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Superintegrability in classical mechanics

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Superintegrable Hamiltonians in three degrees of freedom possess more than three functionally independent globally defined and single-valued integrals of motion. Some familiar examples, such as the Kepler problem and the harmonic oscillator, have been known since the time of Laplace. Here, a classification theorem is given for superintegrable potentials with invariants that are quadratic polynomials in the canonical momenta. Such systems must possess separable solutions to the Hamilton-Jacobi equation in more than one coordinate system. There are 11 coordinate systems for which the Hamilton-Jacobi equation separates in $\mathbb{R}^3$. One coordinate system may be arbitrarily rotated or translated with respect to the other, yielding 66 distinct cases. In each case, the differential equations for separability in the two coordinates are integrated to give a complete list of all superintegrable potentials with four or five quadratic integrals. The tables—which may be consulted independently of the main body of the paper—list the distinct superintegrable potentials, the separating coordinates, and the isolating integrals of the motion. If there exist five isolating integrals, then all finite classical trajectories are closed; if only four, then the trajectories are restricted to a two-dimensional surface. An extraordinary consequence of the work is the discovery of perturbations to both the Kepler problem and the harmonic oscillator that do not destroy the fragile degeneracy. The perturbed systems still have five isolating integrals of the motion.

I. INTRODUCTION

The Kepler problem and harmonic oscillator possess properties that have special interest about them—for example, all finite classical trajectories are closed and all quantum eigenenergies are multiply degenerate. This is because the potentials admit separable solutions to the Hamilton-Jacobi equation[1,2] in more than one coordinate system, which manifests itself in the existence of additional isolating integrals of the motion. For example, it has long been known[3–6] that the Kepler problem possesses five functionally independent isolating integrals. These are generated by separating the Hamilton-Jacobi equation in spherical polar and rotational parabolic coordinates.[7]

The conditions for separability of the Hamilton-Jacobi equation in orthogonal coordinates were first published by Stäckel[8] and are repeated in Goldstein.[9] Robertson[10] showed that the Schrödinger equation possesses a separable solution if the Hamilton-Jacobi equation does, provided the Ricci tensor diagonalizes. For three-dimensional flat space, the 11 possible coordinate systems in which separation may take place were deduced in a paper by Eisenhart[11] and are listed in Morse and Feshbach.[12] Some—such as rectangular Cartesian or spherical polar—are very familiar, others—such as rotational parabolic or elliptic cylindrical—much less so. They are all obtainable as degenerations of the confocal ellipsoidal coordinates.[13] For each of the coordinates, Eisenhart[14] determined the form of the potential that permits separation of variables.

These potentials, designated Stäckel or separable potentials, played a crucial role in Hamiltonian mechanics before the development of more qualitative geometric methods for differential equations. For example, Jacobi's study of the geodesics of a triaxial ellipsoid[15,16] or Neumann's investigation of a particle moving on a sphere under the action of a linear force[17,18] exploited the separability of the Hamilton-Jacobi equation to solve the equations of motion by quadratures. Stäckel potentials are still widely used in many branches of physics—for example, in the calculation of an intermediary drag-free orbit of an artificial satellite orbiting an oblate planet,[19,20] in the elucidation of the structure and dynamics of elliptical galaxies[21–23] and in the determination of the wave functions of the hydrogen atom and molecule.[24–26]

Arnold Sommerfeld[27] in his classic *Atomic Structure and Spectral Lines* seems to have been the first to note that if a potential is separable in more than one coordinate system, it possesses additional isolating functionally independent integrals, i.e., is superintegrable. The first systematic inquiry into this problem was begun by Winternitz and co-workers. They[28,29] found every potential in two degrees of freedom that is separable in more than one way. They isolated the dynamical symmetry group by identifying the degeneracy of the energy levels of the quantum system with the dimensions of all irreducible representations of a Lie group. Subsequently, they[30] extended this to three degrees of freedom by finding every potential separable in spherical polars and at least one additional system. The remaining possibilities do not appear to have been investigated and are given here.

In Sec. II it is shown that a superintegrable Hamiltonian in three degrees of freedom with invariants quadratic in the canonical momenta must admit a separable solution to the Hamilton-Jacobi equation in at least two coor-

TABLE I. A Hamiltonian system in three degrees of freedom is maximally superintegrable if it admits five globally defined and single-valued integrals of the motion. The table gives a complete list of all maximally superintegrable natural Hamiltonians in a three-dimensional flat space with integrals that are quadratic polynomials in the canonical momenta. Such systems must admit separable solutions to the Hamilton-Jacobi equation in at lead two coordinate systems. Rotational parabolic coordinates (ζ,η,ϕ) are defined in (3.8), spherical polars (r,θ,ϕ) in (3.18), and parabolic cylindricals $(\zeta',\eta' z)$ in (3.19). Note that $\mathbf{P}$ is the linear momentum, $\mathbf{L}$ is the angular momentum, and the components (P_1,P_2,P_3) and (L_1,L_2,L_3) always refer to the Cartesian frame.

Potential	Separating coordinates	Isolating integrals
$k(x^2+y^2+z^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$	Rectangular Cartesian, spherical polar, cylindrical polar, elliptic cylindrical, confocal ellipsoidal, conical, oblate spheroidal, prolate spheroidal	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+k(x^2+y^2+z^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$, $I_1=\frac{1}{2}P_1^2+kx^2+\frac{k_1}{x^2}$, $I_2=\frac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2}$, $I_3=\frac{1}{2}\lvert\mathbf{L}\rvert^2+\frac{k_1}{\sin^2\theta\cos^2\phi}+\frac{k_2}{\sin^2\theta\sin^2\phi}+\frac{k_3}{\cos^2\theta}$, $I_4=\frac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}$
$-\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}$	Rotational parabolic, conical, spherical polar	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2-\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}$, $I_1=\frac{1}{2}L^2+\frac{k_1}{\sin^2\theta\cos^2\phi}+\frac{k_2}{\sin^2\theta\sin^2\phi}$, $I_2=\frac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}$, $I_3=\frac{1}{2}L_2^2+\frac{k_1\cos^2\theta}{\sin^2\theta\cos^2\phi}$, $I_4=L_1P_2-P_1L_2+(\zeta-\eta)\times\left[-\frac{k}{\zeta+\eta}+\frac{k_1}{\zeta\eta\cos^2\phi}+\frac{k_2}{\zeta\eta\sin^2\phi}\right]$
$\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$	Spherical polar, parabolic cylindrical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$, $I_1=\frac{1}{2}\lvert\mathbf{L}\rvert^2+\frac{k_1\cos\phi+k_2}{\sin^2\theta\sin^2\phi}+\frac{k_3}{\cos^2\theta}$, $I_2=\frac{1}{2}L_3^2+\frac{k_1\cos\phi+k_2}{\sin^2\phi}$, $I_3=\frac{1}{2}P_3^2+\frac{k_3}{z^2}$, $I_4=L_3P_2+\frac{(k_1+k_2)\zeta'^2+(k_1-k_2)\eta'^2}{\zeta'\eta'(\zeta'+\eta')}$
$\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+k_3z$	Rotational parabolic, parabolic cylindrical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+k_3z$, $I_1=L_1P_2-P_1L_2-\frac{k_3\zeta\eta}{2}+\frac{(\zeta-\eta)(k_1\cos\phi+k_2)}{\zeta\eta\sin^2\phi}$, $I_2=\frac{1}{2}L_3^2+\frac{k_1\cos\phi+k_2}{\sin^2\phi}$, $I_3=\frac{1}{2}P_3^2+k_3z$, $I_4=L_3P_2+\frac{(k_1+k_2)\zeta'^2+(k_1-k_2)\eta'^2}{\zeta'\eta'(\zeta'+\eta')}$

TABLE I. (*Continued*).

Potential	Separating coordinates	Isolating integrals
$k(x^2+y^2)+4kz^2+\frac{k_1}{x^2}+\frac{k_2}{y^2}$	Rectangular Cartesian, rotational parabolic, elliptic cylindrical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+k(x^2+y^2)+4kz^2+\frac{k_1}{x^2}+\frac{k_2}{y^2}$, $I_1=\frac{1}{2}P_1^2+kx^2+\frac{k_1}{x^2}$, $I_2=\frac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2}$, $I_3=\frac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}$, $I_4=L_1P_2-P_1L_2-(\zeta-\eta)\left[k\zeta\eta-\frac{k_1}{\zeta\eta\cos^2\phi}-\frac{k_2}{\zeta\eta\sin^2\phi}\right]$

dinate systems. There are 11 such coordinate systems, which may be arbitrarily translated or rotated. The most general result is obtained when the symmetry of the separating coordinates is maximized by taking the origin and axes of the Euclidean frames to coincide. Additionally, a potential may separate in the same coordinate system rotated or translated with respect to itself. In each of the 66 (i.e., ${}^{11}C_2+11$) cases, the differential equations for separability in the two coordinates are integrated. Very often, as the symmetry of the separating coordinates is diminished, the result is a special case of a more general superintegrable potential and so the 66 cases yield only 12 distinct superintegrable potentials. Details of some of the calculations are given in Sec. III and the results are collected in the tables. This finishes the program of Winternitz and generates a complete list, up to

TABLE II. A Hamiltonian system in three degrees of freedom is minimally superintegrable if it admits four globally defined and single-valued integrals of the motion. The table gives a complete list of all minimally superintegrable natural Hamiltonians in a three-dimensional flat space with integrals that are quadratic polynomials in the canonical momenta. Such systems must admit separable solutions to the Hamilton-Jacobi equation in at least two coordinate systems. Rotational parabolic coordinates (ζ,η,ϕ) are defined in (3.8), spherical polars (r,θ,ϕ) in (3.18), cylindrical polars (R,ϕ,z) in (3.45), and parabolic cylindricals (ζ',η',z) and (λ',μ',z) in (3.19) and (3.51), respectively. Note that $\mathbf{P}$ is the linear momentum, $\mathbf{L}$ is the angular momentum, and the components (P_1,P_2,P_3) and (L_1,L_2,L_3) always refer to the Cartesian frame. F denotes an arbitrary function of the indicated argument.

Potential	Separating coordinates	Isolating integrals
$F(r)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$	Spherical polar, conical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+F(r)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$, $I_1=\frac{1}{2}L_1^2+\frac{k_2\cos^2\theta}{\sin^2\theta\sin^2\phi}+\frac{k_3\sin^2\phi\sin^2\theta}{\cos^2\theta}$, $I_2=\frac{1}{2}L_2^2+\frac{k_1\cos^2\theta}{\sin^2\theta\cos^2\phi}+\frac{k_3\cos^2\phi\sin^2\theta}{\cos^2\theta}$, $I_3=\frac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}$
$k(x^2+y^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+F(z)$	Rectangular Cartesian, cylindrical polar, elliptic cylindrical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+k(x^2+y^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+F(z)$, $I_1=\frac{1}{2}P_1^2+kx^2+\frac{k_1}{x^2}$, $I_2=\frac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2}$, $I_3=\frac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}$
$4kx^2+ky^2+\frac{k_2}{y^2}+F(z)$	Rectangular Cartesian, parabolic cylindrical,	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+4kx^2+ky^2+\frac{k_2}{y^2}+F(z)$, $I_1=\frac{1}{2}P_1^2+4kx^2$, $I_2=\frac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2}$, $I_3=L_3P_2-k\zeta'\eta'(\zeta'-\eta')+\frac{k_2(\zeta'-\eta')}{\zeta'\eta'}$

TABLE II. (*Continued*).

Potential	Separating coordinates	Isolating integrals
$\frac{k}{(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+\frac{k_1 x}{y^2(x^2+y^2)^{1/2}}+F(z)$	Cylindrical polar, parabolic cylindrical	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+\frac{k}{(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+\frac{k_1 x}{y^2(x^2+y^2)^{1/2}}+F(z),$ $I_1=\frac{1}{2}L_3^2+\frac{k_1\cos\phi+k_2}{\sin^2\phi},$ $I_2=\frac{1}{2}P_3^2+F(z),$ $I_3=L_3P_2+\frac{1}{\zeta'+\eta'}\times\left[k(\zeta'-\eta')+\frac{\eta'(k_1-k_2)}{\zeta'}+\frac{\zeta'(k_1+k_2)}{\eta'}\right]$
$k(x^2+y^2+z^2)+\frac{k_3}{z^2}+\frac{F(y/x)}{x^2+y^2}$	Spherical polar, cylindrical polar, prolate spheroidal, oblate spheroidal	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+k(x^2+y^2+z^2)+\frac{k_3}{z^2}+\frac{F(y/x)}{x^2+y^2},$ $I_1=\frac{1}{2}P_3^2+kz^2+\frac{k_3}{z^2},$ $I_2=\frac{1}{2}\lvert\mathbf{L}\rvert^2+\frac{k_3}{\cos^2\theta}+\frac{F(\tan\phi)}{\sin^2\theta},$ $I_3=\frac{1}{2}L_3^2+F(\tan\phi)$
$k(x^2+y^2)+4kz^2+\frac{F(y/x)}{x^2+y^2}$	Cylindrical polar, rotational parabolic	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+k(x^2+y^2)+\frac{F(y/x)}{x^2+y^2},$ $I_1=\frac{1}{2}P_3^2+4kz^2,$ $I_2=\frac{1}{2}L_3^2+F(\tan\phi),$ $I_3=L_1P_2-P_2L_1-k\zeta\eta(\zeta-\eta)+\frac{(\zeta-\eta)F(\tan\phi)}{\zeta\eta}$
$-\frac{k}{r}+\frac{k_1 z}{r(x^2+y^2)}+\frac{F(y/x)}{x^2+y^2}$	Spherical polar, rotational parabolic	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2-\frac{k}{r}+\frac{k_1 z}{r(x^2+y^2)}+\frac{F(y/x)}{x^2+y^2},$ $I_1=\frac{1}{2}\lvert\mathbf{L}\rvert^2+\frac{k_1\cos\theta+F(\tan\phi)}{\sin^2\theta},$ $I_2=\frac{1}{2}L_3^2+F(\tan\phi),$ $I_3=L_1P_2-P_1L_2-\frac{k(\zeta-\eta)}{\zeta+\eta}+\frac{k_1(\zeta^2+\eta^2)}{\zeta\eta(\zeta+\eta)}+\frac{(\zeta-\eta)F(\tan\phi)}{\zeta\eta}$
$\frac{k}{R}+\frac{k_1\sqrt{R+y}}{R}+\frac{k_2\sqrt{R-y}}{R}+F(z)$	Mutually orthogonal parabolic cylindricals	$E=\frac{1}{2}\lvert\mathbf{P}\rvert^2+\frac{k}{R}+\frac{k_1\sqrt{R+y}}{R}+\frac{k_2\sqrt{R-y}}{R}+F(z),$ $I_1=\frac{1}{2}P_3^2+F(z),$ $I_2=L_3P_1+\frac{1}{\lambda'+\mu'}[k(\lambda'-\mu')-k_1\sqrt{\lambda'}\mu'+k_2\sqrt{\mu'}\lambda'],$ $I_3=L_3P_2+\frac{1}{\zeta'+\eta'}\left[k(\zeta'-\eta')-(k_1+k_2)\left[\frac{\zeta'}{2}\right]^{1/2}\eta'+(k_1-k_2)\left[\frac{\eta'}{2}\right]^{1/2}\zeta'\right].$

the equivalence class of linear transformations, of all potentials with four or five isolating functionally independent integrals quadratic in the momenta.

The superintegrable potentials share the interesting properties of the Kepler problem and the harmonic oscillator. The classical trajectories always fill a surface of less dimensions than the number of degrees of freedom. In a companion paper,[31] the quantum wave functions are found by solving the Schrödinger equation and the degeneracy is used to isolate the dynamical symmetry group. Representations of the groups are constructed in terms of annihilation and creation operators of energy quanta.

Tables I and II—which may be consulted independently of the main body of the paper—list the distinct superintegrable potentials, the separating coordinates, and the isolating integrals of the motion.

II. SUPERINTEGRABILITY AND SEPARABILITY

A d-dimensional Hamiltonian is *superintegrable* if there exist more than d functionally independent globally defined and single-valued integrals. Not all the integrals of a superintegrable system can be in involution, but they must be functionally independent otherwise the extra invariants are trivial. To test for functional independency of a set of n integrals in d degrees of freedom, the $n \times 2d$ Jacobian

$$\frac{\partial(I_1,\ldots,I_n)}{\partial(x_i,p_i)} \tag{2.1}$$

is constructed. If it possesses rank n, then the integrals are all nontrivial. In this paper the emphasis is on natural Hamiltonian systems in three degrees of freedom in flat space i.e.,

$$H=\tfrac{1}{2}|\mathbf{P}|^2+V(\mathbf{x}) . \tag{2.2}$$

We prove that the the problem of finding all superintegrable potentials with quadratic invariants is exactly equivalent to finding all potentials separable in more than one way in the confocal ellipsoidal coordinates and their degenerations.

First, we show that two commuting quadratic integrals can exist if and only if the potential separates in the confocal ellipsoidal coordinates or their degenerations. This is a straightforward extension to classical mechanics of a result already known in quantum mechanics.[30] All invariants must be either even or odd in the momenta and so a quadratic integral may be taken without loss of generality as

$$K_1=\phi_{ik}(\mathbf{x})P_iP_k+g_1(\mathbf{x}) . \tag{2.3}$$

The vanishing of the Poisson bracket of H and K_1 leads to the system of ten equations

$$\frac{\partial\phi_{ik}}{\partial x_j}P_iP_jP_k=0 . \tag{2.4}$$

Solving, it follows that any quadratic integral must be a symmetric bilinear polynomial in the generators of the Euclidean group $E(3)$, i.e.,

$$K_1=a_{ik}L_iL_k+2b_{ik}L_iP_k+c_{ik}P_iP_k+g_1(\mathbf{x}) , \tag{2.5}$$

where $L_i=\epsilon_{ijk}x_jP_k$, $a_{ik}=a_{ki}$, and $c_{ik}=c_{ki}$. Rotations may be used to set $a_{ik}=\mathrm{diag}(a_1,a_2,a_3)$. By hypothesis, there exists a further quadratic integral K_2, namely,

$$K_2=\alpha_{ik}L_iL_k+2\beta_{ik}L_iP_k+\gamma_{ik}P_iP_k+g_2(\mathbf{x}) . \tag{2.6}$$

Requiring the Poisson bracket of K_1 and K_2 to vanish implies that $\alpha_{ik}=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3)$ and leads to the same systems of equations given in Ref. 30 and labeled (A), (B), (C), (A'), (B'), (C'), and (D'). The solutions in all cases are separable in the confocal ellipsoidal coordinates or their degenerations.

The commuting integrals K_1 and K_2 are identified with the commuting operators of the Cartan subalgebra of a semisimple Lie algebra. The Cartan-Weyl basis[32–34] is

$$\begin{aligned}&[K_i,K_j]=0 \quad (i,j=1,\ldots,l) ,\\&[K_i,I_\alpha]=\alpha_iI_\alpha ,\\&[I_\alpha,I_\beta]=N_{\alpha\beta}I_{\alpha+\beta} , \quad \text{if } \alpha+\beta\neq 0 ,\\&[I_\alpha,I_{-\alpha}]=\alpha^iK_i .\end{aligned} \tag{2.7}$$

Here, $N_{\alpha\beta}=0$ if $\alpha+\beta$ is not a root, l is the rank of the algebra and α_i are the covariant components of the root vector. A semisimple Lie algebra decomposes into a direct sum of the Cartan subalgebra and one-dimensional root spaces which are generated by the root vectors I_α. The important point is that if α and β are roots, then there is a finite string of roots[35]

$$\beta+j\alpha,\beta+(j-1)\alpha,\ldots,\beta,\ldots,\beta-(k-1)\alpha,\beta-k\alpha , \tag{2.8}$$

where j and k are positive integers. For any I_β, choosing γ equal to $(j+1)\alpha$ and using (2.7) implies that there exists a commuting operator

$$[I_\beta,I_\gamma]=0 . \tag{2.9}$$

Now, superintegrable systems possess at least one additional integral I_1. By (2.9), there then always exists a further integral I_2 (not necessarily distinct from or functionally independent of the other integrals) which commutes with I_1. This may be verified in three degrees of freedom by examining the root spaces of the four locally distinct semisimple Lie algebras of rank two, namely, su(3), so(5), sp(4), and so(4), as was done in Ref. 30. A superintegrable system therefore possesses two pairs of two commuting integrals (other than the Hamiltonian). By hypothesis, the integrals are all quadratic and so the potential must separate in the confocal ellipsoidal coordinates and their degenerations in at least two different ways.

III. HAMILTONIANS SEPARABLE IN MORE THAN ONE COORDINATE SYSTEM

There are 11 coordinate systems in which the three-dimensional Hamilton-Jacobi equation separates, namely, rectangular Cartesian, spherical polar, cylindrical polar, rotational parabolic, parabolic cylindrical, elliptic cylindrical, oblate and prolate spheroidal, conical, paraboloidal and confocal ellipsoidal.[14,30] We wish to find every potential that separates in at least two coordinate systems, which may be rotated or translated with respect to each other. The most general result is always obtained when the coordinate systems are arranged *canonically*, that is, with origin and axes coinciding. Additionally, a potential may separate in two coordinate systems of the same type, but displaced or rotated with respect to each other. In the 11 cases when one of the separating coordinates is spherical polars, the solutions are given in Ref. 30; the remaining 55 cases are investigated in Ref. 36. In this section we give details of the calculations in five cases to illustrate the general methods. The remaining results are reported in the tables, which give a complete list of all distinct superintegrable potentials with quadratic invariants. Note that, in what follows, A, B, C, D, E, and F always denote the arbitrary functions in Stäckel or separable potentials and k_1, k_2, k_3, etc. are numerical constants.

A. Conical and rotational parabolic coordinates

Conical coordinates (r,μ,ν) are given in terms of rectangular Cartesians (x,y,z) by

$$r^2=x^2+y^2+z^2\ , \tag{3.1}$$

$$\frac{x^2}{\tau+\alpha}+\frac{y^2}{\tau+\beta}+\frac{z^2}{\tau+\gamma}=0\ , \tag{3.2}$$

where $\tau=\mu$ or ν and $-\gamma\le\nu\le-\beta\le\mu\le-\alpha$ with α, β, and γ constants. Surfaces of constant r are spheres, while surfaces of constant μ or ν define cones of elliptic cross section. In terms of conical coordinates, rectangular Cartesians (x,y,z) are

$$\begin{aligned} x^2&=\frac{r^2(\mu+\alpha)(\nu+\alpha)}{(\alpha-\gamma)(\alpha-\beta)}\ ,\\ y^2&=\frac{r^2(\mu+\beta)(\nu+\beta)}{(\beta-\alpha)(\beta-\gamma)}\ ,\\ z^2&=\frac{r^2(\mu+\gamma)(\nu+\gamma)}{(\gamma-\alpha)(\gamma-\beta)}\ . \end{aligned} \tag{3.3}$$

The transformation from (x,y,z) is unique. The converse is not true—each choice $(r,,\mu,\nu)$ corresponds in general to eight different points $(\pm x,\pm y,\pm z)$. The Stäckel or separable potential in conical coordinates[14,30] has the form

$$V=A(r^2)+\frac{B(\mu)+C(\nu)}{r^2(\mu-\nu)}\ . \tag{3.4}$$

A necessary and sufficient condition for V to separate in conical coordinates is

$$\frac{\partial^2}{\partial\mu\,\partial r}(r^2V)=0\ , \tag{3.5}$$

$$\frac{\partial^2}{\partial\nu\,\partial r}(r^2V)=0\ , \tag{3.6}$$

$$\frac{\partial^2}{\partial\mu\,\partial\nu}[(\mu-\nu)V]=0\ . \tag{3.7}$$

Rotational parabolic coordinates (ζ,η,ϕ) are defined as

$$\zeta=r+z,\quad \eta=r-z,\quad \phi=\arctan\left[\frac{y}{x}\right]\ , \tag{3.8}$$

where ζ and η take values from 0 to ∞. Surfaces of constant ζ or η are confocal paraboloids of revolution with the z axis as the axis of symmetry. The point with rotational parabolic coordinates (ζ,η,ϕ) has rectangular Cartesian coordinates (x,y,z) given by

$$x=\sqrt{\zeta\eta}\cos\phi,\quad y=\sqrt{\zeta\eta}\sin\phi,\quad z=\tfrac{1}{2}(\zeta-\eta)\ . \tag{3.9}$$

The Stäckel or separable potential in rotational parabolic coordinates[7,14] may be taken as

$$V=\frac{D(\zeta)+E(\eta)}{\zeta+\eta}+\frac{F(\tan^2\phi)}{\zeta\eta}\ . \tag{3.10}$$

If the potential is to separate in both coordinates, then (3.5) implies that

$$\frac{\partial^2}{\partial\nu\,\partial r}\left[rD\left[r+\frac{r(\nu+\gamma)^{1/2}(\mu+\gamma)^{1/2}}{(\gamma-\alpha)^{1/2}(\gamma-\beta)^{1/2}}\right]+rE\left[r-\frac{r(\nu+\gamma)^{1/2}(\mu+\gamma)^{1/2}}{(\gamma-\alpha)^{1/2}(\gamma-\beta)^{1/2}}\right]\right]=0\ . \tag{3.11}$$

This simplifies to give

$$2D'(\zeta)-2E'(\eta)+\zeta D''(\zeta)-\eta E''(\eta)=0\ . \tag{3.12}$$

Separating and solving, we find the solutions can be taken without loss of generality as

$$D(\zeta)=\frac{k_3}{\zeta}-2k,\quad E(\eta)=\frac{k_4}{\eta}\ . \tag{3.13}$$

Equation (3.6) yields no additional information, but (3.7) implies $k_3=k_4=0$ and

$$\tan^2\phi\sec^4\phi F''(\tan^2\phi)+2\sec^2\phi F'(\tan^2\phi)-2F=0\ . \tag{3.14}$$

The ordinary differential equation (3.14) is easily solved to obtain

$$F(\tan^2\phi)=\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}\ . \tag{3.15}$$

So, the most general potential separable in conical and rotational parabolic coordinates is

$$V=-\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}\ . \tag{3.16}$$

By separating the Hamilton-Jacobi equation, the five independent isolating integrals may be taken as

$$\begin{aligned} E&=\tfrac{1}{2}|\mathbf{P}|^2-\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}\ ,\\ I_1&=\tfrac{1}{2}L^2+\frac{k_1}{\sin^2\theta\cos^2\phi}+\frac{k_2}{\sin^2\theta\sin^2\phi}\ ,\\ I_2&=\tfrac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}\ ,\\ I_3&=\tfrac{1}{2}L_2^2+\frac{k_1\cos^2\theta}{\sin^2\theta\cos^2\phi}\ ,\\ I_4&=L_1P_2-P_1L_2\\ &\quad+(\zeta-\eta)\left[-\frac{k}{\zeta+\eta}+\frac{k_1}{\zeta\eta\cos^2\phi}+\frac{k_2}{\zeta\eta\sin^2\phi}\right]\ , \end{aligned} \tag{3.17}$$

where (r,θ,ϕ) are familiar spherical polar coordinates defined as

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta\ . \tag{3.18}$$

The potential is additionally separable in spherical polar coordinates, but is not given in Ref. 30 as there are more general potentials separable in either spherical polar and conical or spherical polar and rotational parabolic coordinates (cf. Table II). Keplerian motion is recovered in the special case $k_1=k_2=0$. The integral I_4 then becomes the z component of the Runge-Lenz vector[7] and the integrals I_1, I_2, and I_3 imply conservation of angular momentum.

B. Rotational parabolic and parabolic cylindrical coordinates

Parabolic cylindrical coordinates (ζ',η',z) are defined as

$$\zeta'=(x^2+y^2)^{1/2}+x,\quad \eta'=(x^2+y^2)^{1/2}-x\ , \tag{3.19}$$

where ζ' and η' can take values from 0 to ∞. Surfaces of constant ζ' or η' are confocal parabolic cylinders. Each choice (ζ',η',z) corresponds to two different points $(x,\pm y,z)$ given by

$$x=\tfrac{1}{2}(\zeta'-\eta'),\quad y=\sqrt{\zeta'\eta'}\ . \tag{3.20}$$

The Stäckel or separable potential in parabolic cylindrical coordinates[14,30] is

$$V=\frac{D(\zeta')+E(\eta')}{\zeta'+\eta'}+F(z^2)\ . \tag{3.21}$$

A necessary and sufficient condition for V to separate in parabolic cylindrical coordinates is

$$\frac{\partial^2 V}{\partial\zeta'\partial z}=0\ , \tag{3.22}$$

$$\frac{\partial^2 V}{\partial\eta'\partial z}=0\ , \tag{3.23}$$

$$\frac{\partial^2}{\partial\zeta'\partial\eta'}[(\zeta'+\eta')V]=0\ . \tag{3.24}$$

The Stäckel potential in rotational parabolic coordinates (ζ,η,ϕ) is

$$V=\frac{A(\zeta)+B(\eta)}{\zeta+\eta}+\frac{C(\tan^2\phi)}{\zeta\eta}\ . \tag{3.25}$$

From (3.22), it can be deduced that

$$(\zeta+\eta)^2[\zeta A''(\zeta)-\eta B''(\eta)]+6(\zeta-\eta)[A(\zeta)+B(\eta)]+2(\zeta+\eta)[(\eta-2\zeta)A'(\zeta)+(2\eta-\zeta)B'(\eta)]=0\ . \tag{3.26}$$

In particular, (3.26) must hold on $\eta=0$. Choosing $B(0)$ and $B'(0)$ to be zero without loss of generality, then $A(\zeta)$ satisfies the equation

$$\zeta^2A''(\zeta)-4\zeta A'(\zeta)+6A(\zeta)=0\ , \tag{3.27}$$

which has the solution

$$A(\zeta)=k_4\zeta^3+\tfrac{1}{2}k_3\zeta^2\ . \tag{3.28}$$

Substituting into (3.26) enables $B(\eta)$ to be found as

$$B(\eta)=k_4\eta^3-\tfrac{1}{2}k_3\eta^2\ . \tag{3.29}$$

Equation (3.23) gives no further information, but (3.24) implies $k_4=0$ and

$$2C(\tan^2\phi)-4\tan^2\phi\sec^4\phi C''(\tan^2\phi)-2\sec^2\phi(4+3\tan^2\phi)C'(\tan^2\phi)=0\ . \tag{3.30}$$

This can be integrated to give

$$C(\tan^2\phi)=\frac{k_1\cos\phi+k_2}{\sin^2\phi}\ . \tag{3.31}$$

Accordingly, the solution for the most general potential separable in rotational parabolic and parabolic cylindrical coordinates is

$$V=\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+k_3z\ . \tag{3.32}$$

By separating the Hamilton-Jacobi equation, the five isolating independent integrals may be taken as

$$\begin{aligned}
E&=\tfrac{1}{2}|\mathbf{P}|^2+\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+k_3z\ ,\\
I_1&=L_1P_2-P_1L_2-\frac{k_3\zeta\eta}{2}+\frac{(\zeta-\eta)(k_1\cos\phi+k_2)}{\zeta\eta\sin^2\phi}\ ,\\
I_2&=\tfrac{1}{2}L_3^2+\frac{k_1\cos\phi+k_2}{\sin^2\phi}\ ,\\
I_3&=\tfrac{1}{2}P_3^2+k_3z\ ,\\
I_4&=L_3P_2+\frac{(k_1+k_2)\zeta'^2+(k_1-k_2)\eta'^2}{\zeta'\eta'(\zeta'+\eta')}\ .
\end{aligned} \tag{3.33}$$

C. Rotational parabolic and rectangular Cartesian coordinates

Using the form of the Stäckel potentials in the two coordinate systems given in Refs. 14 and 30, we have that

$$\begin{aligned}
V&=\frac{A(\zeta)+B(\eta)}{\zeta+\eta}+\frac{C(\tan^2\phi)}{\zeta\eta}\\
&=D(x^2)+E(y^2)+F(z^2)\ ,
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
x&=\sqrt{\zeta\eta}\cos\phi\ ,\\
y&=\sqrt{\zeta\eta}\sin\phi\ ,\\
z&=\tfrac{1}{2}(\zeta-\eta)\ .
\end{aligned} \tag{3.35}$$

To separate in rectangular Cartesians, the potential must satisfy

$$\frac{\partial^2 V}{\partial x\,\partial z}=0\ , \tag{3.36}$$

$$\frac{\partial^2 V}{\partial y\,\partial z}=0\ , \tag{3.37}$$

$$\frac{\partial^2 V}{\partial x\,\partial y}=0\ . \tag{3.38}$$

Equation (3.36) leads to exactly (3.26) which has been shown to admit solutions

$$A(\zeta)=k\zeta^3+\frac{k_3\zeta^2}{2},\quad B(\eta)=k\eta^3-\frac{k_3\eta^2}{2}\ . \tag{3.39}$$

To determine the function C, (3.38) is used to deduce

$$\sec^4\phi\tan^2\phi C''(\tan^2\phi)+2\sec^2\phi C'(\tan^2\phi)-2C(\tan^2\phi)=0\ . \tag{3.40}$$

This is readily integrated to give

$$C(\tan^2\phi)=\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}\ . \tag{3.41}$$

Accordingly, the most general potential separable in rotational parabolic and Cartesian coordinates is

$$V=k(x^2+y^2)+4kz^2+\frac{k_1}{x^2}+\frac{k_2}{y^2}+k_3z\ . \tag{3.42}$$

The linear term in z may be made to vanish by performing a real translation and so without loss of generality $k_3=0$. By separating the Hamilton-Jacobi equation, the five isolating integrals are found to be

$$\begin{aligned}
E&=\tfrac{1}{2}|\mathbf{P}|^2+k(x+y^2)+4kz^2+\frac{k_1}{x^2}+\frac{k_2}{y^2}\ ,\\
I_1&=\tfrac{1}{2}P_1^2+kx^2+\frac{k_1}{x^2}\ ,\\
I_2&=\tfrac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2}\ ,\\
I_3&=\tfrac{1}{2}L_3^2+\frac{k_1}{\cos^2\phi}+\frac{k_2}{\sin^2\phi}\ ,\\
I_4&=L_1P_2-P_1L_2-(\zeta-\eta)\left[k\zeta\eta-\frac{k_1}{\zeta\eta\cos^2\phi}-\frac{k_2}{\zeta\eta\sin^2\phi}\right]\ .
\end{aligned} \tag{3.43}$$

The anisotropic harmonic oscillator with rational frequency ratio 1:1:2 is included as the special case $k_1=k_2=0$.

D. Cylindrical polar and parabolic cylindrical coordinates

By assumption, the potential has structure

$$V=A(R)+\frac{B(\phi)}{R^2}+F(z)=\frac{D(\zeta')+E(\eta')}{\zeta'+\eta'}+F(z)\ , \tag{3.44}$$

where

$$\begin{aligned}
x&=\tfrac{1}{2}(\zeta'-\eta')=R\cos\phi\ ,\\
y&=\sqrt{\zeta'\eta'}=R\sin\phi\ .
\end{aligned} \tag{3.45}$$

To separate in cylindrical polars, the potential must satisfy

$$\frac{\partial^2}{\partial R\,\partial\phi}(R^2V)=0\ , \tag{3.46}$$

which can be simplified to yield the ordinary differential equation

$$2D'(\zeta')-2E'(\eta')+\zeta'D''(\zeta')-\eta'E''(\eta')=0\ . \tag{3.47}$$

The solutions can be taken without loss of generality as

$$D(\zeta')=k+\frac{k_2-k_1}{\zeta'},\quad E(\eta')=k+\frac{k_1+k_2}{\eta'}\ , \tag{3.48}$$

to give the potential

$$V=\frac{k}{R}+\frac{k_1\cos\phi+k_2}{R^2\sin^2\phi}+F(z)\ . \tag{3.49}$$

The isolating integrals may be found by separating the Hamilton-Jacobi equation to be

$$\begin{aligned}
E&=\tfrac{1}{2}|\mathbf{P}|^2+\frac{k}{(x^2+y^2)^{1/2}}+\frac{k_1x}{y^2(x^2+y^2)^{1/2}}+\frac{k_2}{y^2}+F(z)\ ,\\
I_1&=\tfrac{1}{2}L_3^2+\frac{k_1\cos\phi+k_2}{\sin^2\phi}\ ,\\
I_2&=\tfrac{1}{2}P_3^2+F(z)\ ,\\
I_3&=L_3P_2+\frac{1}{\zeta'+\eta'}\left[k(\zeta'-\eta')+\frac{\eta'(k_1-k_2)}{\zeta'}+\frac{\zeta'(k_1+k_2)}{\eta'}\right]\ .
\end{aligned} \tag{3.50}$$

If the separating coordinate systems possess a common coordinate—such as z for the cylindrical polar (R,ϕ,z) and parabolic cylindrical (ζ',η',z) systems—the solution for the potential always contains an unspecified function. The superintegrable potential then admits four, not five, isolating independent integrals.

E. Two mutually orthogonal parabolic cylindrical coordinate systems

Consider two parabolic cylindrical coordinate systems (ζ',η',z) and (λ',μ',z) related by

$$\begin{aligned}
x&=\tfrac{1}{2}(\zeta'-\eta')=\sqrt{\lambda'\mu'}\ ,\\
y&=\sqrt{\zeta'\eta'}=\tfrac{1}{2}(\lambda'-\mu')\ .
\end{aligned} \tag{3.51}$$

The potential separates in both coordinate systems if

$$\begin{aligned}
V&=\frac{A(\zeta')+B(\eta')}{\zeta'+\eta'}+F(z)\\
&=\frac{D(\lambda')+E(\mu')}{\lambda'+\mu'}+F(z)\ .
\end{aligned} \tag{3.52}$$

Constructing the equation

$$\frac{\partial^2}{\partial\zeta'\partial\eta'}[(\zeta'+\eta')V]=0\ , \tag{3.53}$$

we find that $D(\lambda')$ and $E(\mu')$ must satisfy

$$D'(\lambda')-E'(\mu')+2\lambda'D''(\lambda')-2\mu'E''(\mu')=0\ . \tag{3.54}$$

Separating, the solutions can be taken without loss of generality as

$$D(\lambda')=k+k_1\sqrt{\lambda'},\quad E(\mu')=k+k_2\sqrt{\mu'}\ , \tag{3.55}$$

to give the superintegrable system

$$V=\frac{k}{R}+\frac{k_1\sqrt{R+y}}{R}+\frac{k_2\sqrt{R-y}}{R}+F(z)\ , \tag{3.56}$$

where $R^2=x^2+y^2$. This is the three-dimensional analog of the plane system found by Fris *et al.*[28] and subsequently investigated by Sen.[37] The four functionally independent integrals may be taken as

$$\begin{aligned}
E&=\tfrac{1}{2}|\mathbf{P}|^2+\frac{k}{R}+\frac{k_1\sqrt{R+y}}{R}+\frac{k_2\sqrt{R-y}}{R}+F(z)\ ,\\
I_1&=\tfrac{1}{2}P_3^2+F(z)\ ,\\
I_2&=L_3P_1+\frac{1}{\lambda'+\mu'}[k(\lambda'-\mu')-k_1\sqrt{\lambda'}\mu'+k_2\sqrt{\mu'}\lambda']\ ,\\
I_3&=L_3P_2+\frac{1}{\zeta'+\eta'}\left[k(\zeta'-\eta')-(k_1+k_2)\left[\frac{\zeta'}{2}\right]^{1/2}\eta'+(k_1-k_2)\left[\frac{\eta'}{2}\right]^{1/2}\zeta'\right]\ .
\end{aligned} \tag{3.57}$$

It may be verified by direct calculation that this is the only new superintegrable system when the separating coordinates are of the same type but rotated or translated with respect to each other. For example, the most general potential separable in a pair of arbitrarily rotated or translated rotational parabolic coordinates is always a special case of (3.16).

IV. CLASSICAL EQUATIONS OF MOTION

It is entirely characteristic of superintegrable systems that the classical trajectories always fill a surface of less dimensions than the number of degrees of freedom. This is because the intersection of the level sets of integrals determines the trajectory in phase space. In three degrees of freedom, the existence of five globally defined and functionally independent integrals (hereafter called *maximal superintegrability)* immediately implies all finite trajectories are closed. If there are four integrals (*minimal superintegrability*), the generic trajectory densely fills a 2-surface and not the entire three-dimensional space. This was already well known to Born.[38]

A. Maximal superintegrability: The generalized Kepler potential

As a first example, we consider the superintegrable Hamiltonian given by

$$H=\tfrac{1}{2}|\mathbf{P}|^2-\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}\ . \tag{4.1}$$

The constants k, k_1, and k_2 are taken as positive, so the Hamiltonian is perfectly physical and motion is confined to, say, the quarter-space $x>0$ and $y>0$. Using the expressions given for the integrals in (3.17), the actions are found to be

$$\begin{aligned}
J_r&=\oint p_r dr=2\pi\left[-\sqrt{2I_1}-\frac{k}{\sqrt{-2E}}\right]\ ,\\
J_\theta&=\oint p_\theta d\theta=2\pi(\sqrt{2I_1}-\sqrt{2I_2})\ ,\\
J_\phi&=\oint p_\phi d\phi=2\sqrt{2}\pi(\sqrt{I_2}-\sqrt{k_1}-\sqrt{k_2})\ .
\end{aligned} \tag{4.2}$$

Solving for E in terms of the actions yields the result

$$E\equiv H=\frac{-2\pi^2k^2}{[J_r+J_\theta+J_\phi+2\sqrt{2}\pi(\sqrt{k_1}+\sqrt{k_2})]^2}\ . \tag{4.3}$$

As a consequence of the three action variables appearing only in the form $J_r+J_\theta+J_\phi$, it follows that all the frequencies are equal and the motion is completely degenerate. The degenerate frequencies can be removed by a canonical transformation with generating function

$$F=(w_\phi-w_\phi)J_1+(w_\theta-w_r)J_2+w_rJ_3\ , \tag{4.4}$$

where (w_r,w_θ,w_ϕ) are the old angles and (J_1,J_2,J_3) are the new action variables. In the new action-angle coordinates, the Hamiltonian is simply

$$H=\frac{-2\pi^2k^2}{[J_3+2\sqrt{2}\pi(\sqrt{k_1}+\sqrt{k_2})]^2}\ . \tag{4.5}$$

It is now a function of the only action variable for which the corresponding frequency is nonzero. We note that in the limit $k_1=k_2=0$, we recover the well-known result for the Kepler problem. By solving the Hamilton-Jacobi equation in rotational parabolic coordinates, the trajectories are given in terms of a parameter τ by

$$\begin{aligned}
\zeta&=\frac{-\gamma_1}{2E}+\left[\frac{\gamma_1^2}{4E^2}+\frac{I_3}{E}\right]^{1/2}\sin(2\sqrt{-2E}\,\tau+C_1)\ ,\\
\eta&=\frac{-\gamma_2}{2E}+\left[\frac{\gamma_2^2}{4E^2}+\frac{I_3}{E}\right]^{1/2}\sin(2\sqrt{-2E}\,\tau+C_2)\ ,\\
\sin^2\phi&=\frac{\gamma_3}{2I_3}-\left[\frac{\gamma_3^2}{4I_3^2}-\frac{k_2}{I_3}\right]^{1/2}\times\frac{2(I_2-I_3)\zeta\eta-I_3(\eta-\zeta)^2}{2(I_2-I_3)\zeta\eta}\ ,
\end{aligned} \tag{4.6}$$

where γ_i and C_i are constants. For trapped orbits, these are closed fourth-degree curves. Note that in distinction to true Keplerian motion, the orbits are not confined to a plane.

B. Maximal superintegrability: the Winternitz system

The three degrees of freedom Hamiltonian system given by

$$H=\tfrac{1}{2}|\mathbf{P}|^2+k(x^2+y^2+z^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$$

was first identified as superintegrable in Ref. 30. The obvious extension to N degrees of freedom is known as the Winternitz system and possesses $2N-1$ globally defined single-valued and functionally independent integrals, as shown in Ref. 31.

The actions for the Winternitz system with $N=3$ are

$$J_x=\oint p_x dx=2\sqrt{2}\pi\left[\frac{I_1}{2\sqrt{k}}-\sqrt{k_1}\right],$$
$$J_y=\oint p_y dy=2\sqrt{2}\pi\left[\frac{I_2}{2\sqrt{k}}-\sqrt{k_2}\right],\qquad (4.7)$$
$$J_z=\oint p_z dz=2\sqrt{2}\pi\left[\frac{I_3}{2\sqrt{k}}-\sqrt{k_3}\right],$$

where I_1, I_2, and I_3 are the integrals given by

$$I_1=\tfrac{1}{2}P_1^2+kx^2+\frac{k_1}{x^2},$$
$$I_2=\tfrac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2},\qquad (4.8)$$
$$I_3=\tfrac{1}{2}P_3^2+kz^2+\frac{k_3}{z^2}.$$

Writing the Hamiltonian as a function of the actions, we find that

$$H=\frac{1}{\pi}\left[\frac{k}{2}\right]^{1/2}(J_x+J_y+J_z)+2\sqrt{k}(\sqrt{k_1}+\sqrt{k_2}+\sqrt{k_3}).\qquad (4.9)$$

The action variables appear only in the form $J_x+J_y+J_z$ and so the motion is completely degenerate. The classical equations of motion may be solved to give the trajectories

$$x^2=\frac{I_1}{2k}+\left[\frac{I_1^2}{4k^2}-\frac{k_1}{k}\right]^{1/2}\sin(\sqrt{8k}\,t+C_1),$$
$$y^2=\frac{I_2}{2k}+\left[\frac{I_2^2}{4k^2}-\frac{k_2}{k}\right]^{1/2}\sin(\sqrt{8k}\,t+C_2),\qquad (4.10)$$
$$z^2=\frac{I_3}{2k}+\left[\frac{I_3^2}{4k^2}-\frac{k_3}{k}\right]^{1/2}\sin(\sqrt{8k}\,t+C_3).$$

Here, C_1, C_2, and C_3 are constants of integration which depend on the initial conditions. For $k>0$ and $k_i>0$, a trapped partial ($E<0$) is confined to an octant of $\mathbb{R}^3$ and traces out a closed curve of the fourth degree.

C. Minimal superintegrability

As a final example, we briefly consider the Hamiltonian

$$H=\tfrac{1}{2}|\mathbf{P}|^2+4kx^2+ky^2+\frac{k_2}{y^2}+F(z),\qquad (4.11)$$

which separates in rectangular Cartesian and parabolic cylindrical coordinates and possesses integrals

$$I_1=\tfrac{1}{2}P_1^2+4kx^2,$$
$$I_2=\tfrac{1}{2}P_2^2+ky^2+\frac{k_2}{y^2},\qquad (4.12)$$
$$I_3=L_3P_2-k\zeta'\eta'(\zeta-\eta')+\frac{k_2(\zeta'-\eta')}{\zeta'\eta'},$$

in addition to the energy. The trajectories are

$$x=\frac{1}{2}\left[\frac{I_1}{k}\right]^{1/2}\sin(\sqrt{8k}\,t+C_1),$$
$$y^2=\frac{I_2}{2k}+\left[\frac{I_2^2}{4k^2}-\frac{k_2}{k}\right]^{1/2}\sin(\sqrt{8k}\,t+C_2).\qquad (4.13)$$

So, the projected motion on the (x,y) plane is closed and traces out the path of a fourth-degree curve. In separable systems, the trajectory is bounded by the coordinate surfaces and so a trapped particle oscillates in z between turning points z_+ and z_-. The generic orbit densely fills the 2-surface defined by (4.13) between the turning points.

V. CONCLUSION

The main result of the paper is the construction of a complete list, up to the equivalence class of linear transformations, of all superintegrable systems in three degrees of freedom which possess invariants linear or quadratic in the momenta. The Kepler problem, spherically symmetric potentials, the isotropic harmonic oscillator and the anisotropic oscillator with rational frequency ratio 1:1:2 are included as special cases of more general results. There are three known superintegrable potentials that do not appear in the tables. The most obvious omission is the anisotropic harmonic oscillator with rational frequency ratio $l:m:n$ where $l+m+n\geq 5$, i.e.,

$$H=\tfrac{1}{2}|\mathbf{P}|^2+l^2x^2+m^2y^2+n^2z^2.\qquad (5.1)$$

The potential separates in rectangular Cartesians and possesses two commuting quadratic integrals. There are two additional integrals which may be taken as polynomials of degree $l+m-1$ and $l+n-1$ (see, e.g., Ref. 39). The Calogero potential in a harmonic well,

$$H=\tfrac{1}{2}|\mathbf{P}|^2+k(x^2+y^2+z^2)+\frac{k_1}{(x-y)^2}+\frac{k_1}{(y-z)^2}+\frac{k_1}{(z-x)^2},\qquad (5.2)$$

was found to be superintegrable by the method of Lax pairs in Ref. 40. On introducing cylindrical polar coordinates (R,ϕ,z') such that

$$R\cos\phi=(2z-x-y)/\sqrt{3},$$
$$R\sin\phi=x-y,\qquad (5.3)$$
$$z'=(z+y+z)\sqrt{6}/3,$$

then (5.2) may be scaled to give a separable potential. There are two commuting quadratic integrals—the two remaining invariants are polynomials of degree 6 in the momenta. Finally, the potential discovered by Thompson[41] readily generalizes to give the superintegrable Hamiltonian in three degrees of freedom

$$H=\tfrac{1}{2}|\mathbf{P}|^2+x^{-2a}+F(z),\qquad (5.4)$$

where $a=1/(2l+1)$, $l=1,2,\ldots$. The system separates in Cartesians but admits an additional polynomial invariant of degree $2(l+1)$.

Obviously, by restricting ourselves to superintegrable systems with integrals that are linear or quadratic polynomials in the momenta, we lose part of the results. A point of interest is to generalize the classification theorem of superintegrable systems to the cases listed above that separate once but not twice. In three degrees of freedom, the theory of linear and quadratic invariants is very complete. Although a number of investigations have been made using simple *Ansätze*,[42,43] there is no analogous theory that predicts the existence of higher-degree polynomial invariants. The realization though that the additional integrals occur in commuting pairs may still enable progress to be made. Finally, we note that there are no known superintegrable systems that do not separate in at least one of the confocal ellipsoidal coordinates or their degenerations. It is likely that such system cannot exist, but nothing has been proved.

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