as discussed earlier, so that Eq. (8) becomes

$$\langle \hat{a}_{n+1} \rangle = r(\langle \hat{a}_n \rangle - |\langle \hat{a}_n \rangle|^2) - r\langle \delta \hat{a}_n^\dagger \delta \hat{a}_n \rangle . \qquad (9)$$

We can derive an equation for $\langle \delta \hat{a}_n^\dagger \delta \hat{a}_n \rangle$ from the Heisenberg equation of motion for $\delta \hat{a}$. This gives us an equation in which third-order quantum corrections appear. The following set of equations results when higher-order correlations than $\langle \delta \hat{a} \delta \hat{a} \rangle$, $\langle \delta \hat{a}^\dagger \delta \hat{a} \rangle$, and their Hermitian conjugates are neglected:

$$x_{n+1} = r(x_n - |x_n|^2) - r y_n , \qquad (10a)$$

$$y_{n+1} = -y_n e^{-2\beta} + e^{-\beta} r [(2 - x_n - x_n^*) y_n - x_n z_n^* - x_n^* z_n] , \qquad (10b)$$

$$z_{n+1} = -z_n e^{-2\beta} + e^{-\beta} r [2(1 - x_n^*) z_n - 2 x_n y_n - x_n] , \qquad (10c)$$

where $x = \langle \hat{a} \rangle$, $y = \langle \delta \hat{a}^\dagger \delta \hat{a} \rangle$, and $z = \langle \delta \hat{a} \delta \hat{a} \rangle$. In what follows we iterate the map (10) with x_0, y_0, and z_0 real, so that x_n, y_n, and z_n are real for all n. We take $y_0 = z_0 = 0$, as is the case if the oscillator is initially in a coherent state.

Equations (10) reduce to the classical, one-dimensional logistic map when the quantum corrections $y_n, z_n \to 0$. This occurs when the limit $\hbar \to 0$ is taken, although $\hbar$ does not appear explicitly in (10) because we have found it more convenient to work with $\hat{a}$ and $\hat{a}^\dagger$ (satisfying $[\hat{a}, \hat{a}^\dagger] = 1$) than with the coordinate and momentum variables $\hat{q}$ and $\hat{p}$ (satisfying $[\hat{q}, \hat{p}] = i\hbar$). To see quite generally that the quantum corrections vanish in the limit $\hbar \to 0$, consider the commutator $[F(\hat{q}, \hat{p}), G(\hat{q}, \hat{p})]$, where F and G are arbitrary functions of $\hat{q}$ and $\hat{p}$. From Taylor series expansions of F and G about $\langle \hat{q} \rangle, \langle \hat{p} \rangle$, and the commutator $[\delta \hat{q}, \delta \hat{p}] = i\hbar$, one readily obtains

$$[F(\hat{q},\hat{p}), G(\hat{q},\hat{p})] = i\hbar \{F, G\} + \tfrac{1}{2} i\hbar \left[\left(\frac{\partial^2 F}{\partial q^2} \frac{\partial^2 G}{\partial p^2} - \frac{\partial^2 F}{\partial p^2} \frac{\partial^2 G}{\partial q^2} \right) \langle \delta \hat{q} \delta \hat{p} + \delta \hat{p} \delta \hat{q} \rangle + \left(\frac{\partial^2 F}{\partial q^2} \frac{\partial^2 G}{\partial q \, \partial p} - \frac{\partial^2 G}{\partial q^2} \frac{\partial^2 F}{\partial q \, \partial p} \right) \langle \delta \hat{q}^2 \rangle - \left(\frac{\partial^2 F}{\partial p^2} \frac{\partial^2 G}{\partial q \, \partial p} - \frac{\partial^2 G}{\partial p^2} \frac{\partial^2 F}{\partial q \, \partial p} \right) \langle \delta \hat{p}^2 \rangle \right] + \cdots \qquad (11)$$

where $\{F, G\} = \{F(q,p), G(q,p)\}$ is the usual Poisson bracket of F and G with respect to $q \equiv \langle \hat{q} \rangle$ and $p \equiv \langle \hat{p} \rangle$. Now if $G(\hat{q}, \hat{p})$ is the Hamiltonian $H(\hat{q}, \hat{p})$, then (11) gives the time variation of F via the Heisenberg equation of motion $\dot{F}(\hat{q}, \hat{p}) = (1/i\hbar)[F(\hat{q}, \hat{p}), H(q,p)]$. The classical limit is defined by the Hamiltonian equation of motion $\dot{F}(\langle q \rangle, \langle p \rangle) = (1/i\hbar)^{-1} \{F(\langle q \rangle, \langle p \rangle), H(\langle q \rangle, \langle p \rangle)\}$, and this limit is obtained only if *all* quantum corrections, including the lowest-order corrections $\langle \delta \hat{q}^2 \rangle$, $\langle \delta \hat{p}^2 \rangle$, and $\langle \delta \hat{q} \delta \hat{p} + \delta \hat{p} \delta \hat{q} \rangle$ in (11), vanish in the limit $\hbar \to 0$. It follows in particular that the quantum corrections appearing in (10) must vanish in the classical limit $\hbar \to 0$.

Equations (10) include only the very lowest-order quantum corrections. If we include third- and higher-order corrections, then (10) is replaced by a larger set of equations. Indeed the full quantum mechanics of the problem involves an infinite number of coupled equations. However, one can show quite generally that the contribution of a higher quantum correction in our model goes as $\exp(-m\beta)$, where m is the order of the correction.[14] That is, the classical limit in our model can also be obtained by taking $\beta \to \infty$, since in this limit all quantum corrections are negligible after a few iterations and (10) reduces to the classical logistic map.

Equation (10a) has the same form as the logistic map with additive noise.[15–17] The similarity is not exact in that the "noise" represented by the last term in (10a) is derived from and coupled to the dynamics of the system and is not a random external perturbation as in previous work. *In fact the noise here is a measure of the strength of the quantum correlations.* By increasing the dissipation parameter β in the system we reduce the strength of the quantum correlations. We can observe the effect of the increase in dissipation by plotting a β-bifurcation diagram of x, as in Fig. 1 for $r = 3.8$. For small values of β we see stable behavior; the map is on a fixed point. As β is increased, however, we see a period-doubling transition to chaos. This period doubling should not be confused with the period-doubling transition with increasing r in the classical logistic map.[15–17]

Since the strong dissipation limit $\beta \to \infty$ of the "quantum logistic map" (10) gives the classical map, we should not be surprised that the high-dissipation orbits seen in Fig. 1 are chaotic, given the value of r for that figure. In Fig. 2 we present a β-bifurcation diagram for $r = 3.5$, corresponding to a period-four orbit in the classical map. We see a period doubling to the period-four behavior of the classical map. More generally we see a *period-doubling transition to the classical behavior as the dissipation parameter is varied.*

This is clearly illustrated in Fig. 3, which shows a β-bifurcation diagram for a value of r where the classical

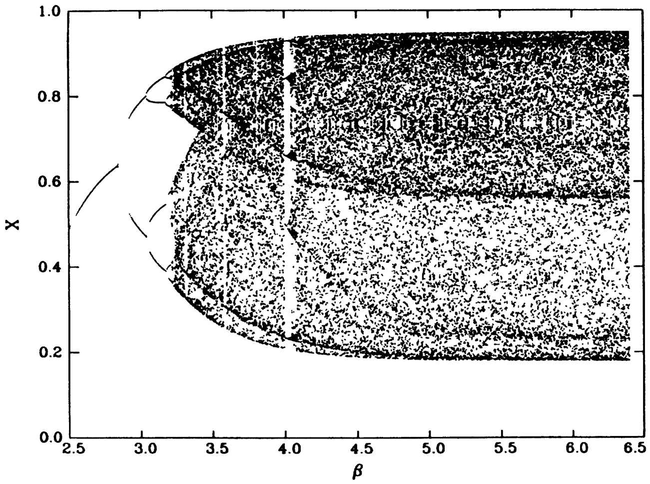

FIG. 1. β-bifurcation diagram for x with $r=3.8$.

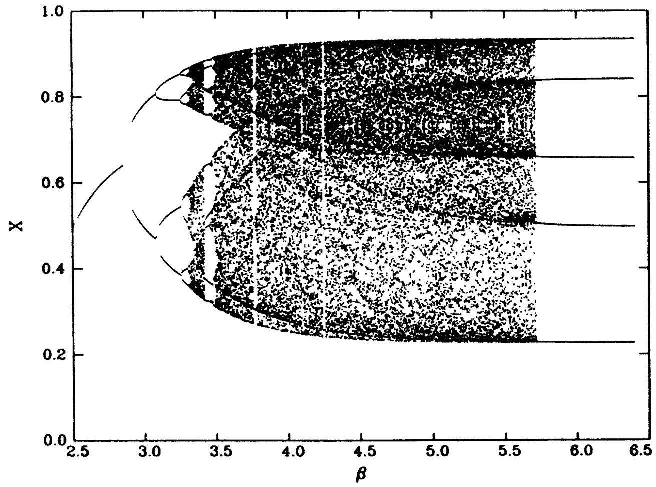

FIG. 3. As in Fig. 1 except that $r=3.74$.

limit is a period-five orbit embedded in the chaotic region. The period doubling to chaos, with increasing β, is followed by an abrupt transition to the classically predicted period-five orbit. In this case there exists an intermediate regime of dissipation in which the quantum correlations destabilize, rather than stabilize, the map. This pattern is generic to periodic orbits embedded in the chaotic regime of the classical map. Figures 1–3 present a variety of behavior that may be understood in terms of the following arguments.

It is clear from the equations (10) governing the map with lowest-order quantum corrections that, for sufficiently large β, the period doubling to chaos follows that of the classical map. For intermediate values of β, however, there are important differences. For such intermediate values of β the values of r where successive bifurcations occur are larger in the quantum map than in the classical map. This may be regarded as a manifestation of the enhanced stability accompanying the quantization of classically chaotic systems. In other words, the bifurcation diagram with respect to r is shifted towards $r=4.0$. A consequence of this shift is that we see a period-doubling transition as a function of r to the highest-period orbit allowed for that particular value of β. In addition, the chaotic orbits span a smaller portion of the unit interval at these intermediate values of the dissipation parameter.

This last result implies greater stability for intermediate dissipation than for large dissipation, which would appear to contradict what is observed in Fig. 3 for periodic orbits embedded in the chaotic regime of the classical map. The resolution lies in the shift of the critical values of r. The value of r corresponding, say, to the period-five orbit at intermediate β values is larger than that for the classical limit. With reference to the classical map, this shifts the dynamics, for intermediate β, from the narrow window of stability associated with the period-five orbit to the chaotic regime lying outside (corresponding to larger values of r). With increasing dissipation, the shift in the critical r values goes to zero and the dynamics of the classical map

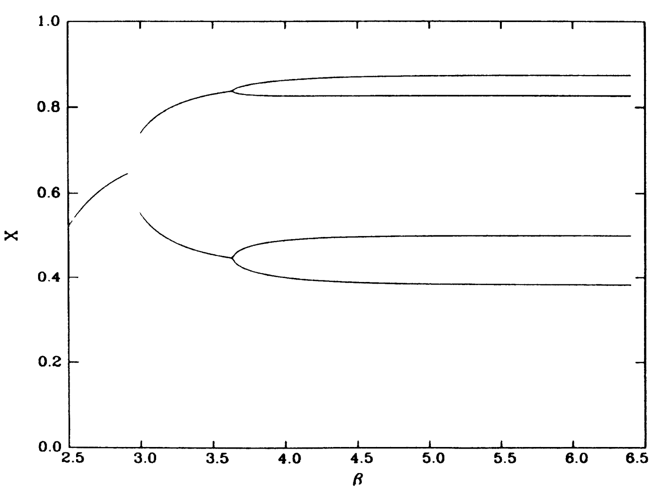

FIG. 2. As in Fig. 1 except that $r=3.5$.

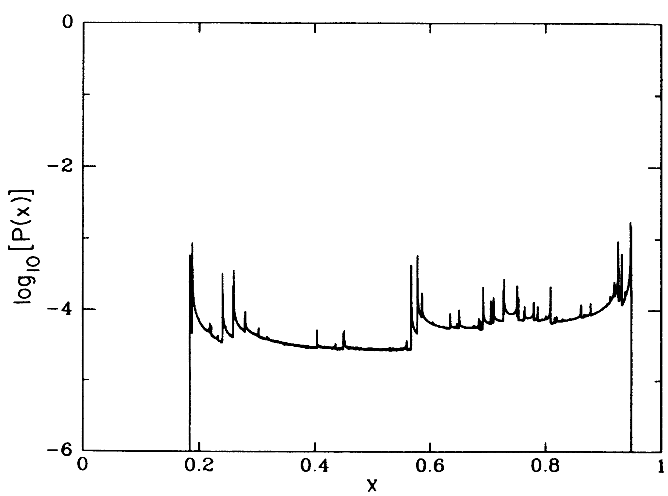

FIG. 4. Invariant probability distribution $P(x)$ calculated by dividing the x axis into 20000 bins and counting the number of occurrences of an x value in each bin during 10^7 iterations of the map with $\beta=5.0$ and $r=3.8$.

are recovered.

We wish to emphasize that the map (10), where the noise is self-consistently generated, is essentially different from the logistic map with additive random noise. This is clearly illustrated in Fig. 4, where we show a typical invariant probability distribution $P(x)$, obtained at fixed values of r and β. The similarity of the peak structure to that obtained for the classical map is striking. It is also clear that $P(x)$ does not exhibit the broadening of peaks seen in the case of the logistic map with external additive noise.[15–17] (Compare Fig. 4 with Figs. 11 and 12 of Ref. 16).

It should also be emphasized that our approach involves no factorization approximations.[4,18] Successively higher-order quantum corrections are included by increasing the dimensionality of the map, as discussed earlier. Although it is not as general as the approach of Graham,[5] for instance, it does enjoy great simplicity, and is easily generalized to continuous flows and conservative systems.

In summary, we have presented a straightforward technique for studying the effects of quantum correlations on a dissipative system. By doing this for the example of the logistic-map system we have found a period-doubling transition to the classical behavior as the dissipation parameter is varied.

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