Divergences in the iterative and perturbative methods for computing Hannay's angle

I. Gjaja and A. Bhattacharjee
Department of Applied Physics, Columbia University, New York, New York 10027
(Received 29 September 1989; revised manuscript received 27 December 1989)

A classical analog is obtained for Berry's nonperturbative scheme of adiabatic iteration which computes corrections to Berry's phase (or Hannay's angle) for finite values of the adiabatic parameter ϵ. The iterative method is compared to the Lie version of adiabatic perturbation theory. Both approaches show a divergence of $k!\epsilon^k$, where k is the order of iteration. It is argued that the divergences are a mathematical artifact of the asymptotic methods used, not related to the physical effect of transitions, nor to the nonconservation of the action variables.

I. INTRODUCTION

It was found by Berry[1] that if the parameters of a Hamiltonian specifying a quantum system are taken infinitesimally slowly around a circuit in parameter space, then the system acquires a geometric phase in addition to the familiar dynamical phase $-\int_{-\infty}^{+\infty} E_n dt$. The classical analog of Berry's phase was subsequently elucidated by Hannay[2] and Berry[3] ("Hannay's angle"). To see the realization of Hannay's angle, we recall that the motion of an integrable classical system undergoing an adiabatic excursion is confined to tori given by constant values of the action variables. Then Hannay's angle appears as an anholonomy in the angle variables, completely specified by the path in parameter space.

More recent work has relaxed the restriction of the infinitesimal rate of change of the parameters, allowing instead the rate to be small but finite. In particular, adiabatic perturbation theory has been employed to calculate corrections to several orders to Berry's phase and Hannay's angle for some representative systems.[4-6] Berry, on the other hand, has developed a nonperturbative, iterative procedure for calculating corrections to the phase.[7] Applying it to the spin-$\frac{1}{2}$ system, he found that the sequence of corrections begins to diverge after $N \sim \epsilon^{-1}$ iterations, where ϵ is the adiabatic parameter. The divergence was universal, i.e., almost always independent of the initial cycle in the parameter space.

In this paper we bridge the gap between the iterative and perturbative schemes. We use the classical analog of Berry's iterative procedure and compare it to Lie perturbation theory. The two methods are shown to diverge in the same order and for the same reason for a broad class of Hamiltonians; only the computational details are different. The analysis is restricted to integrable systems, away from resonances (so as to avoid resonant denominators).

In Sec. II of this paper we review Berry's iterative procedure and obtain its classical analog. In Sec. III we describe the adiabatic perturbation theory using Lie operators, as given, for example, in Refs. 8 and 9. We work out the example of a generalized harmonic oscillator and show that the iterative and perturbative schemes agree if at each iteration the ϵ dependence is expanded in a series. We then take up a very general Hamiltonian, and show how the divergence occurs in the adiabatic perturbation theory. Section IV is devoted to the study of the divergence of the iterative procedure. At each iteration, the explicit ϵ dependencies of the generating functions are expanded in a Lie perturbation series. The terms in the series from subsequent iterations show a divergence which is analogous to the one found in Sec. III. An example of the divergence of the iterative scheme is given in Appendix A for both the quantum-mechanical and the classical version of the generalized harmonic oscillator. Concluding remarks are presented in Sec. V.

II. ADIABATIC ITERATION

The iterative scheme consists of a sequence of unitary transformations which diagonalize the Hamiltonian with changing parameters. At each step, however, the diagonalization is not complete because of the explicit time dependence of the unitary operators.

Berry considers a smooth cyclic evolution of the Hamiltonian H_0, $H_0(-\infty)=H_0(+\infty)$, with all the time derivatives vanishing at $t=\pm\infty$. He takes $\{|N\rangle\}$ to be a (complete) set of initial eigenstates of H_0, the phases of which are arbitrary but fixed. The adiabatic iteration then provides a prescription for calculating the total phase that the state $|\Psi_0\rangle$ accumulates at $t=+\infty$ with respect to $|N\rangle$, if $|\Psi_0(-\infty)\rangle=|N\rangle$, and

$$i\frac{\partial}{\partial t}|\Psi_0(t)\rangle = H_0(t)|\Psi_0(t)\rangle \ . \tag{2.1}$$

The rate of change of H_0 is assumed to be small, $\partial H_0/\partial t = O(\epsilon)$, $\epsilon \ll 1$, so that $|\langle \Psi_0(+\infty)|N\rangle| - 1 \ll 1$. [It is actually of the order of $\exp(-1/\epsilon)$ (Ref. 10).] We thus assume that all explicit time dependence of H_0 occurs in the combination ϵt. This then holds for the eigenvalues and eigenstates of H_0 as well.

If $|n_0(t)\rangle$ denotes the eigenstate of $H_0(t)$, $H_0|n_0\rangle = E_0(n)|n_0\rangle$, then the choice of its phase can be made unique by requiring that

$$\left\langle n_0 \left| \frac{\partial}{\partial t} \right| n_0 \right\rangle = 0 \ . \tag{2.2}$$

This condition is known as parallel transport, and is taken for definiteness, as well as computational simplicity.[7]

The total phase accumulated by $|\Psi_0\rangle$ is independent of Eq. (2.2).[1] In Ref. 7 the total phase of $|\Psi_0(+\infty)\rangle$ is written as $\gamma - \int_{-\infty}^{+\infty} E_0(n)dt$, and the calculations are presented for γ. Thus

$$\gamma = \mathrm{Im}[\ln\langle N|\Psi_0(+\infty)\rangle] + \int_{-\infty}^{+\infty} E_0(n)dt \ . \tag{2.3}$$

Adiabatic iteration now stipulates the introduction of a unitary operator U_0, such that $|n_0(t)\rangle = U_0(t)|N\rangle$. As shown, for example, in Refs. 11 and 12, then

$$U_0(+\infty)|N\rangle = e^{i\gamma_0}|N\rangle \ , \tag{2.4}$$

where γ_0 is exactly the original Berry's geometric phase.[1] Berry uses the operator U_0 to construct a new state $|\Psi_1(t)\rangle$, $|\Psi_1(t)\rangle = U_0^\dagger(t)|\Psi_0(t)\rangle$. With this definition

$$\gamma = \gamma_0 + \mathrm{Im}[\ln\langle N|\Psi_1(+\infty)\rangle] + \int_{-\infty}^{+\infty} dt\, E_0(n) \ , \tag{2.5}$$

and $|\Psi_1(t)\rangle$ satisfies

$$i\frac{\partial}{\partial t}|\Psi_1(t)\rangle = H_1(t)|\Psi_1(t)\rangle \ , \tag{2.6a}$$

with

$$H_1 = U_0^\dagger H_0 U_0 - i\epsilon U_0^\dagger \dot{U}_0 \ . \tag{2.6b}$$

Here the overdot denotes $\partial/\partial\tau$ and $\tau \equiv \epsilon t$. Equation (2.6b) shows that the off-diagonal elements of H_1 in the $\{|N\rangle\}$ representation are of order ϵ [and $-i\epsilon U_0^\dagger \dot{U}_0$ does not have diagonal elements because of condition (2.2)]. Then in the adiabatic approximation we can neglect the term $-i\epsilon U_0^\dagger \dot{U}_0$, which yields

$$|\Psi_1(+\infty)\rangle = \exp\left[-i\int_{-\infty}^{+\infty} E_0(n)dt\right]|N\rangle \ ,$$

and $\gamma = \gamma_0$. This is Berry's original result.[1] [Note that if there had been diagonal terms of $O(\epsilon)$ in H_1, then they could not have been neglected, as they would have yielded a correction to γ of $O(1)$.] For ϵ finite, Eqs. (2.5) and (2.6) provide a first step in the adiabatic iteration.

The procedure can now be easily repeated: we let $H_1|n_1\rangle = E_1(n)|n_1\rangle$ with $\langle n_1|\dot{n}_1\rangle = 0$; then $|n_1(\tau)\rangle = U_1(\tau)|N\rangle$, followed by $|\Psi_2\rangle = U_1^\dagger|\Psi_1\rangle$, etc.

$$H_{k+1} = U_k^\dagger H_k U_k - i\epsilon U_k^\dagger \dot{U}_k \ , \tag{2.7a}$$

$$U_k|N\rangle = |n_k\rangle, \quad H_k|n_k\rangle = E_k|n_k\rangle, \quad \langle n_k|\dot{n}_k\rangle = 0 \ . \tag{2.7b}$$

As given in Ref. 7, in the $\{|N\rangle\}$ representation, the matrix elements of H_{k+1} are

$$\langle M|H_{k+1}|N\rangle = E_k(n)\delta_{MN} - \frac{\epsilon(1-\delta_{MN})i\langle M|U_k^\dagger \dot{H}_k U_k|N\rangle}{E_k(n) - E_k(m)} \ . \tag{2.8}$$

The off-diagonal elements come from the second term in Eq. (2.7a).

To obtain the kth approximant to γ, one can neglect the off-diagonal terms of H_{k+1}, on account that they are smaller than the diagonal ones by a factor of ϵ. Then

$$|\Psi_{k+1}\rangle \simeq \exp\left[-i\int_{-\infty}^{+\infty} E_k dt\right]|N\rangle \ ,$$

and

$$\gamma \simeq \gamma^{(k)} = \sum_{i=1}^{k} \gamma_i(n) + \int_{-\infty}^{+\infty} [E_0(n,\tau) - E_k(n,\tau)]\frac{1}{\epsilon}d\tau \ . \tag{2.9}$$

As emphasized by Berry, the procedure described above is not perturbative since each of the γ_k's contains ϵ to infinite order. We note, however, that at each step we are required to solve exactly the equations of motion keeping the parameters of H constant (i.e., we need to find a suitable U for each iteration).

We now proceed to give the classical analog of the iterative procedure. It is useful to introduce the usual time evolution operator U^k for the Hamiltonian H_k,

$$U^k(-\infty,\tau) = \mathcal{T}\exp\left[-\frac{i}{\hbar\epsilon}\int_{-\infty}^{\tau} H_k(\tau')d\tau'\right] \ . \tag{2.10}$$

$\mathcal{T}$ here denotes time ordering, and we have introduced $\hbar$ explicitly. Notice that in contrast to the unitary operators given in Eqs. (2.6) and (2.7), in the time evolution operator the index appears as a superscript. In this notation the first step of Berry's iterative procedure can be written as

$$U^0 = U_0 U^1 \ , \tag{2.11a}$$

and the $(k+1)$th as

$$U^k = U_k U^{k+1} \ , \tag{2.11b}$$

leading to

$$U^0 = U_0 U_1 \cdots U_k U^{k+1} \ . \tag{2.11c}$$

To obtain a classical analog of Eq. (2.11a), we use a form for U_0 inferred, for example, from Appendix A of Ref. 7, or directly by differentiating with respect to τ the defining equation for U_0 [$|n_0\rangle = U_0|N\rangle$],

$$U_0 = \mathcal{T}\exp\left[\int_{-\infty}^{\tau}\sum_{n_0}|\dot{n}_0\rangle\langle n_0|d\tau\right] \ . \tag{2.12}$$

Next, we rewrite Eq. (2.11a) as $U^1 = U_0^\dagger U^0$, and examine the commutator of the arguments of $U_0^\dagger$ and U^0,

$$\left[\int_{-\infty}^{\tau} d\tau' \sum_{n_0}|\dot{n}_0(\tau')\rangle\langle n_0(\tau')|, \frac{1}{\hbar\epsilon}\int_{-\infty}^{\tau} d\tau'' H_0(\tau'')\right] \ . \tag{2.13}$$

Using $\sum_{n_0}|\dot{n}_0\rangle\langle n_0| = -\sum_{n_0}|n_0\rangle\langle\dot{n}_0|$, and $H_0 = \sum_{m_0} E_{m_0}|m_0\rangle\langle m_0|$, in the position representation the integrand from Eq. (2.13), with $1/\hbar\epsilon$ factored out, yields

$$\sum_{m_0,n_0}\left[E_{m_0}(\tau'')\int dr\,dr'dr''|r\rangle\langle r|\dot{n}_0(\tau')\rangle\langle n_0(\tau')|r'\rangle\langle r'|m_0(\tau'')\rangle\langle m_0(\tau'')|r''\rangle\langle r''|+\text{H.c.}\right]. \tag{2.14}$$

Here H.c. stands for the Hermitian conjugate.

To proceed to the limit $\hbar\to 0$, we use for $\langle r|n_0\rangle$ the semiclassical wave function associated with the torus in phase space, the actions of which are quantized according to the Bohr-Sommerfeld rule $I(n_0)=(n_0+\sigma)\hbar$.[13,14] (σ is a constant which need not concern us at present.)

$$\langle r|n_0(\tau)\rangle=\sum_s a_s(r,I(n_0),\tau)\exp[(i/\hbar)G_0^s(r,I(n_0),\tau)]\,. \tag{2.15}$$

Here $a_s(r,I(n_0),\tau)$ is independent of $\hbar$, and in r it varies slowly compared to $G_0^s/\hbar$; $G_0^s(r,I(n_0),\tau)$ is a generating function for the canonical transformation $(r,p)\to(\theta_0,I(n_0))$; and s labels the branches of the multivalued generating function. Substituting Eq. (2.15) into Eq. (2.14) yields to lowest order in $\hbar$

$$\sum_{m_0,n_0}\left[E_{m_0}(\tau'')\int dr\,dr'dr''|r\rangle\langle r''|\sum_{s,s',s'',s'''}\frac{i}{\hbar}\frac{\partial}{\partial\tau'}G_0^s(r,I(n_0),\tau')\right.$$
$$\times a_s(r,I(n_0),\tau')a_{s'}^*(r',I(n_0),\tau')a_{s''}(r',I(m_0),\tau'')a_{s'''}^*(r'',I(m_0),\tau'')$$
$$\times\exp\left\{\frac{i}{\hbar}[G_0^s(r,I(n_0),\tau')-G_0^{s'}(r',I(n_0),\tau')+G_0^{s''}(r',I(m_0),\tau'')\right.$$
$$\left.\left.-G_0^{s'''}(r'',I(m_0),\tau'')]\right\}+\text{H.c.}\right]. \tag{2.16}$$

As $\hbar\to 0$, we can also consider the sums over n_0 and m_0 as integrals [$\sum_{n_0}\leftrightarrow 1/\hbar\int dI(n_0)$]. Keeping to the lowest order in $\hbar$, the integral in $I(n_0)$ gives $\delta_{s,s'}\delta(r-r')$, and the integral over $I(m_0)$, $\delta_{s'',s'''}\delta(r'-r'')$. Otherwise, the rapid oscillations of the exponentials send the integrals (with $\hbar^{-3}$ factored out) to zero, linearly in $\hbar$ if $G_0^s(I)$ have no stationary points on the interval $I\in[0,\infty]$, or as $\hbar^{1/p}$ if there is a stationary point of order p (see, for example, Ref. 15). Consequently, to lowest order in $\hbar$, Eq. (2.16) becomes

$$\sum_{m_0,n_0}E_{m_0}(\tau'')\int dr|r\rangle\langle r|\left[\frac{i}{\hbar}\sum_{s,s'}\frac{\partial}{\partial\tau'}G_0^s(r,I(n_0),\tau')|a_s(r,I(n_0),\tau')|^2|a_{s'}(r,I(m_0),\tau'')|^2+\text{H.c.}\right]=0\,. \tag{2.17}$$

Hence the commutator of Eq. (2.13) is

$$\int_0^\tau\int_0^\tau d\tau'd\tau''\int dr\,dr'|r\rangle\langle r'|\sum_{m_0,n_0}O(\hbar^\mu),\quad \mu>-1$$

which is of higher order in $\hbar$ than either of the terms forming the commutator. (This can be easily verified by writing either term in the position representation.) To lowest order in $\hbar$ we then have

$$U^1=\mathcal{T}\exp\left[-\frac{i}{\hbar\epsilon}\int_{-\infty}^{\tau}d\tau'\left[H_0(\tau')-i\hbar\epsilon\sum_{n_0}|\dot{n}_0(\tau')\rangle\langle n_0(\tau')|\right]\right]. \tag{2.18}$$

Differentiating both sides with respect to τ gives

$$H_1=H_0-i\hbar\epsilon\sum_{n_0}|\dot{n}_0\rangle\langle n_0|\,. \tag{2.19}$$

We now take the matrix element of Eq. (2.19) between $\langle r|$ and $|m_0\rangle$ and obtain

$$\int dr'\langle r|H_1|r'\rangle\langle r'|m_0\rangle$$
$$=E_{m_0}(\tau)\langle r|m_0\rangle-i\hbar\epsilon\frac{\partial}{\partial\tau}\langle r|m_0\rangle\,. \tag{2.20}$$

With $\langle r|H_1|r'\rangle=H_1(r,\hbar/i\partial_r,\tau)\delta(r-r')$, and using again the semiclassical expression for the wave functions $\langle r|m_0\rangle$, to lowest order in $\hbar$, Eq. (2.20) becomes [$I(m_0)\equiv I_0$],

$$\sum_s a^s(r,I_0,\tau)\exp\left[\frac{i}{\hbar}G_0^s(r,I_0,\tau)\right]$$
$$\times\left[H_1\left[r,\frac{\partial G_0(r,I_0,\tau)}{\partial\tau},\tau\right]-H_0(I_0,\tau)\right.$$
$$\left.-\epsilon\frac{\partial}{\partial\tau}G_0^s(r,I_0,\tau)\right]=0\,. \tag{2.21}$$

Observing that in the limit $\hbar\to 0$ the terms in the sum become orthogonal for different s, and expressing the dynamical variables in terms of the canonically conjugate pair (I_0,θ_0), yields

$$H_1(I_0,\theta_0,R_i(\tau)) = H_0(I_0,R_i(\tau)) + \epsilon\dot{R}_i(\tau)\frac{\partial G_0[r,I_0,R_j(\tau)]}{\partial R_i}\bigg|_{r=r(I_0,\theta_0,\tau)} . \quad (2.22)$$

We have dropped the explicit index s on G_0 (the last term is single valued anyway after evaluating r). In writing Eq. (2.22), we have emphasized that the explicit time dependence of the Hamiltonian comes from a set of time-dependent parameter R_i, the rate of change of which, in t, is of $O(\epsilon)$. The summation over i is assumed. H_1 is thus simply a result of a canonical transformation $(r,p)\rightarrow(I_0,\theta_0)$, where I_0 and θ_0 are the appropriate action-angle variables for $R_i=\text{const}$. As R_i's are an explicit function of time, the second term on the right-hand side of Eq. (2.22) is nonzero, and H_1 is also a function of θ_0.

The condition for parallel transport can be inferred directly from Eq. (2.2). We write $\langle n_0|\dot{n}_0\rangle$ in position representation and use the semiclassical wave function for $|n_0\rangle$. To lowest order in $\hbar$ the result is

$$\sum_{s,s'}\int dr\, a_s(r,I_0,R_i)a_{s'}^*(r,I_0,R_j)\dot{R}_i\frac{\partial G_0^s(r,I_0,R_j)}{\partial R_i} \times\exp\left[\frac{i}{\hbar}[G_0^s(r,I_0,R_i)-G_0^{s'}(r,I_0,R_i)]\right]=0 . \quad (2.23)$$

As $\hbar\rightarrow 0$, the exponent produces $\delta_{s,s'}$. To proceed further, we identify[14] $|a_s|^2$ with $1/2\pi\ d\theta_0^s(r,I_0,R_i)/dr$, yielding

$$\dot{R}_i\left\langle\frac{\partial G_0(r,I_0,R_j)}{\partial R_i}\bigg|_{r=r(I_0,\theta_0,R_i)}\right\rangle_{\theta_0}=0 , \quad (2.24)$$

where the angular brackets indicate an average over θ_0. Equation (2.24) removes the indeterminancy of $G_0(r,I_0,R_i)$ to within an arbitrary function of I_0 and R_i, which corresponds to the action and parameter-dependent specification of the origin with respect to which θ_0 is measured. [Recall that $\theta_0(I_0,r,R_i)=\partial G_0(I_0,r,R_i)/\partial I_0$.] In particular, then, the anholonomy in the angle variable appears as a nonzero shift in the origin as the cycle in parameter space is traversed. Based on Eq. (2.11b), we can now repeat the procedure for arbitrary k. $|n_k\rangle$ is expressed in the θ_{k-1} representation (which is fine since I_{k-1}, θ_{k-1} form a canonically conjugate pair). U_k is again taken to be of the form (2.12) (with the appropriate indices on n), and all the steps are repeated. The result is

$$H_k(I_k,R_i)+\epsilon\dot{R}_i\frac{\partial G_k(\theta_{k-1},I_k,R_i)}{\partial R_i}\bigg|_{\theta_{k-1}=\theta_{k-1}(\theta_k,I_k,R_i)} = H_{k+1}(\theta_k,I_k,R_i) , \quad (2.25)$$

where at each step the generating functions satisfy the condition for parallel transport.

The iteration can be terminated after k steps by writing $H_k(I_k,R_i)\cong H_{k+1}(I_k,R_i)$. Then $H_k(I_k,R_i)$ corresponds to $E_k(n)$ in Eq. (2.9) and the accumulated shifts of the origin with respect to which θ_k is measured correspond to $\sum_{i=0}^k\gamma_i(n)$, with $\Delta\theta_k=-\hbar\partial\gamma_k/\partial I_k$.[3]

The shifts in the origin are sometimes denoted as "geometric" contributions to Hannay's angle, whereas E_k-E_0 gives the "dynamical" contribution. This splitting is dependent on the condition given by Eq. (2.24). If we were to change it, the relative dynamical and geometric contributions would change. The total change in θ due to the variation of parameters, however, is given by the sum of the dynamical and geometric contributions [see Eq. (2.9)], and is independent of condition (2.24).[6]

While we have outlined the derivation above for one degree of freedom, it is obvious that it can be extended to any number by simply adding indices on I_k and θ_k [in the step preceding Eq. (2.24) $|a_s|^2$ is identified with $(2\pi)^{-N}\det|d\theta_i(r_k,I_{0_k},R_k)/dr_j|$]. In that case, of course, the caveat of integrability and not being at resonance applies.

III. ADIABATIC PERTURBATION THEORY USING LIE OPERATORS

A. Review of Lie perturbation theory

For the sake of completeness, we give here a brief description of the Lie version of adiabatic perturbation theory. More complete accounts can be found, for example, in Refs. 8 and 9.

We denote by $\mathbf{z}$ all the variables of canonically conjugate pairs $\mathbf{z}=(q_i,p_i)$. For $2N$ degrees of freedom, for example, $z_j=q_j$ and $z_{j+N}=p_j$. The adiabatic perturbation theory is then applied to Hamiltonians of the form

$$h=\sum_{n=0}^{\infty}\epsilon^n h_n(\mathbf{z},\epsilon t) , \quad (3.1)$$

where ϵ is taken to be the small parameter. We note that ϵ appears both as a free parameter, multiplying h_n, and as the adiabatic parameter, multiplying t. From now on, we revert to the variable τ, $\tau=\epsilon t$.

We assume that we can solve for the motion of $\mathbf{z}$ when $\epsilon=0$. The aim of the perturbation theory is to obtain a canonical transformation, ordered in powers of ϵ, from $\mathbf{z}$ to $\bar{\mathbf{z}}$, such that the motion of $\bar{\mathbf{z}}$ under $K(\bar{\mathbf{z}},\tau)$, can also be solved (in powers of ϵ). Of course, $\lim_{\epsilon\rightarrow 0}\bar{\mathbf{z}}=\mathbf{z}$. The generator of this transformation is $w(\mathbf{z},\tau,\epsilon)$ and it obeys

$$\frac{\partial\bar{\mathbf{z}}}{\partial\epsilon}=\{\bar{\mathbf{z}},w\}\equiv -L(\bar{\mathbf{z}}) . \quad (3.2)$$

The curly brackets denote the Poisson bracket, and L is the Lie operator $L\equiv\{w,\ \}$. The canonical transformation operator is defined as T, $Tf(\mathbf{z},\tau)=f(\bar{\mathbf{z}}(\mathbf{z},\tau),\tau)$. T and its inverse obey the equations

$$\frac{\partial T}{\partial\epsilon}=-TL,\quad \frac{\partial T^{-1}}{\partial\epsilon}=LT^{-1} , \quad (3.3)$$

with the formal solution

$$T=\mathcal{G}\exp\left[-\int_0^{\epsilon}d\epsilon' L(\epsilon')\right] ,$$

and correspondingly for T^{-1}. $\mathcal{E}$ stands for the ϵ ordering of the expanded exponential since L's at different values of ϵ do not commute. The transformed Hamiltonian is now given by

$$K = T^{-1}h + T(\epsilon)^{-1}\int_0^{\epsilon} d\epsilon' T(\epsilon')\epsilon' \frac{\partial w(\epsilon')}{\partial \tau} . \tag{3.4}$$

Equation (3.4) is a functional equality, i.e., the same dummy variable is used on both sides of the equation.

We can expand T, w, and K in powers of ϵ:

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \epsilon^n T_n , \\ w &= \sum_{n,k=0}^{\infty} \epsilon^{n+k} w_{n+1,k} , \\ K &= \sum_{n,k=0}^{\infty} \epsilon^{n+k} K_{n,k} . \end{aligned} \tag{3.5}$$

The double sums in Eq. (3.5) occur because ϵ is used both as a free small parameter, and as the adiabatic parameter. Substituting Eq. (3.5) into Eq. (3.4), and using (3.1), we obtain the recursion relations

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\epsilon^k \{ w_{n,k}, h_0 \} + \epsilon \frac{\partial w_{n,k}}{\partial \tau} \right] \\ &\quad = n \left[\sum_{k=0}^{\infty} \epsilon^k K_{n,k} - h_n \right] \\ &\qquad - \sum_{m=1}^{n-1} \left[\sum_{k=0}^{\infty} \epsilon^k L_{n-m} K_{m,k} + m T_{n-m}^{-1} h_m \right] , \end{aligned} \tag{3.6a}$$

$$T_n = -\frac{1}{n} \sum_{m=0}^{n-1} T_m L_{n-m} , \tag{3.6b}$$

$$T_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_m^{-1} . \tag{3.6c}$$

Equations (3.6) contain additional implicit ϵ dependencies in L_n's (and hence in T_n's), which come from the k expansions in Eq. (3.5). These need to be taken into account when computing any specific $w_{n,k}$. It should also be noted that up to now no specific prescriptions were given for choosing the $K_{n,k}$'s that enter the right-hand side of Eq. (3.6a). We use this freedom to choose $K_{n,k}$ such that the right-hand side of Eq. (3.6a) is free of secularities. This means that $K_{n,k}$ is set equal to the orbit average of the other terms on the right-hand side of Eq. (3.6a).

In what follows, we restrict ourselves to systems with one degree of freedom (not counting the explicit τ dependence). The results can be easily transcribed to integrable systems of many degrees of freedom away from resonances, by simply adding indices on the variables of conjugate pairs, and performing summations where they are required. In this paper we consider systems in which the initial transformation to action-angle variables has been done, i.e., $h_0 = h_0(I,\tau)$. Hence, regardless of whether we are doing iteration or perturbation, the first step of adiabatic iteration, specified by Eq. (2.22), will be performed. For systems with Hamiltonians of the form $H(q,p,R_i(\tau)) \equiv H(q,p,\tau)$ (i.e., Hamiltonians with slowly varying parameters) the first iteration yields

$$h(I,\theta,\tau) = h_0(I,\tau) + \epsilon h_1(I,\theta,\tau) . \tag{3.7}$$

This is exact, of course. Therefore, in Eq. (3.6a), $\{w_{n,k}, h_0\}$ can be replaced by $\omega_0(I)(\partial w_{n,k}/\partial\theta)$, with $\omega_0 \equiv (\partial h_0/\partial I)$. The condition of smoothness, on the other hand, requires that $\partial_\tau^n R_i(\pm\infty) = 0$ for all n, where R_i's are given in Eq. (2.22). Hence in Eq. (3.7) $h_1(\pm\infty) = 0$. We also remark, in passing, that the lowest-order approximation to Hannay's angle (the original anholonomy[2,3]) is independent of whether h_0 depends on τ or not.[16]

B. Iterative and perturbative methods applied to the harmonic oscillator

Before examining divergences in the adiabatic perturbation theory, we take up a specific Hamiltonian and compare the anholonomies computed using the perturbation theory and the iterative method. The system under consideration is the generalized harmonic oscillator,

$$H = \tfrac{1}{2}[X(\tau)q^2 + 2Y(\tau)qp + Z(\tau)p^2] . \tag{3.8}$$

Neglecting for the moment condition (2.24), the first step of iteration yields[6]

$$H_1'(I_0',\theta_0',\tau) = \omega_0 I_0' + \epsilon\omega_0 I_0' \left[A(\tau)\sin^2\theta_0' - \frac{B(\tau)}{2}\sin(2\theta_0') \right] . \tag{3.9}$$

Here the terms multiplying ϵ come from $\dot{G}_0'$, and the primes denote that the condition of parallel transport has not yet been satisfied. Further, $\omega_0^2 = XZ - Y^2$, $A = 1/(\omega_0^2 Z)(Y\dot{Z} - Z\dot{Y})$, and $B = \dot{Z}/Z - \dot{\omega}_0/\omega_0$. From the condition of smoothness, A, B, and all of their derivatives vanish at $\tau = \pm\infty$. [We note that the term multiplying $\sin\theta\cos\theta$ in Eq. (32) of Ref. 6 should have the opposite sign. This change should be carried through all subsequent calculations.] As a specific example we take $\omega_0 = \text{const}$, and choose a path in parameter space such that $X + Z = \text{const} \equiv a$. With this, $X = \frac{1}{2}(a + b\cos\alpha)$, $Y = (b/2)\sin\alpha$, $Z = \frac{1}{2}(a - b\cos\alpha)$, and $a = 2\omega_0\cosh\beta$, $b = 2\omega_0\sinh\beta$. We have thus expressed X, Y, and Z in terms of ω_0 and two angles α and β; of the three, only α is time dependent. Geometrically, the cycle in parameter space is a circle on a sheet of a hyperboloid with constant transverse and conjugate axes.

Perturbation theory has already been applied to Eq. (3.9) to $O(\epsilon^2)$.[6] The computations are straightforward, if tedious, and to $O(\epsilon^3)$ we obtain

$$\begin{aligned} K &= h_0 + \epsilon K_{1,0} + \epsilon^2(K_{1,1} + K_{2,0}) + \epsilon^3(K_{1,2} + K_{2,1} + K_{3,0}) + O(\epsilon^4) \\ &= \omega_0 \bar{I} + \tfrac{1}{2}\epsilon\omega_0 \bar{I} A - \tfrac{1}{8}\epsilon^2\omega_0 \bar{I}(A^2 + B^2) + \tfrac{1}{32}\epsilon^3 \bar{I}[(\dot{A}B - 3A\dot{B}) + 2\omega_0 A(A^2 + B^2)] + O(\epsilon^4) . \end{aligned} \tag{3.10}$$

$\bar{I}$ is the transformed action. The total shift in $\bar{\theta}$ due to the change of parameters can now be denoted as $\Delta\bar{\theta}=1/\epsilon\int_{-\infty}^{+\infty}d\tau(\partial K/\partial\bar{I}-\omega_0)$, yielding

$$\Delta\bar{\theta}=\int_{-\infty}^{+\infty}d\tau\{\tfrac{1}{2}\omega_0 A-\tfrac{1}{8}\epsilon\omega_0(A^2+B^2)+\tfrac{1}{32}\epsilon^2[(\dot{A}B-3A\dot{B})+2\omega_0 A(A^2+B^2)]\}\ . \tag{3.11}$$

For the path we have chosen

$$\Delta\bar{\theta}=-\pi(1-\cosh\beta)-\frac{\epsilon\sinh^2\beta}{8\omega_0}\int_{-\infty}^{+\infty}d\tau\,\dot{\alpha}^2+\frac{\epsilon^2\sinh^2\beta\cosh\beta}{16\omega_0^2}\int_{-\infty}^{+\infty}d\tau\,\dot{\alpha}^3\ . \tag{3.12}$$

As we see, this result makes no distinction between geometric and dynamical contributions to $\Delta\bar{\theta}$. A clarifying remark is in order here. We have defined, and computed, Hannay's angle and its corrections in terms of the transformed angle variable $\bar{\theta}$. Because of the condition of smoothness of h, however, the result is the same as the shift in the original angle variable θ_0', given in Eq. (3.9). At $\tau=\pm\infty$, $H_1'(I_0')=K(\bar{I})|_{\bar{I}=I_0'}$, and the canonical transformation $(I_0',\theta_0')\to(\bar{I},\bar{\theta})$ reduces to the identity transformation. This is most easily seen from the fact that $w_{n,k}$, and hence T, does not contain any quantities that are integrated in τ [$T(\tau)$ depends only on A, B, and their derivatives at τ]. Then setting A, B, and their derivatives equal to zero simply produces the identity transformation. So the definitions of (I_0',θ_0') and $(\bar{I},\bar{\theta})$ coincide at $\tau=\pm\infty$, and the total accumulated shift in $\bar{\theta}$ is the same as the one in θ_0'. This is, of course, not true for the accumulated shift up to an arbitrary time.

To apply the iterative method to Eq. (3.9), we first implement the condition given by Eq. (2.24). To this end we perform a canonical transformation $(I_0',\theta_0')\to(I_0,\theta_0)$, where $I_0=I_0'$, $\theta_0=\theta_0'-\nu_0$, and $\nu_0=\frac{1}{2}\omega_0\int_{-\infty}^{\tau}d\tau' A(\tau')$. Equation (3.9) becomes

$$H_1(I_0,\theta_0,\tau)=\omega_0 I_0\left[1-\frac{\epsilon}{2}\{A\cos[2(\theta_0+\nu_0)]+B\sin[2(\theta_0+\nu_0)]\}\right]\ . \tag{3.13}$$

The expression above could have been obtained, of course, in one step from Eq. (3.8). It constitutes the first iteration. We see now that if we stopped at this iteration, $\Delta\theta_d=0$, and

$$\Delta\theta_g=\frac{\omega_0}{2}\int_{-\infty}^{+\infty}d\tau\,A(\tau)=-\pi(1-\cosh\beta)$$

(subscripts d and g stand for "dynamical" and "geometric" parts of $\Delta\theta$). In order to verify agreement with higher-order terms from perturbation theory, we need to perform additional iterations.

We proceed with the second iteration again in two steps: first we transform $(I_0,\theta_0)\to(I_1',\theta_1')$ without applying parallel transport, and second, we redefine the origin of the angle variable so that Eq. (2.24) is satisfied:

$$\begin{aligned}I_1'&=\frac{1}{2\pi}\int_0^{2\pi}d\theta_0 I_0(\theta_0)\\&=\frac{H_1}{2\pi\omega_0}\int_0^{2\pi}d\theta_0\left[1-\frac{\epsilon}{2}\{A\cos[2(\theta_0+\nu_0)]+B\sin[2(\theta_0+\nu_0)]\}\right]^{-1}\\&=\frac{H_1}{\omega_0 R}\ ,\end{aligned} \tag{3.14}$$

where $R\equiv[1-(\epsilon^2/4)(A^2+B^2)]^{1/2}$. To find the generating function $G_1'(\theta_0,I_1',\tau)$, we use $\partial G_1'/\partial\theta_0=I_0(\theta_0,I_1',\tau)$. With Eq. (3.14),

$$G_1'(\theta_0,I_1',\tau)=I_1'\tan^{-1}\left\{\frac{1}{R(\tau)}\left[\left[1+\frac{\epsilon A(\tau)}{2}\right]\times\tan[\theta_0+\nu_0(\tau)]-\frac{\epsilon B(\tau)}{2}\right]\right\}\ . \tag{3.15}$$

We also write θ_1' as a function of I_1', θ_0, and τ by using $\theta_1'=[\partial G_1'(\theta_0,I_1',r)]/\partial I_1'$,

$$\theta_1'(\theta_0,I_1',\tau)=\tan^{-1}\left\{\frac{1}{R}\left[\left[1+\frac{\epsilon A}{2}\right]\tan(\theta_0+\nu_0)-\frac{\epsilon B}{2}\right]\right\}\ . \tag{3.16}$$

To obtain $\partial G_1'/\partial\tau|_{\theta_0(\theta_1',I_1',\tau)}$ we differentiate Eq. (3.15) with respect to τ, and invert Eq. (3.16). The first step of the second iteration then produces

$$H_2'(I_1',\theta_1',\tau)=H_1(I_1',\tau)+\epsilon\left.\frac{\partial G_1'(\theta_0,I_1',\tau)}{\partial\tau}\right|_{\theta_0(\theta_1',I_1',\tau)}\ , \tag{3.17}$$

where $H_1(I_1',\tau)=\omega_0 R I_1'$. Rather than displaying explicitly the second term on the right, we proceed immediately with the second step, $(I_1',\theta_1')\to(I_1,\theta_1)$, where $I_1=I_1'$ and

$$\theta_1=\theta_1'-\frac{1}{\epsilon}\int_{-\infty}^{\tau}d\tau'\left\langle\left.\frac{\partial G_1'}{\partial\tau}\right|_{\theta_0(\theta_1',I_1',\tau)}\right\rangle_{\theta_1'}\ .$$

The result is

$$H_2(I_1,\theta_1,\tau)=\omega_0 R I_1+\epsilon I_1\{C_1\sin[2(\theta_1+\nu_1)]+C_2\cos[2(\theta_1+\nu_1)]\}\ , \tag{3.18}$$

where

$$\nu_1(\tau)=\frac{1}{\epsilon}\int_{-\infty}^{\tau}d\tau'\frac{1}{2R}\left[\frac{\dot{A}B}{4(1+A/2)}-\frac{\dot{B}}{2}+(1+A/2)\dot{\nu}_0+\frac{\dot{\nu}_0R^2}{(1+A/2)}+\frac{\dot{\nu}_0B^2}{4(1+A/2)}\right]\ ,$$

and C_1 and C_2 are functions of τ which need not concern us at present. If we stop at this iteration, we obtain $\Delta\theta_d = 1/\epsilon \int_{-\infty}^{+\infty} d\tau\, \omega_0 (R-1)$ and $\Delta\theta_g = \nu_1(+\infty)$. Expanding these results to $O(\epsilon^2)$ [which is $O(\epsilon^3)$ for the Hamiltonian], for our specific path in parameter space,

$$\Delta\theta_d = -\frac{\epsilon \sinh^2\beta}{8\omega_0} \int_{-\infty}^{+\infty} d\tau\, \dot{\alpha}^2 , \tag{3.19a}$$

$$\Delta\theta_g = -\pi(1-\cosh\beta) + \frac{\epsilon^2 \sinh^2\beta \cosh\beta}{16\omega_0^2} \int_{-\infty}^{+\infty} d\tau\, \dot{\alpha}^3 . \tag{3.19b}$$

We see that $\Delta\theta_d + \Delta\theta_g$ is in agreement with the result of the perturbation theory. Again we comment on the question of canonical variables. The transformation $(I_0', \theta_0') \to (I_1, \theta_1)$ consists of three canonical transformations, one of which reduces to the identity transformation at $\tau = +\infty$ (smoothness of h) $[(I_0, \theta_0) \to (I_1', \theta_1')]$, and two which do not for the angle variable, but do for the action variable $[(I_0', \theta_0') \to (I_0, \theta_0)$ and $(I_1', \theta_1') \to (I_1, \theta_1)]$. Then the shift in θ_0' at $\tau = +\infty$ (Hannay's angle) is given by the accumulated shift in θ_1 (dynamical part), corrected for the change in the origin with respect to which θ_1 is measured (geometric part). (On the basis of this one might argue that the phase shift computed by perturbation theory is entirely dynamical.) Equations (3.19a) and (3.19b) are analogous to the explicit calculations in Ref. 7 where the quantum-mechanical version of the adiabatic iteration was applied to the spin-$\frac{1}{2}$ system. We also note, although we do not present the explicit results, that if condition (2.24) had not been satisfied, then $\Delta\theta$ would have remained the same, but $\Delta\theta_d$ and $\Delta\theta_g$ would have been different.

C. Divergences in perturbation theory

We now turn to the study of divergences in high-order terms of perturbation theory. The equations we use are given by (3.6a)–(3.6c), and the Hamiltonian is given in Eq. (3.7). We introduce the notation $\tilde{h}_1 = h_1 - \langle h_1 \rangle_\theta$. Then,

$$\tilde{h}_1(I, \theta, \tau) = \sum_{l \neq 0} C_l(I, \tau) \exp(i\theta l) . \tag{3.20}$$

(In parallel transport $\langle h_1 \rangle_\theta = 0$.)

Let us first examine $w_{1,k}$. We set $K_{1,0} = \langle h_1 \rangle_\theta$. From Eq. (3.6a) then,

$$w_{1,0} = -\frac{1}{\omega_0} \int d\theta\, \tilde{h}_1 . \tag{3.21}$$

This expression is determined up to a function of I and R_j. Let us choose this function so that $\langle \partial w_{1,0}/\partial\tau \rangle_\theta = 0$. Then $K_{1,1} = 0$,

$$w_{1,1} = -\frac{1}{\omega_0} \int d\theta \frac{\partial w_{1,0}}{\partial\tau} = \frac{1}{\omega_0} \int d^2\theta \frac{\partial(\tilde{h}_1/\omega_0)}{\partial\tau} . \tag{3.22}$$

Again we can choose the function of I and R_j such that $\langle \partial w_{1,1}/\partial\tau \rangle_\theta = 0$. Then $K_{1,2} = 0$. Repeating the procedure, we obtain

$$\begin{aligned} w_{1,k} &= \frac{(-1)^{k+1}}{\omega_0} \int d^{k+1}\theta \left\{ \frac{\partial}{\partial\tau} \left[\frac{1}{\omega_0} \frac{\partial}{\partial\tau} \left[\frac{1}{\omega_0} \cdots \frac{\partial(\tilde{h}_1/\omega_0)}{\partial\tau} \right] \cdots \right] \right\} \\ &\equiv \frac{(-1)^{k+1}}{\omega_0} \int d^{k+1}\theta \left[\frac{\partial}{\partial\tau} \frac{1}{\omega_0} \right]^k \tilde{h}_1 . \end{aligned} \tag{3.23}$$

From the condition of smoothness of h, $h_1(\pm\infty) = 0$, $\tilde{h}_1(\pm\infty) = 0$, and $\omega_0(\pm\infty)$ is finite; hence $\tilde{h}_1/\omega_0(\pm\infty) = 0$. If we take these conditions to imply analyticity of $\tilde{h}_1$, ω_0, and $\tilde{h}_1/\omega_0$ in a strip about the real τ axis,[7] then $w_{1,k}$ as given by Eq. (3.23) is divergent. To see this, we first consider the case when ω_0 is time independent. For any τ_0 on the real axis, $\tilde{h}_1$ can be expanded in a convergent Taylor series about τ_0. Then we can use the Cauchy-Hadamard formula (see, for instance, Ref. 17), which yields

$$\limsup_{k\to\infty} \left\{ \left| \frac{1}{k!} \frac{\partial^k \tilde{h}_1(\tau_0)}{\partial\tau^k} \right| \right\}^{1/k} = \frac{1}{R} , \tag{3.24}$$

where R is the radius of convergence of the Taylor series (distance to the nearest singularity). For simplicity, in what follows, we assume that the sequence $\{[|(1/k!)\partial_\tau^k \tilde{h}_1(\tau_0)|]^{1/k}\}$ has only one limit point (and so we drop sup). Thus we have $|\partial_\tau^k \tilde{h}_1(\tau_0)| \to k! R^{-k}$ as $k \to \infty$. For finite R, R^{-k} is subdominant to $k!$, and Eq. (3.23) gives the asymptotic behavior for $\epsilon^k w_{1,k}$ as

$$\epsilon^k w_{1,k} \sim \epsilon^k k! , \tag{3.25}$$

for all finite τ. The factors from the θ integrations are also of the form m^{-k} (m is an integer), and therefore subdominant to $k!$.

Three things should be noticed here. First, if the singularities of $\tilde{h}_1$ are all at a finite distance from $\tau = 0$, then the factor R^{-k} guarantees that $w_{1,k}$ is zero at $\tau = \pm\infty$. Second, we emphasize the general nature of the result (3.25). *Any* singularity of $\tilde{h}_1$ in the finite domain of the complex τ plane (and not on the real axis) leads to this form for $w_{1,k}$ (poles or isolated essential singularities, branch points, natural boundaries, etc.). Third, we keep in mind that the divergence in Eq. (3.25) really goes as $(\epsilon/mR)^k k!$, where m is the integer of the term in Eq. (3.20) for which mR is smallest. Thus the series starts to diverge after mR/ϵ terms, rather than after $1/\epsilon$ terms. A similar comment also applies to the discussions that follow.

Returning to the case when ω_0 is a function of time, we

write the integrand in Eq. (3.23) in the form

$$\frac{1}{\omega_0^{k+1}}\frac{\partial^k \tilde{h}_1}{\partial \tau^k}-\frac{\partial \omega_0}{\partial \tau}\frac{1}{\omega_0^{k+2}}\frac{\partial^{k-1}\tilde{h}_1}{\partial \tau^{k-1}} + \cdots - \left[\frac{\partial \omega_0}{\partial \tau}\right]^k \frac{1}{\omega_0^{k+2}}\tilde{h}_1 . \quad (3.26)$$

If both $\tilde{h}_1$ and ω_0 have singularities in the finite τ plane, then the first few and the last few terms of Eq. (3.26) are dominant, behaving as $k!$. Then, unless there is a cancellation to $O(1/k!)$ between leading order terms for all finite τ, Eq. (3.26) will still have the dominant behavior of $k!$ for some regions of finite τ. Whereas the possibility of cancellation for all τ cannot be ruled out *a priori*, we consider this a pathological case. Thus, in general, even when ω_0 is time dependent, $\epsilon^k w_{1,k}$ will have the dominant behavior given by Eq. (3.25), at least for some region of finite τ.

As a word of caution, we remark that simply eliminating the singularities of ω_0 and $\tilde{h}_1$ in the complex τ plane does not necessarily remove the divergence. We take, for example, $\omega_0=1$ and $\tilde{h}_1 \propto \exp(-\tau^2)$, both of which are entire functions with all derivatives vanishing at $\tau=\pm\infty$. Then it is easy to show, by considering, for example, the Fourier transform of $w_{1,k}$, that $\epsilon^k w_{1,k} \sim \epsilon^k k!!$, which is divergent.

Next, we give the dominant behavior of $w_{n,k}$ for arbitrary n and k, satisfying $n \gg 1$ and $k \gg 1$, for the case when the only singularities of h_1 and ω_0 are finite-order poles (note that we do not require that $k \gg n$). The steps of the derivation are outlined in Appendix B. The result is

$$\epsilon^{n+k-1} w_{n,k} \sim \epsilon^{n+k}\frac{(n+k)!}{n!} , \quad (3.27)$$

for some region of finite τ. (If h_1 and ω_0 have other types of singularities in the complex τ plane, then the analysis is more complicated. A brief mention of some possibilities is made in Appendix B.)

As can be seen from this expression, the divergence occurs because of the second index k on $w_{n,k}$, i.e., because of the compounding of the τ derivatives on functions containing singularities in the complex τ plane. It also follows that for a given power of $\epsilon, N, N \gg 1$, the most divergent term is the one for which $k \gg n$. Thus $w_{n,k} \sim N!$ is the most divergent multiplier of ϵ^N. This also shows that $w_{1,N}$ is of the order of the most divergent term multiplying ϵ^N (hence, to see the full divergence, one needs to calculate only $w_{1,k}$ to high order). These results agree with Berry's study of the spin-$\frac{1}{2}$ case, where he took the zeroth-order function to have only simple poles. He found the divergence to behave as $m!\epsilon^m$.[7]

IV. DIVERGENCE OF THE ITERATIVE PROCEDURE

In this section we examine the divergence of the method of adiabatic iteration. An example is provided in Ref. 7 for the Hamiltonian $H=\mathbf{B}(\tau)\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ are the Pauli spin matrices [note that the right-hand side of Eq. (47) in Ref. 7 should be multiplied by $(-i)$]. Another example is given in Appendix A, where the iteration is applied to the Hamiltonian given by Eq. (3.8), with $2qp$ replaced by $(qp+pq)$. The unitary operator and the canonical transformation method yield the same results, which, furthermore, are analogous to the example studied in Ref. 7.

In our approach, we use the time-independent version of Lie perturbation theory to find the generating function for each iteration. The advantage of this method is that it offers a straightforward computational prescription for computing all quantities after many iterations, for any Hamiltonian. In addition, we do not insist on parallel transport. At $\tau=+\infty$ (and $H_{1,1}=0$) the transformation operator T will reduce to the identity operator, in analogy with adiabatic perturbation theory. The first task, then, to which we now turn, is to obtain an equation governing the νth term of the expansion after the αth iteration (we denote the subscripts and superscripts by Greek indices to distinguish them from the quantities in adiabatic perturbation theory). Once we have the required expressions we can determine how the divergence occurs. We assume that the first iteration has been performed, and the Hamiltonian H_1 (notation of Sec. II) obtained,

$$H_1(I_0,\theta_0,\tau)=H_{1,0}(I_0,\tau)+\epsilon H_{1,1}(I_0,\theta_0,\tau) . \quad (4.1)$$

Here $H_{1,0}=H_0$ and

$$H_{1,1}=\left.\frac{\partial G_0(r,I_0,\tau)}{\partial \tau}\right|_{r(\theta_0,I_0,\tau)} .$$

The second index on H is introduced to denote powers of ϵ. Equation (4.1) is the same as Eq. (3.7) in a slightly changed notation.

The next iteration, which, for clarity in notation, we denote by 1, consists of bringing H_1 into a canonical form (independent of the angle variables) by neglecting, at first, the explicit dependencies on τ. Using the canonical transformation operators from Sec. III, we write the transformed Hamiltonian as

$$K^{(1)}=T^{(1)^{-1}}(\epsilon)H_1 . \quad (4.2)$$

But H_1, T, and w do in fact depend on τ, and hence the correct transformed Hamiltonian H_2 is given by Eq. (3.4),

$$H_2=K^{(1)}+\epsilon T^{(1)^{-1}}(\epsilon)\int_0^{\epsilon} d\epsilon' T^{(1)}(\epsilon')\frac{\partial w^{(1)}(\epsilon')}{\partial \tau} . \quad (4.3)$$

The superscripts indicate the order of iteration. We note carefully that the small parameter arising from differentiating $w^{(1)}$ with respect to τ is ϵ, not ϵ'. It can therefore be taken out of the integral. As stipulated in the method of adiabatic iteration, we need to find a generating function (or equivalently $T^{(1)}$ and $w^{(1)}$) which satisfies Eq. (4.2), and then substitute it in Eq. (4.3). We solve for $w^{(1)}$ and $T^{(1)}$ by using perturbation theory,

$$T^{(1)}=\sum_{\nu=0}^{\infty}\epsilon^{\nu}T_{\nu}^{(1)}\ ,$$
$$w^{(1)}=\sum_{\nu=0}^{\infty}\epsilon^{\nu}w_{\nu+1}^{(1)}\ , \tag{4.4}$$
$$K^{(1)}=\sum_{\nu=0}^{\infty}\epsilon^{\nu}K_{\nu}^{(1)}\ .$$

In contrast to Eq. (3.5), only single sums occur in $w^{(1)}$ and $K^{(1)}$; ϵ multiplying t is not expanded. Substituting Eq. (4.4) into Eq. (4.2) and using Eq. (3.3), we obtain[8]

$$\omega_0\frac{\partial w_{\nu}^{(1)}}{\partial\theta}=\nu(K_{\nu}^{(1)}-H_{1,\nu})-\sum_{\mu=1}^{\nu-1}(L_{\nu-\mu}^{(1)}K_{\mu}^{(1)}+\mu T_{\nu-\mu}^{(1)^{-1}}H_{1,\mu})\ . \tag{4.5}$$

Compared to Eq. (3.6a), Eq. (4.5) does not contain derivatives with respect to τ, and all the quantities have only one index ν, which has been explicitly solved for (the superscripts have no relation to the power of ϵ).

Suppose the set of equations (4.5) has been solved and substituted into Eq. (4.3) [$w^{(1)}$ as calculated from Eq. (4.5) will in general converge[8]]. We bring H_2 into the canonical form by neglecting the explicit τ dependences:

$$K^{(2)}=T^{(2)^{-1}}(\epsilon)H_2\ , \tag{4.6}$$

which yields the analog of Eq. (4.5),

$$\omega_0\frac{\partial w_{\nu}^{(2)}}{\partial\theta}=\nu(K_{\nu}^{(2)}-H_{2,\nu})-\sum_{\mu=1}^{\nu-1}(L_{\nu-\mu}^{(2)}K_{\mu}^{(2)}+\mu T_{\nu-\mu}^{(2)^{-1}}H_{2,\mu})\ . \tag{4.7}$$

We have used the fact that $H_{2,0}=H_{1,0}=H_0$. H_2 is given by

$$H_2=\sum_{\nu=0}^{\infty}\epsilon^{\nu}(K_{\nu}^{(1)}+Q_{\nu}^{(1)})\ , \tag{4.8}$$

where

$$Q^{(1)}=\epsilon T^{(1)^{-1}}\int_0^{\epsilon}d\epsilon' T^{(1)}\frac{\partial w^{(1)}}{\partial\tau}\ . \tag{4.9}$$

We now notice that if $Q_{\nu}^{(1)}=0$ for $\nu\le N$, then Eq. (4.7) reads

$$\omega_0\frac{\partial w_{\nu}^{(2)}}{\partial\theta}=\nu(K_{\nu}^{(2)}-K_{\nu}^{(1)})-\sum_{\mu=1}^{\nu-1}(L_{\nu-\mu}^{(2)}K_{\mu}^{(2)}+\mu T_{\nu-\mu}^{(2)^{-1}}K_{\mu}^{(1)})\ , \tag{4.10}$$

which can be easily solved (as $K_{\nu}^{(1)}$ is independent of θ):

$$\omega_0\frac{\partial w_0^{(2)}}{\partial\theta}=0\rightarrow w_0^{(2)}=0\ , \tag{4.11a}$$

$$\omega_0\frac{\partial w_1^{(2)}}{\partial\theta}=K_1^{(2)}-K_1^{(1)}\rightarrow K_{\nu}^{(2)}=K_{\nu}^{(1)},\quad w_1^{(2)}=0 \tag{4.11b}$$

$$\omega_0\frac{\partial w_2^{(2)}}{\partial\theta}=2(K_2^{(2)}-K_2^{(1)})-(L_1^{(2)}K_1^{(2)}+T_1^{(2)^{-1}}K_1^{(1)})=2(K_2^{(2)}-K_2^{(1)})\rightarrow K_2^{(2)}=K_2^{(1)},\quad w_2^{(2)}=0 \tag{4.11c}$$

etc. In Eq. (4.11c), the second term in parentheses on the right-hand side is zero because $w_1^{(2)}=0$. Hence the general solution for $\nu\le N$ is $w_{\nu}^{(2)}=0$, $K_{\nu}^{(2)}=K_{\nu}^{(1)}$, $T_{\nu}^{(2)}=0$ (but $T_0^{(2)}=1$). The observations given in this paragraph will be very useful shortly.

We now determine $Q_{\nu}^{(1)}$ explicitly. Substituting the expansions for $w^{(1)}$, $T^{(1)}$, and $T^{(1)^{-1}}$ in Eq. (4.9), and performing the integration in ϵ yields

$$Q_{\nu}^{(1)}=\sum_{\substack{\mu'=0\\ \mu'+\mu''\le\nu-2}}^{\nu-2}\sum_{\mu''=0}^{\nu-2}\frac{1}{\nu-\mu''-1}T_{\mu''}^{(1)^{-1}}T_{\mu'}^{(1)}\frac{\partial w_{\nu-1-\mu'-\mu''}^{(1)}}{\partial\tau}\ . \tag{4.12}$$

By requiring that for nonzero values the subscript of $w^{(1)}$ is greater than 1 [see Eq. (4.4)], we obtain the condition that $Q_{\nu}^{(1)}=0$ for $\nu\le 1$. From the discussion in the preceding paragraph, this leads to $T_1^{(2)}=0$, $w_{0,1}^{(2)}=0$.

We can continue the process for subsequent iterations. The general result for αth iteration is

$$\omega_0\frac{\partial w_{\nu}^{(\alpha)}}{\partial\theta}=\nu(K_{\nu}^{(\alpha)}-K_{\nu}^{(\alpha-1)})-\sum_{\mu=1}^{\nu-\alpha}(L_{\nu-\mu}^{(\alpha)}K_{\mu}^{(\alpha)}+\mu T_{\nu-\mu}^{(\alpha)^{-1}}K_{\mu}^{(\alpha-1)})-\nu Q_{\nu}^{(\alpha-1)}-\sum_{\mu=\alpha}^{\nu-\alpha}\mu T_{\nu-\mu}^{(\alpha)^{-1}}Q_{\mu}^{(\alpha-1)}\ , \tag{4.13a}$$

with

$$Q_{\nu}^{(\alpha-1)}=\sum_{\substack{\mu'=0\\ \mu'+\mu''\le\nu-2}}^{\nu-2}\sum_{\mu''=0}^{\nu-2}\frac{1}{\nu-\mu''-1}T_{\mu''}^{(\alpha-1)^{-1}}\times T_{\mu'}^{(\alpha-1)}\frac{\partial w_{\nu-1-\mu'-\mu''}^{(\alpha-1)}}{\partial\tau} \tag{4.13b}$$

and

$$Q_{\nu}^{(\alpha-1)}=0\ \text{ for }\nu\le\alpha-1\ ,\qquad w_{\nu}^{(\alpha)}=0\ \text{ for }\nu\le\alpha-1\ . \tag{4.13c}$$

An application of the condition (4.13c) has been used to restrict sums appearing in (4.13a). Similarly, in Eq. (4.13b) the terms in the sum with μ' and $\mu''\neq 0$ and μ' or $\mu''<\alpha-1$ vanish. For $\alpha=1$ Eq. (4.13) should be supplemented with $Q_1^{(0)}=H_{1,1}$, $Q_{\gamma}^{(0)}=0$ for $\gamma\ge 2$, $K_0^{(0)}=H_{1,0}$, $K_{\beta}^{(0)}=0$ for $\beta\ge 1$.

As an example, we can apply the procedure described above to the Hamiltonian of Eq. (3.9). After two iterations, to $O(\epsilon^3)$,

$$K^{(2)}=\omega_0 I_3^{(2)}+\tfrac{1}{2}\epsilon\omega_0 I_3^{(2)}A-\tfrac{1}{8}\epsilon^2\omega_0 I_3^{(2)}(A^2+B^2)+\tfrac{1}{16}\epsilon^3 I_3^{(2)}[(\dot{A}B-A\dot{B})+\omega_0 A(A^2+B^2)]+O(\epsilon^4)\ . \tag{4.14}$$

The third iteration does not produce corrections to the Hamiltonian to this order. It is clear from this expression that $\Delta\theta_3^{(2)}$ is the same as $\Delta\bar{\theta}$ given by Eq. (3.12). The shifts accumulated up to an arbitrary time, however, are not the same. This is due to the fact that $\theta_3^{(2)}$ and $\bar{\theta}$ are different variables which coincide only at $\tau=\pm\infty$ [see comment following Eq. (2.12)].

We now turn to the question of divergences of $w_\nu^{(\alpha)}$. First we examine the divergence in the lowest nonvanishing order of ϵ in each iteration, $\nu=\alpha$. From Eq. (4.13b) we get

$$Q_\alpha^{(\alpha-1)}=\frac{1}{\alpha-1}\frac{\partial w_{\alpha-1}^{(\alpha-1)}}{\partial\tau}\ , \tag{4.15}$$

leading to

$$\omega_0\frac{\partial w_\alpha^{(\alpha)}}{\partial\theta}=\alpha(K_\alpha^{(\alpha)}-K_\alpha^{(\alpha-1)})-\frac{\alpha}{\alpha-1}\frac{\partial w_{\alpha-1}^{(\alpha-1)}}{\partial\tau}\ . \tag{4.16}$$

This gives $w_1^{(1)}=-1/\omega_0\int d\theta\,\tilde{H}_{1,1}$, where $\tilde{H}_{1,1}=H_{1,1}-\langle H_{1,1}\rangle_\theta$. Choosing the constant of integration such that $\langle\partial w_1^{(1)}/\partial\tau\rangle_\theta=0$, for $\alpha=2$ we get $K_2^{(2)}=K_2^{(1)}$. Repeating the procedure, $K_\alpha^{(\alpha)}=K_\alpha^{(\alpha-1)}$, and Eq. (4.16) becomes

$$\omega_0\frac{\partial w_\alpha^{(\alpha)}}{\partial\theta}=-\frac{\alpha}{\alpha-1}\frac{\partial w_{\alpha-1}^{(\alpha-1)}}{\partial\tau}\ . \tag{4.17}$$

This is easily solved, yielding

$$w_\alpha^{(\alpha)}=\frac{(-1)^\alpha\alpha}{\omega_0}\int d^\alpha\theta\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^{\alpha-1}\tilde{H}_{1,1}\ . \tag{4.18}$$

But this expression is the same as Eq. (3.23) ($\tilde{H}_{1,1}\equiv\tilde{h}_1$). Hence, for $\alpha\gg1$, $\epsilon^{\alpha-1}w_\alpha^{(\alpha)}\sim\epsilon^\alpha\alpha!$.

In general, we consider the case $w_{\alpha+\beta}^{(\alpha)}$, when the only singularities of $H_{1,1}$ and ω_0 in the complex τ plane are finite-order poles. For $\alpha\gg1$ and $\beta\gg1$, we obtain the asymptotic relation

$$\epsilon^{\alpha+\beta-1}w_{\alpha+\beta}^{(\alpha)}\sim\epsilon^{\alpha+\beta}\frac{(\alpha+\beta)!}{\beta!}\ , \tag{4.19}$$

for some region of finite τ. The steps of the derivation are outlined in Appendix C.

Therefore, for a fixed power of $\epsilon,N,N\gg1$, the most divergent term has $\alpha\gg\beta$ (i.e., the divergence increases with the increasing order of iteration), and the divergence occurs as $\epsilon^N N!$. We also see that $w_\alpha^{(\alpha)}$, which is the lowest nonvanishing term after the αth iteration, is of the order of the most divergent terms. This divergence, $\epsilon^N N!$, has already been seen in perturbation theory. It is also in agreement with the results of the spin-$\frac{1}{2}$ problem from Ref. 7. In that case, however, the only explicit results were given, in our notation, for $\beta=0$ (see also Appendix A).

V. CONCLUSION

We have seen how a divergence occurs in the iterative and perturbative approaches for calculating corrections to Hannay's angle. For the method of iteration, we have expanded the αth step in powers of ϵ; the most divergent term, for $\alpha\gg1$, behaved as $\epsilon^\alpha\alpha!$. In perturbation theory, on the other hand, for a given power of $\epsilon,N,N\gg1$, the most divergent term was $\epsilon^N N!$. Both of these occurrences were the result of taking α (or N) derivatives of $H_{1,1}$ (or h_1) with respect to τ. It can also be seen that the details of the path in parameter space are not important, as long as ω_0 and $H_{1,1}$ (or h_1) possess singularities in the finite domain of the complex τ plane. Thus the divergence is "universal." Though this paper has not dealt with the question, we expect these results to remain valid for the quantum-mechanical versions of the iterative and perturbative methods (see Appendix A for an example). Another remark is in order: While we have obtained the results in Secs. III and IV in order to understand the behavior of corrections to Hannay's angle, there is nothing specific to that problem in the derivations. Hence the divergences obtained are to be seen as germane properties of adiabatic perturbation theory, or adiabatic iteration, applied to Hamiltonians given by Eq. (3.7), for integrable systems away from resonances. They occur whenever h_0 or h_1 have singularities in the finite domain of the complex τ plane.

Finally, on the physical significance of divergences. It was suggested in Ref. 7 that the divergences are linked to nonadiabatic effects of transitions between eigenstates of the Hamiltonian. This proposal is reinforced by a study of WKB methods applied to harmonic-oscillator-like problems in one dimension.[18,19] Similarly, transitions seem to occur under conditions closely parallel to those for divergences.[10] We caution, however, against the inference that the transitions and the divergences in the iterative and perturbative methods are connected in a universal way. In Appendix D we present an example in which the evolving state returns at $\tau=+\infty$ exactly to the initial state, yet the iteration and perturbation procedures seem to diverge.

ACKNOWLEDGMENTS

We thank Professor J. M. Luttinger for a useful discussion and Professor M. V. Berry for a stimulating correspondence. This research is supported by the U.S. Department of Energy, Grant No. DE-FG02-86ER53222, and by Brookhaven National Laboratory, Grant No. BNL 35 9628-S.

APPENDIX A

The Hamiltonian

$$H=\tfrac{1}{2}[X(\tau)q^2+Y(\tau)(qp+pq)+Z(\tau)p^2]\ , \tag{A1}$$

where $[q,p]=i$, can be written in terms of generators of SO(2,1),[4]

$$H_0=(r_0)_i g_{ij}T_j\ . \tag{A2}$$

Here $T_1=-\frac{1}{4}(q^2-p^2)$, $T_2=-\frac{1}{4}(qp+pq)$, $T_3=-\frac{1}{4}(q^2+p^2)$, $x_0=Z-X$, $y_0=-2Y$, $z_0=Z+X$, $\mathbf{r}_0\equiv(x_0,y_0,z_0)$ and

$$g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}.$$

The summation over repeated indices, except k, is assumed. $T_{1,2,3}$ satisfy

$$[T_i,T_j]=i\epsilon_{ijm}g_{mn}T_n\ , \tag{A3}$$

and we assume that T_3 is diagonal.

The path in parameter space is chosen as in Sec. III ($z_0^2-x_0^2-y_0^2\equiv\rho_0^2=$const, and $X+Z=$const). The kth iteration is given by

$$H_k\rightarrow H_{k+1}=(r_{k+1})_i g_{ij}T_j\ . \tag{A4}$$

First we find a recursion relation relating $\mathbf{r}_{k+1}$ and $\mathbf{r}_k$. U_k of Eqs. (2.7) is given by

$$U_k=\mathcal{T}\exp\left[-iT_n g_{nl}\int_{-\infty}^{\tau}\frac{d\tau'}{\epsilon}[\Omega_{\|k}(\tau')]_l\right], \tag{A5}$$

where $\mathbf{\Omega}_{\|k}$ is the velocity of parallel transport for the kth iteration,

$$\mathbf{\Omega}_{\|k}=\frac{\epsilon}{|\rho_k|^2}(z_k\dot{y}_k-y_k\dot{z}_k,x_k\dot{z}_k-z_k\dot{x}_k,x_k\dot{y}_k-y_k\dot{x}_k)\ . \tag{A6}$$

We can now substitute these expressions into Eq. (2.8) and get the desired recursion relations. We proceed, however, only to the lowest expected order in ϵ, and set $U_k=1$. Defining $\zeta_k\equiv x_k+iy_k$, with $E_k=\pm\rho_k,0$ [in a three-dimensional representation of the SO(2,1) algebra], we obtain

$$\zeta_{k+1}=\frac{i\epsilon}{2}\dot{\zeta}_k\ , \tag{A7a}$$

$$z_{k+1}=\rho_k\ . \tag{A7b}$$

We have taken $\rho_0=2$. Equation (A7a) is analogous to Eq. (47) in Ref. 7. If ζ_0 has poles in the complex τ plane, then $\zeta_k=(i/2)^k\epsilon^k d^k\zeta_0/d\tau^k$ diverges as $\epsilon^k k!$. Also, replacing U_k by 1 is justified for $k\lesssim 1/\epsilon$. The divergence of ζ_k leads to the divergence of γ_k, since[7]

$$\gamma_k=\frac{\epsilon}{4}\int_{-\infty}^{\infty}d\tau\,\mathrm{Im}(\zeta_k^*\dot{\zeta}_k)\langle N|T_3|N\rangle\ . \tag{A8}$$

This formula is valid in the approximation $\rho_k=2\ \forall k$. We see that the divergence happens in the off-diagonal elements in Eq. (2.8).

The same divergence occurs in the sequence of canonical transformations described in Sec. II. The Hamiltonian is given by Eqs. (3.8) and (3.13),

$$H_1(I_0,\theta_0)=I_0(1-\epsilon\{A_0\cos[2(\theta_0+\nu_0)]+B_0\sin[2(\theta_0+\nu_0)]\})\ , \tag{A9}$$

and we assume that the parameters change in the manner described in Sec. III. With respect to Eq. (3.13), $A_0=A/2$, $B_0=B/2$, and ω_0 has been set equal to 1. The subsequent iteration produces, Eq. (3.18),

$$H_2(I_1,\theta_1,\tau)=I_1\{R_1-\epsilon[A_1\cos2(\theta_1+\nu_1)+B_1\sin2(\theta_1+\nu_1)]\}\ . \tag{A10}$$

The changes from Eq. (3.18) include the addition of a subscript 1 on R and the renaming of C_1 and C_2 to A_1 and B_1. The iteration can be repeated, yielding

$$H_{k+1}(I_k,\theta_k,\tau)=I_k(R_k-\epsilon\{A_k\cos[2(\theta_k+\nu_k)]+B_k\sin[2(\theta_k+\nu_k)]\})\ , \tag{A11}$$

with the recursion relations

$$R_k=[R_{k-1}^2-\epsilon^2(A_{k-1}^2+B_{k-1}^2)]^{1/2}\ , \tag{A12a}$$

$$B_k=-\frac{1}{2}\left[\epsilon\left[-\frac{\dot{R}_k}{R_k}+\frac{\dot{R}_{k-1}+\dot{A}_{k-1}}{R_{k-1}+A_{k-1}}\right]+2\frac{\dot{\nu}_{k-1}B_{k-1}}{R_{k-1}+A_{k-1}}\right], \tag{A12b}$$

$$A_k=-\frac{1}{2}\left[\epsilon\left[\frac{(\dot{R}_{k-1}+\dot{A}_{k-1})B_{k-1}}{(R_{k-1}+A_{k-1})R_k}-\frac{\dot{B}_{k-1}}{R_k}\right]+\frac{\dot{\nu}_{k-1}(R_{k-1}+A_{k-1})}{R_k}+\frac{\dot{\nu}_{k-1}B_{k-1}^2}{R_k(R_{k-1}+A_{k-1})}-\frac{\dot{\nu}_{k-1}R_k}{R_{k-1}+A_{k-1}}\right], \tag{A12c}$$

$$\dot{\nu}_k=\frac{1}{2}\left[\epsilon\left[\frac{(\dot{R}_{k-1}+\dot{A}_{k-1})B_{k-1}}{(R_{k-1}+A_{k-1})R_k}-\frac{\dot{B}_{k-1}}{R_k}\right]+\frac{\dot{\nu}_{k-1}(R_{k-1}+A_{k-1})}{R_k}+\frac{\dot{\nu}_{k-1}B_{k-1}^2}{R_k(R_{k-1}+A_{k-1})}+\frac{\dot{\nu}_{k-1}R_k}{R_{k-1}+A_{k-1}}\right]. \tag{A12d}$$

We attempt an asymptotic simplification with $R_k=1$, $\dot{\nu}_k, A_k, B_k\ll 1$. Equations (A12a)–(A12c) reduce to

$$R_k=1\quad\forall k\ , \tag{A13a}$$

$$B_k=-\tfrac{1}{2}\epsilon\dot{A}_{k-1}\ , \tag{A13b}$$

$$A_k=\tfrac{1}{2}\epsilon\dot{B}_{k-1}\ . \tag{A13c}$$

For Eq. (A12d), we need to keep one higher-order term, the reason for which will become apparent shortly,

$$\dot{\nu}_k=\dot{\nu}_{k-1}-\frac{\epsilon}{2}\dot{B}_{k-1}+\frac{\epsilon}{2}\dot{A}_{k-1}B_{k-1}\ . \tag{A13d}$$

Defining $C_k=A_k-iB_k$, Eqs. (A13b) and (A13c) become

$$C_k=\tfrac{1}{2}i\epsilon\dot{C}_{k-1}\ . \tag{A14}$$

The change in Hannay's angle due to the kth iteration is

$$\Delta\theta_{(k)}=\int_{-\infty}^{\infty}d\tau(\dot{\nu}_k-\dot{\nu}_{k-1})$$
$$=\frac{\epsilon}{2}\int_{-\infty}^{\infty}d\tau(\dot{A}_{k-1}B_{k-1}-\dot{B}_{k-1})\ . \tag{A15}$$

Because of the assumed smoothness of the Hamiltonian $A_k(\pm\infty)=B_k(\pm\infty)=0$, and

$$\Delta\theta_{(k)}=\frac{\epsilon}{4}\int_{-\infty}^{\infty}d\tau(\dot{A}_{k-1}B_{k-1}-A_{k-1}\dot{B}_{k-1})$$
$$=\frac{\epsilon}{4}\int_{-\infty}^{\infty}d\tau\,\mathrm{Im}(C_{k-1}^{*}\dot{C}_{k-1})\ . \tag{A16}$$

If $C_0(\tau)$ has poles, C_k, as given by Eq. (A14), diverges as $\epsilon^k k!$ and the asymptotic approximation on Eq. (A12) is consistent for $k\lesssim 1/\epsilon$. $\Delta\theta_{(k)}$ then also diverges in a manner completely analogous to γ_k [see Eq. (A8)].

APPENDIX B

We determine the dominant behavior of $w_{n,k}$ for arbitrary n and k. First we look at $w_{n,0}$. For $n=1$, $w_{1,0}$ is given by Eq. (3.21). As far as the explicit time dependence is concerned, $w_{1,0}\sim\tilde{h}_1/\omega_0$. For $n\geq 2$, we set the following preliminaries: $L_{n-m}K_{m,k}=L(w_{n-m,l})K_{m,k}$ which corresponds to a power of ϵ of $n-m-1+l+m+k=n-1+l+k$. Then to $O(\epsilon^{n-1})$ (which is the order of $w_{n,0}$) we set $l=k=0$. So

$$\omega_0\frac{\partial w_{n,0}}{\partial\theta}=nK_{n,0}-\sum_{m=1}^{n-1}L(w_{n-m,0})K_{m,0}+T_{n-1}^{-1}h_1\ . \tag{B1}$$

For future reference, as well as for the present calculation, we now write the explicit form for T_{n-1}^{-1}, obtained by iterating (3.6c),[8]

$$T_{n-1}^{-1}=\sum_{\substack{m_1\cdots m_r\\ n-1>m_1>\cdots>m_r}}\frac{1}{(n-1)m_1\cdots m_r}L(w_{n-1-m_1,l_1})$$
$$\times L(w_{m_1-m_2,l_2})\cdots L(w_{m_r,l_r})\ . \tag{B2}$$

Equation (B2) corresponds to the power of ϵ of $n-2-m_1+m_1-m_2+\cdots-m_r+m_r+l_1+\cdots+l_r=n-2+l_1+l_2+\cdots+l_r$. To $O(\epsilon^{n-1})$ for $T_{n-1}^{-1}h_1$, then $l_1=l_2=\cdots=l_r=0$. Therefore the system of equations (B1) is closed for $w_{n,0}$, i.e., $w_{n,0}$ does not depend on $w_{m,k}$ for $k\geq 1$. In addition, there are no time derivatives in Eq. (B1). Hence, for the computation of $w_{n,0}$, τ can be treated as constant. From the assumed form of h_1 [Eq. (3.20) and the comment above it], we see that as far as the τ dependence is concerned $w_{n,0}\sim h_1^{n-1}$. $w_{n,0}$ may also contain ω_0 to a power less than or of the order of n; this, however, is not relevant for the argument that follows. (The order and the location of the singularities of h_1 and ω_0 in τ should be independent of I so as to insure that the derivatives with respect to I do not affect the singular points. This is, however, what is realized if the parameters vary in time in a *prescribed* fashion, independent of the value of the action variable.)

We now turn to $w_{n,k}$. From (3.6a),

$$\omega_0\frac{\partial w_{n,k}}{\partial\theta}=-\frac{\partial w_{n,k-1}}{\partial\tau}+f(n,k)\ , \tag{B3a}$$

where

$$f(n,k)=nK_{n,k}-\sum_{m=1}^{n-1}\sum_{\substack{l,l'=0\\ l+l'=k}}^{k}L(w_{n-m,l})K_{m,l'}+T_{n-1,k}^{-1}h_1\ . \tag{B3b}$$

In the last term of Eq. (B3b), we have added the index k on T_{n-1}^{-1}, where $k=l_1+\cdots+l_r$ and $l_1,\ldots,l_r$ are given in Eq. (B2). We can iterate Eq. (B3a) in k for fixed n. The result is

$$w_{n,k}=\frac{(-1)^k}{\omega_0}\int d^k\theta\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^k\omega_0 w_{n,0}$$
$$-\sum_{j=1}^{k}\frac{(-1)^j}{\omega_0}\int d^j\theta\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^{j-1}f(n,k-j+1)\ . \tag{B4}$$

Before we proceed further we establish the following useful result: All terms in $w_{n,k}$ contain exactly k derivatives with respect to τ. Here the derivatives are taken of h_1, or of some power of h_1, and their number does not specify how they are distributed [e.g., $h_1\partial_\tau^2 h_1$ and $(\partial_\tau h_1)^2$ both contain two derivatives]. The proposition above is valid under the following condition: As $w_{n,k}$'s are given in terms of indefinite integrals over θ, we choose the constant of integration so that it has the same number of derivatives as the integrated part. Of course, an arbitrary constant of integration would not affect the end physical result; a term canceling it would appear at a subsequent step. However, the systematics of relating the second index on $w_{n,k}$ with the number of derivatives would be lost. The simplest way to show the relation between k and the number of derivatives is by a variant of the induction principle on n (first index of $w_{n,k}$). This we call induction 1, in contrast to induction 2 to be introduced later.

For $n=1$ we have seen that $w_{1,k}$ contains k derivatives, for all k. Similarly, $K_{1,k}$ is either zero or it also contains k derivatives for all k ($K_{1,k}=\langle w_{1,k}\rangle$). Next we assume that for all $n\leq r$, $w_{n,k}$ and $K_{n,k}$ have k derivatives for all k (or $K_{n,k}$ is zero). For $n=r+1$ we then have

$$\omega_0\frac{\partial w_{r+1,k}}{\partial\theta}=-\frac{\partial w_{r+1,k-1}}{\partial\tau}+(r+1)K_{r+1,k}$$
$$-\sum_{m=1}^{r}\sum_{\substack{l,l'=0\\ l+l'=k}}^{k}L(w_{n-m,l})K_{m,l'}+T_{r,k}^{-1}h_1\ . \tag{B5}$$

The last two terms in this expression contain k derivatives by assumption [see Eq. (B2)]. (The taking of the Poisson bracket does not alter the number of derivatives.) $K_{r+1,k}$, on the other hand, is given by the orbit average of the other terms on the right-hand side of Eq. (B5).

Therefore it has the same number of derivatives as those terms. It remains to see how many derivatives $\partial_r w_{r+1,k-1}$ contains. This is most easily established by doing an induction in k (induction 2). [Equation (B4) does not help because of terms

$$\sum_{j=1}^{k-1}(-1)^j\omega_0^{-1}\int d^j\theta(\partial_\tau\omega_0^{-1})^{j-1}K_{r+1,k-j}$$

for which the hypothesis of induction 1 does not hold.]

For $k=1$ we have

$$\omega_0\frac{\partial w_{r+1,1}}{\partial\theta}=-\frac{\partial w_{r+1,0}}{\partial\tau}+(r+1)K_{r+1,1}+g(r,1)\ , \tag{B6a}$$

where $g(n-1,k)\equiv f(n,k)-nK_{n,k}$. By hypothesis of induction 1 $g(r,1)$ has one derivative, and from the discussion following Eq. (B2) $w_{r+1,0}$ has none. Hence $K_{r+1,1}$ and $w_{r+1,1}$ both have one derivative. Next we assume that for $k=l$, $w_{r+1,l}$ and $K_{r+1,l}$ contain l derivatives. Then $k=l+1$ gives

$$w_{r+1,l+1}=\frac{1}{\omega_0}\int d\theta\left[-\frac{\partial w_{r+1,l}}{\partial\tau}+(r+1)K_{r+1,l+1}+g(r,l+1)\right]. \tag{B6b}$$

By hypothesis of induction 1 $g(r,l+1)$ contains $l+1$ derivatives, and by hypothesis of induction 2 $w_{r+1,l}$ contains l. Hence $w_{r+1,l+1}$ and $K_{r+1,l+1}$ [orbit average of the other terms on the right-hand side of Eq. (B6b)] both contain $l+1$ derivatives. Therefore induction 2 gives that $w_{r+1,k-1}$ contains $k-1$ derivatives. This result substituted into Eq. (B5) immediately completes the first induction. Hence the assertion that all terms in $w_{n,k}$ contain exactly k derivatives holds.

We return to the divergence of $w_{n,k}$, and examine the case $n\gg1$, $k\gg1$. [The two limiting cases are simple: For n finite and $k\gg1$, Eq. (3.24) and the discussion following Eq. (3.26) apply and $\epsilon^{n+k-1}w_{n,k}\sim\epsilon^{n+k-1}k!$; for k finite and $n\gg1$, $\epsilon^{n+k-1}w_{n,k}\sim(n\pm k)!/n!\epsilon^{n+k-1}$, which does not diverge for sufficiently small ϵ.] First, we limit ourselves to the case where the only singularities of h_1 and ω_0 in the complex τ plane are poles of finite order. Then the most divergent terms in $w_{n,k}$ are the ones which take the highest derivative of the highest power of h_1. (This can be seen most easily by writing h_1 and ω_0 in a partial fraction expansion.) From the discussion in the preceding paragraph, and from Eq. (B4), the highest derivative is the kth one, and it can act on the highest power of h_1, which is $n-1$. This is manifest in the first term in Eq. (B4), but it also appears in the remaining terms. In $f(n,k-j+1)$, the highest derivative is the $(k-j+1)$th, and it acts on h_1^{n-1} by choosing the indices as follows: $m=1$, $l=k-j+1$, $l'=0$ (or $l'=k-j+1$, $l=0$) for the sum term of $f(n,k-j+1)$ [see Eq. (B3b)], and $l_1=k-j+1$, $l_2,\ldots=0$ and $m_1\cdots m_r=0$ for the last term [see Eq. (B2)]. [For this case, then, $T^{-1}_{n-1,k-j+1}=1/(n-1)L(w_{n-1,k-j+1})$.] Hence the remaining terms of Eq. (B4) contain

$$\frac{\partial^{j-1}f(n,k-j+1)}{\partial\tau^{j-1}}\sim\frac{\partial^{j-1}[(\partial^{k-j+1}h_1^{n-1})/\partial\tau^{k-j+1}]}{\partial\tau^{j-1}}\sim\frac{(n+k)!}{n!}h_1^{n+k}\ . \tag{B7a}$$

Similarly,

$$\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^k\omega_0 w_{n,0}\sim\frac{1}{\omega_0^{k-1}}\frac{\partial^k w_{n,0}}{\partial\tau^k}\sim\frac{(n+k)!}{n!}h_1^{n+k}\ . \tag{B7b}$$

We see that ω_0 raised to some power comparable to n does not affect this result [see discussion following Eq. (3.26)]. Therefore, including the explicit ϵ dependence, the asymptotic form of $w_{n,k}$ is

$$\epsilon^{n+k-1}w_{n,k}\sim\epsilon^{n+k}\frac{(n+k)!}{n!}\ . \tag{B8}$$

We briefly comment on the case when h_1 and ω_0 possess singularities other than poles. If h_1 and ω_0 have, in addition to poles, branch points at which the functions do not tend to infinity [e.g., $\sin(\tau^2-\tau_0^2)^{1/2}$], then Eq. (B8) very likely still holds. The other cases are less clear, but one may conjecture the following: If all singularities of h_1 and ω_0 are branch points at which the functions do not blow up, then $\partial_\tau^k h_1^{n-1}$ diverges more slowly than $k!$, for $k\gg1$, $n\gg1$, and k not much greater than n (and then the contributions of other terms in $w_{n,k}$ should be studied). On the other hand, if h_1 or ω_0 have at least one essential singularity, or a branch point at which the functions blow up, or a sequence of nonisolated poles, then the divergence of $\partial_\tau^k h_1^{n-1}$ is more rapid than $k!$, for $n\gg1$, $k\gg1$, and k not much greater than n (and this is still the most divergent term in $w_{n,k}$). (The part of this conjecture pertaining to $\partial_\tau^k h_1^{n-1}$ can easily be verified in particular cases by using, for example, the symbolic manipulation routine MACSYMA.)

APPENDIX C

We consider $w^{(\alpha)}_{\alpha+\beta}$. First we look for the most divergent term in $Q^{(\alpha-1)}_{\alpha+\beta}$. In Eq. (4.13b), all the terms contain $\partial/\partial\tau$ acting on $w^{(\alpha-1)}$ (which is multiplied by a combination of other terms also containing $w^{(\alpha-1)}$). Of these, the most divergent is the one which takes the derivative of the highest previous derivative of $H_{1,1}$. As to the role of the T's, from the general expression following Eq. (3.3) [or even directly from equations (3.6b) and (3.6c)], we see that for any smooth function h of I and θ, $T^{(\alpha)}_\mu$ or $T^{(\alpha)^{-1}}_\mu$ cannot be more divergent than $w^{(\alpha)}_\mu$. Thus

$$Q^{(\alpha-1)}_{\alpha+\beta}\sim\frac{1}{\alpha+\beta-1}\frac{\partial w^{(\alpha-1)}_{\alpha+\beta-1}}{\partial\tau}\ . \tag{C1}$$

We write Eq. (4.13a) as

$$\omega_0\frac{\partial w^{(\alpha)}_{\alpha+\beta}}{\partial\theta}=-\frac{\alpha+\beta}{\alpha+\beta-1}\frac{\partial w^{(\alpha-1)}_{\alpha+\beta-1}}{\partial\tau}+F(\alpha,\alpha+\beta)\ , \tag{C2a}$$

where

$$F(\alpha,\alpha+\beta)=(\alpha+\beta)(K^{(\alpha)}_{\alpha+\beta}-K^{(\alpha-1)}_{\alpha+\beta}) - \sum_{\mu=1}^{\beta}(L^{(\alpha)}_{\alpha+\beta-\mu}K^{(\alpha)}_{\mu}+\mu T^{(\alpha)^{-1}}_{\alpha+\beta-\mu}K^{(\alpha-1)}_{\mu}) - \sum_{\mu=\alpha}^{\beta}\frac{\mu}{\mu-1}T^{(\alpha)^{-1}}_{\alpha+\beta-\mu}\frac{\partial w^{(\alpha-1)}_{\mu-1}}{\partial\tau} . \quad \text{(C2b)}$$

We use Eq. (C2a) to obtain $w^{(\alpha)}_{\alpha+\beta}$ in terms of w's from previous iterations,

$$w^{(\alpha)}_{\alpha+\beta}=\frac{(-1)^{\alpha-1}(\alpha+\beta)}{\omega_0(1+\beta)}\int d^{\alpha-1}\theta\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^{\alpha-1}\omega_0 w^{(1)}_{\beta+1} + \sum_{\gamma=1}^{\alpha-1}\frac{(-1)^{\alpha-\gamma+1}}{\omega_0}\frac{(\alpha+\beta)}{(\beta+\gamma+1)} \times\int d^{\alpha-\gamma}\theta\left[\frac{\partial}{\partial\tau}\frac{1}{\omega_0}\right]^{\alpha-\gamma-1} \times F(\gamma+1,\beta+\gamma+1) . \quad \text{(C3)}$$

In a manner completely analogous to Appendix B, we can now perform two inductions (first in β, and then in α), and conclude that all terms in $w^{(\alpha)}_{\alpha+\beta}$ contain exactly $\alpha-1$ derivatives with respect to τ, acting on some power of $H_{1,1}$ or ω_0.

We now limit ourselves again to the case when $H_{1,1}$ and ω_0 have only finite-order poles in the complex plane, and deduce the divergence of $w^{(\alpha)}_{\alpha+\beta}$. The dominant contributions come from the highest derivative, $(\alpha-1)$th, acting on the highest power of $H_{1,1}$, which is β (by construction, the calculation of $w^{(1)}_{1+\beta}$ freezes the time dependence, and the powers of ϵ come simply from taking higher powers of $H_{1,1}$). While the function $F(\gamma+1,\beta+\gamma+1)$ in Eq. (C3) is more tedious to analyze than the corresponding function f in Eq. (B4), the considerations above show that the first term in Eq. (C3) contains the full divergence of $w^{(\alpha)}_{\alpha+\beta}$. Hence, for $\alpha\gg1$ and $\beta\gg1$,

$$\epsilon^{\alpha+\beta-1}w^{(\alpha)}_{\alpha+\beta}\sim\epsilon^{\alpha+\beta}\frac{(\alpha+\beta)!}{\beta!} . \quad \text{(C4)}$$

As in the case of perturbation theory, we did not have to take into account the dependence of $w^{(1)}_{\beta+1}$ on powers of ω_0. Similarly, the remarks concerning other types of singularities of $H_{1,1}$ and ω_0 can be taken over from Appendix B.

APPENDIX D

We consider the harmonic oscillator with slowly varying frequency,

$$H=\tfrac{1}{2}[p^2+\omega^2(\tau)q^2] . \quad \text{(D1)}$$

For this Hamiltonian, there is no zeroth-order contribution to $\Delta\theta$. For ϵ finite, however, $\Delta\theta$ is nonzero.[20] The exact transition amplitudes between an initial state and a final state have been calculated for (D1),[21] in terms of a function ρ which is given as a solution of a differential equation. Thus, if at $\tau=-\infty$ the system is in the ground state of H, then the probability of being in state $|m\rangle$ at $\tau=+\infty$ is

$$P_{0m}=\frac{m!}{2^m[(m/2)!]^2}\left[\frac{\cosh\delta-1}{\cosh\delta+1}\right]^{m/2}\left[\frac{2}{\cosh\delta+1}\right]^{1/2} \text{ for } m \text{ even} \quad =0 \text{ for } m \text{ odd} . \quad \text{(D2)}$$

We assume $\omega(\pm\infty)=1$. Then δ is defined by

$$\tau\to+\infty,\quad \rho(\tau)\to\pm[\cosh\delta\pm\sinh\delta\sin(2\tau+\varphi)] , \quad \text{(D3)}$$

where ρ satisfies

$$\epsilon^2\ddot{\rho}+\omega^2(\tau)\rho-\frac{1}{\rho^3}=0 , \quad \text{(D4)}$$

with $\rho(-\infty)=1$. We are free to choose any combination of signs in Eq. (D3), and φ is a real phase constant. Equation (D4) can be shown to be equivalent to the equations of motion for H given by Eq. (D1).[22]

Therefore the question of the example we are after can be posed as follows: Can we find $\omega(\tau)$, satisfying $\omega(\pm\infty)=1$, $\partial^n_\tau\omega(\pm\infty)=0\ \forall n$, and where ω is not an entire function, such that if $\rho(-\infty)=1$, then $\rho(+\infty)=1$. The condition $\rho(+\infty)=1$ insures that $\delta=0$, thereby giving $P_{0,m}=\delta_{0,m}$; on the other hand, if $\omega(\tau)$ is not an entire function, then there exists the possibility of divergence discussed in Secs. III and IV.

We solve Eq. (D4) for ω^2,

$$\omega^2=\frac{1}{\rho^4}(1-\epsilon^2\rho^3\ddot{\rho}) , \quad \text{(D5)}$$

and take a solution for ρ which satisfies the required conditions, for example, $\rho=(\tau^2+2)/(\tau^2+1)$. From Eq. (D5), ω^2, which gives this ρ, is

$$\omega^2=\frac{\tau^{12}+6\tau^{10}+(15-6\epsilon^2)\tau^8+(20-34\epsilon^2)\tau^6+(15-60\epsilon^2)\tau^4+(6-24\epsilon^2)\tau^2+16\epsilon^2+1}{(\tau^2+1)^2(\tau^2+2)^4} , \quad \text{(D6)}$$

and ω is taken to be the positive square root of this expression. It is clear that $\omega(\pm\infty)=1$ and that $\partial^n_\tau\omega(\pm\infty)=0$. We have thus found an ω which is time dependent and has poles in the complex τ plane, yet it gives no transitions at $\tau=+\infty$. We note that H of Eq. (D1), with ω given by Eq. (D6), differs from the Hamiltonians we have treated earlier in one important respect: it contains explicit ϵ dependence (apart from the ϵ multiplying t, which is absorbed in τ). Nevertheless, we can still apply the perturbative and iterative methods to study this problem.

We begin with Eq. (D1) and perform the first iteration.

By using the generating function

$$G_0(q,I_0,\tau)=\pm I_0\sin^{-1}\left[\left(\frac{\omega}{2I_0}\right)^{1/2}q\right] \pm\frac{1}{2}q(2I_0\omega)^{1/2}\left[1-\frac{q^2\omega}{2I_0}\right]^{1/2}, \quad \text{(D7)}$$

we obtain

$$H_1=\omega I_0+\epsilon I_0\frac{\dot\omega}{2\omega}\sin 2\theta_0 . \quad \text{(D8)}$$

This Hamiltonian can now be treated by using either Lie perturbation theory or the adiabatic iteration.

To apply perturbation theory we write $\omega^2=a+\epsilon^2 b$, where, from Eq. (D5), $a=\rho^{-4}$ and $b=-\rho^{-1}\ddot\rho$, and expand Eq. (D8) to $O(\epsilon^2)$,

$$H_1=I_0a^{1/2}+\epsilon\frac{\dot a}{4a}I_0\sin 2\theta_0+\epsilon^2\frac{b}{2a^{1/2}}I_0+O(\epsilon^3) . \quad \text{(D9)}$$

Performing the procedures described in Sec. III ($\omega_0\equiv a^{1/2}$) to $O(\epsilon^2)$, the Hamiltonian above is transformed into

$$K=a^{1/2}\bar I+\frac{\epsilon^2\bar I}{2a^{1/2}}\left[b-\frac{\dot a^2}{16a^2}\right]+O(\epsilon^3) , \quad \text{(D10)}$$

with the frequency $\bar\omega=a^{1/2}+(\epsilon^2/2a^{1/2})[b-(\dot a^2/16a^2)]$ and the shift in the angle coordinate $\Delta\bar\theta=1/\epsilon\int_{-\infty}^{\infty}d\tau(\bar\omega-a^{1/2})$. [As a matter of convenience, we have defined here $\Delta\bar\theta=1/\epsilon\int_{-\infty}^{\infty}d\tau(\bar\omega-\omega_0)$ rather than $1/\epsilon\int_{-\infty}^{\infty}d\tau(\bar\omega-\omega)$, as in Ref. 20.] In particular, we notice that expressing a and b in terms of ρ gives the second term proportional to $-(\rho^{-1}\ddot\rho+\rho^{-2}\dot\rho^2)$, which is not zero for all times.

We can also apply the iterative procedure to the Hamiltonian of Eq. (D8). Following the method described in Sec. II, to $O(\epsilon^2)$ for the Hamiltonian [or $O(\epsilon)$ for $\Delta\theta$] we obtain

$$\Delta\theta_g=0 , \quad \text{(D11a)}$$

$$\Delta\theta_d=\epsilon\int_{-\infty}^{\infty}d\tau\left[\frac{b}{2a^{1/2}}-\frac{\dot a^2}{32a^{5/2}}\right] . \quad \text{(D11b)}$$

Thus, to $O(\epsilon^2)$, the perturbation and iteration methods agree. What is interesting about the Hamiltonian of Eq. (D8), however, is that it has an exact solution. We have built it into the problem by assuming a smooth solution for ρ [see comment following Eq. (D5)]. Then by a canonical transformation given in Ref. 22, the Hamiltonian can be brought into the form

$$K(P,Q)=\rho^{-2}P(\equiv a^{1/2}P) , \quad \text{(D12)}$$

which gives $\Delta Q=\Delta\theta=0$. (Note that in Ref. 22, the time t is the same as τ in this paper.) Thus both the perturbation theory [Eq. (D10)] and the iteration method [Eqs. (D11a) and (D11b)] fail to capture the exact solution to $O(\epsilon^2)$. But this is also sufficient to insure that they never will, for finite and arbitrary ϵ. To see this, we notice that the zeroth-order solution is the exact one, and hence all terms of higher order in ϵ would have to add up to zero. However, we have shown that the term of $O(\epsilon^2)$ is nonvanishing. Then there exists no possibility of adding higher-order terms in such a way as to cancel the term of $O(\epsilon^2)$, for ϵ arbitrary and finite. ($\{\epsilon^k\}$ form a linearly independent polynomial basis.) Therefore neither the perturbation theory nor the iteration method converge to the exact solution.

From the comments above, we see that the two approaches studied could either converge to an incorrect expression for the frequency or result in an asymptotic series. We think the second possibility is more likely, with the optimum truncation being after the first term, which also gives the exact solution. In particular, in Eq. (D8) both ω and $\dot\omega/\omega$ have singularities in the finite domain of the complex τ plane. This, then, suggests that the compounding of derivatives with respect to τ in either the perturbation theory or the iterative method could lead to divergences analogous to the ones discussed in Secs. III and IV. We have thus found an example in which there are no transitions at $\tau=+\infty$, yet the perturbation and iteration schemes seem to lead to divergences.

It is interesting to consider an alternative iteration formula, given by Eqs. (69) and (70) of Ref. 20,

$$\Delta\theta^{(k)}=\int_{-\infty}^{\infty}\frac{d\tau}{\epsilon}[\Omega^{(k)}(\tau)-\omega(\tau)] , \quad \text{(D13a)}$$

$$\Omega^{(k+1)}=\left[\omega^2+(\Omega^{(k)})^{1/2}\epsilon^2\frac{\partial^2}{\partial\tau^2}(\Omega^{(k)})^{-1/2}\right]^{1/2} , \quad \text{(D13b)}$$

with $(\Omega^{(0)})^2=\omega^2$. It is not difficult to show that this iteration formula does not terminate. In fact, $(\Omega^{(0)})^2=\rho^{-4}+O(\epsilon^2)$, $(\Omega^{(1)})^2=\rho^{-4}+O(\epsilon^4)$, $(\Omega^{(2)})^2=\rho^{-4}+O(\epsilon^6)$, $(\Omega^{(3)})^2=\rho^{-4}+O(\epsilon^8)$, etc., and the iteration scheme appears to be asymptotic to the exact solution but does not converge to it. (For example, for the first four iterations the leading terms in ϵ at $\tau=0$ are ϵ^2, $60\epsilon^4$, $12\,000\epsilon^6$, and $5\,009\,280\epsilon^8$.) We note that the ϵ dependence of the deviation from the exact solution of the iteration procedure (D13a) and (D13b) is different from the iterative method described earlier in this paper. Both methods appear to give the correct asymptotic behavior, but the divergences of the two methods are generally not the same. The divergences thus appear to be no more than mathematical artifacts.

It may also be interesting to point out that ω^2 specified by Eq. (D5) is a "reflectionless" potential.[23] The solution of the equation of motion for, for example, q which starts at $\tau=-\infty$ as a wave traveling in the direction of increasing τ remains so at *all times*. Hence there are no reflections at any time, a fact which is a consequence of the existence of the solution to Eq. (D4). This, however, should be distinguished from the additional requirements on $\rho(\tau)$ which insure that there are no transitions at $\tau=+\infty$ (see, for example, Ref. 21).

Finally, it may be possible to argue that divergences are linked to transitions between the eigenstates of H at finite times. We show below that those transitions cannot be excluded for any choice of time-dependent ω, and hence this example cannot settle the question. We write the Hamiltonian of Eq. (D1) at any time in terms of

creation and annihilation operators,

$$H=\omega(\tau)(b^{\dagger}b+\tfrac{1}{2}) , \tag{D14a}$$

$$b=\left[\frac{\omega(\tau)}{2}\right]^{1/2}[q+i\omega^{-1}(\tau)p] , \tag{D14b}$$

$$b^{\dagger}=\left[\frac{\omega(\tau)}{2}\right]^{1/2}[q-i\omega^{-1}(\tau)p] , \tag{D14c}$$

with $[b(\tau),b^{\dagger}(\tau)]=1$. The instantaneous eigenstates of H are then defined as the eigenstates of $b^{\dagger}(\tau)b(\tau)$. In particular, if the system is in the ground state at $\tau=-\infty$, then the probability of being in the state which is annihilated by $b(\tau)$, at τ, is given by $|\eta|^{-1}$,[21] where

$$\eta(\tau)=(4\omega)^{-1/2}(\rho^{-1}+\omega\rho-i\epsilon\dot{\rho}) . \tag{D15}$$

The condition that the system always be in the instantaneous ground state of H is then given by $|\eta|^2=1$. This coupled with Eq. (D4) enables one to obtain an equation involving only ρ,

$$\epsilon^2\rho^4(\rho\ddot{\rho}-\dot{\rho}^2)^2+4\rho^2\dot{\rho}^2=0 . \tag{D16}$$

If we require ρ to be a real function of τ,[21] then Eq. (D16) has no other solution than $\rho=$const. Hence there is no ρ, and therefore no ω, other than a constant, which insures that the system stays in the ground state at all times.

[1]M. V. Berry, Proc. R. Soc. London Ser. A **392**, 45 (1984).
[2]J. H. Hannay, J. Phys. A **18**, 221 (1985).
[3]M. V. Berry, J. Phys. A **18**, 15 (1985).
[4]R. Jackiw, Int. J. Mod. Phys. A **3**, 285 (1988).
[5]Ph. De Sousa Gerbert, Ann. Phys. (N.Y.) **189**, 155 (1989).
[6]A. Bhattacharjee and T. Sen, Phys. Rev. A **38**, 4389 (1988).
[7]M. V. Berry, Proc. R. Soc. London Ser. A **414**, 31 (1987).
[8]J. R. Cary, Phys. Rep. **79**, 129 (1981).
[9]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[10]J. T. Hwang and P. Pechukas, J. Chem. Phys. **67**, 4640 (1977).
[11]J. Anandan and L. Stodolsky, Phys. Rev. D **35**, 2597 (1987).
[12]C. A. Mead, Phys. Rev. Lett. **59**, 161 (1987).
[13]I. C. Percival, Adv. Chem. Phys. **36**, 1 (1977).
[14]M. V. Berry, in *Chaotic Behavior of Deterministic Systems*, edited by G. Iooss, R. H. G. Helleman, and R. Stora (North-Holland, Amsterdam, 1983).
[15]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[16]L. Vinet, Phys. Rev. D **37**, 2369 (1988).
[17]A. I. Markushevich, *Theory of Functions of a Complex Variable* (Prentice-Hall, Englewood Cliffs, NJ, 1965).
[18]T. J. Schep, Physica **74**, 397 (1974).
[19]M. V. Berry (unpublished).
[20]M. V. Berry, in *Geometric Phases in Physics*, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1988).
[21]H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[22]H. R. Lewis, J. Math. Phys. **9**, 1976 (1968).
[23]M. V. Berry and C. J. Howls (unpublished).