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First-passage-time noninteger moments for some diffusion and dichotomous processes

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We calculate noninteger moments $\langle t^q \rangle$ of first passage time to trapping, at both ends of an interval $(0,L)$, for some diffusion and dichotomous processes. We find the critical behavior of $\langle t^q \rangle$, as a function of q, for free processes. We also show that the addition of a potential can destroy criticality.

I. INTRODUCTION

Recently noninteger moments (NIM's) of probability distributions have found application in the study of scaling properties for fractal objects and random walks,[1-3] chaos,[4] percolation problems in random resistor networks,[5] and diffusion on hierarchic structures.[6,7] In a recent paper Weiss, Havlin, and Matan[8] have studied first-passage-time noninteger moments $\langle t^q \rangle$, $q<1$, for Wiener processes and they have shown that critical behavior can exist as a function of the parameter q. Specifically, if L is a characteristic length of the system then for large L the qth moment of the first passage time (FPT) is expressed as

$$\langle t^q \rangle \sim L^{\tau(q)} \tag{1}$$

and the derivative of $\tau(q)$ with respect to q is discontinuous, that is,

$$\tau(q) \sim \begin{cases} 0, & q<\frac{1}{2} \\ 2q-1, & q>\frac{1}{2} \end{cases} \tag{2}$$

with a logarithmic dependence on L appearing at $q=\frac{1}{2}$. This critical behavior indicates lack of uniform scaling in the underlying diffusion problem as otherwise expected from the fractal nature of the Wiener process and random walks.[1,2,3,9]

In this paper we will study the properties of first-passage-time NIM's on two absorbing boundaries of random processes driven by dichotomous noise. For the free process we will show that $\langle t^q \rangle$ present the same critical behavior as that of the Wiener process (provided that the initial condition is much less than L). We will also show that the addition of a linear potential destroys the critical behavior of the dichotomous noise driven process and the same happens to the Wiener process, i.e., there is no criticality for a diffusion process in a linear potential field.

II. FREE PROCESSES DRIVEN BY DICHOTOMOUS NOISE

Let $X(t)$ be a random process whose dynamical evolution is governed by the equation

$$\dot{X}=F(t) , \tag{3}$$

where $F(t)$ is the random telegraph signal alternately taking on the values ± 1 with an exponential switching time probability density $\psi(t)=e^{-t}$.[10] We assume that $x=0$ and $x=L$ are absorbing boundaries. The first passage time to those boundaries has a probability density function whose Laplace transform reads[11]

$$\hat{f}(s/x)=\frac{(1+s)\sinh(\beta x)+\beta\cosh(\beta x)+\sinh\beta(L-x)}{(1+s)\sinh(\beta L)+\beta\cosh(\beta L)} , \tag{4}$$

where

$$\beta=[s(s+2)]^{1/2} \tag{5}$$

and x is the initial position of the random process.

We now study the behavior of $\langle t^q \rangle$ $(0<q<1)$ as a function of q under the assumption that L is a characteristic length of the system which can grow indefinitely large and x is held fixed. In terms of the Laplace transform of the first-passage-time probability density function f the noninteger moments $\langle t^q \rangle$ are given by[8]

$$\langle t^q \rangle = \frac{-1}{\Gamma(1-q)} \int_0^\infty ds\, s^{-1} \frac{\partial}{\partial s} \hat{f}(s/x) . \tag{6}$$

In order to evaluate the integral on the right-hand side of Eq. (6) when L is large we distinguish three cases: $q<\frac{1}{2}$, $q>\frac{1}{2}$, and $q=\frac{1}{2}$.

(a) For $q<\frac{1}{2}$ the major contribution to the integral comes from values of s such that $\beta L \gg 1$ and $\beta L \gg \beta x$, hence we may use the approximation

$$\hat{f}(s/x) \simeq \frac{e^{-\beta x}}{1+s+\beta} . \tag{7}$$

Then

$$\langle t^q \rangle \simeq \frac{1}{\Gamma(1-q)} \int_0^\infty d\beta [-1+(1+\beta^2)^{1/2}]^{-q} \times \left[x+\frac{1}{\beta+(1+\beta^2)^{1/2}} \right] \times \frac{e^{-\beta x}}{\beta+(1+\beta^2)^{1/2}} , \tag{8}$$

which converges for $q<\frac{1}{2}$ (although it diverges for $q \geq \frac{1}{2}$). Since the right-hand side of Eq. (9) is independent of L we conclude

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$$\langle t^q \rangle \sim K \quad (q < \tfrac{1}{2}) . \tag{9}$$

(b) When $q > \frac{1}{2}$ the major asymptotic contribution to the integral comes from small values of s. Thus defining

$\sigma = L(s/2)^{1/2}$ and $\theta = x/L$ we have

$$\langle t^q \rangle = \frac{-2^{-q}}{\Gamma(1-q)} L^{2q} \int_0^\infty d\sigma\, \sigma^{2q} \frac{\partial}{\partial\sigma} \hat{f}(\sigma/\theta) , \tag{10}$$

where

$$\hat{f}(\sigma/\theta) = \{ (1+2\sigma^2/L^2)\sinh[\theta\sigma\chi(\sigma)] + (2\sigma/L)\chi(\sigma)\cosh[\theta\sigma\chi(\sigma)] + \sinh[(1-\theta)\sigma\chi(\sigma)]\} / \Delta(\sigma) \tag{11}$$

the function $\Delta(\sigma)$ being defined by

$$\Delta(\sigma) \equiv (1+2\sigma^2/L^2)\sinh[\sigma\chi(\sigma)] + (2\sigma/L)\chi(\sigma)\cosh[\sigma\chi(\sigma)] \tag{12}$$

and

$$\chi(\sigma) \equiv (1+\sigma^2/L^2)^{1/2} . \tag{13}$$

When $L \gg 1$ we have

$$\hat{f}(\sigma/\theta) \simeq \frac{\sinh(\theta\sigma) + \sinh[(1-\theta)\sigma]}{\sinh\sigma} \tag{14}$$

and

$$\langle t^q \rangle \simeq \frac{-2^{-q}}{\Gamma(1-q)} L^{2q} I(\theta) , \tag{15}$$

where

$$I(0) \equiv \int_0^\infty d\sigma\, \sigma^{-2q} \frac{\partial}{\partial\sigma} \frac{\sinh(\theta\sigma) + \sinh[(1-\theta)\sigma]}{\sinh\sigma} . \tag{16}$$

In the Appendix we show that, for $q > \frac{1}{2}$,

$$I(\theta) = -K'\theta + O(\theta^2) \tag{17}$$

whence

$$\langle t^q \rangle \sim L^{2q-1}, \quad q > \tfrac{1}{2} . \tag{18}$$

Finally the case $q = \frac{1}{2}$ is also worked out in detail in the Appendix. The final result reads

$$\langle t^{1/2} \rangle \sim \ln L . \tag{19}$$

Collecting results we write

$$\langle t^q \rangle \sim \begin{cases} K, & q < \frac{1}{2} \\ \ln L, & q = \frac{1}{2} \\ L^{2q-1}, & q > \frac{1}{2} . \end{cases} \tag{20}$$

Therefore in free processes driven by dichotomous noise, the noninteger moments exhibit the same critical behavior as do Wiener processes [cf. Eqs. (1) and (2)].

III. LINEARLY BOUND PROCESSES

In this section we will show that the addition of a force field drastically changes the properties of noninteger moments, both in the Wiener process and in the dichotomous case. Specifically, we will see that a linear potential destroys the critical behavior of $\langle t^q \rangle$. We first study the diffusion case and then the dichotomous case.

A. Diffusion in a linear potential

We consider one-dimensional diffusion in a constant force field [i.e., a linear potential $V(x) = vx$] with absorbing boundaries at both ends of the interval $(0,L)$. This model can be used to represent the diffusion of charged particles in a constant electric field.[12]

The first-passage-time probability density function for the process obeys the Kolmogorov equation

$$\frac{\partial f(t/x)}{\partial t} = -\sigma \frac{\partial f(t/x)}{\partial x} + \frac{1}{2} \frac{\partial^2 f(t/x)}{\partial x^2} \tag{21}$$

with the initial condition

$$f(0/x) = \delta(x - x_0) \tag{22}$$

and boundary conditions

$$f(t/0) = f(t/L) = 0 . \tag{23}$$

In Eq. (21) $\sigma = +1\ (-1)$ if $v > 0\ (v < 0)$.[10]

The Laplace transform of the solution to Eqs. (21)–(23) is given by[13]

$$\hat{f}(s/x) = e^{\sigma x} \frac{\sinh[\beta(L-x)] + e^{-\sigma L}\sinh(\beta x)}{\sinh(\beta L)} , \tag{24}$$

where

$$\beta = (1+2s)^{1/2} . \tag{25}$$

When $L \gg 1$ and $L \gg x$ we find after some algebra

$$\hat{f}(s/x) \simeq e^{\sigma x}[e^{-\beta x} + 2e^{-(\sigma+\beta)L}\sinh(\beta x)] . \tag{26}$$

The substitution of Eq. (26) into Eq. (6) yields

$$\langle t^q \rangle \simeq -\frac{2^q e^{\sigma x}}{\Gamma(1-q)} \int_1^\infty d\beta (\beta^2-1)^{-q} \frac{\partial}{\partial\beta} \times [e^{-\beta x} + e^{-(\sigma+\beta)L}\sinh(\beta x)] . \tag{27}$$

When $\sigma = 1$ (attractive potential towards the origin) the second term in the right-hand side of Eq. (27) is negligible, therefore

$$\langle t^q \rangle \simeq \frac{-2^q e^x}{\Gamma(1-q)} \int_1^\infty d\beta (\beta^2-1)^{-q} \frac{\partial}{\partial\beta} e^{-\beta x} = (2\pi)^{-1/2} e^x x^{q+1/2} K_{1/2-q}(x) \quad (0 < q < 1), \tag{28}$$

where $K_\mu(x)$ is a modified Bessel function.[14] We thus see that in this case $\langle t^q \rangle$ is finite and shows no critical behavior.

For the case of a repulsive potential ($\sigma = -1$), we write

$$\langle t^q \rangle = \langle t^q \rangle_{(+)} + \langle t^q \rangle_{(-)} , \tag{29}$$

where $\langle t^q \rangle_{(+)}$ is given by [cf. Eq. (28)]

$$\langle t^q \rangle_{(+)} \simeq (2\pi)^{-1/2} e^{-x} x^{q+1/2} K_{1/2-q}(x) \tag{30}$$

and [cf. Eq. (27) and Ref. (14)]

$$\langle t^q\rangle_{(-)}\simeq\frac{2^q e^{-x}}{\Gamma(1-q)}L\int_1^\infty d\beta(\beta^2-1)^{-q}e^{(1-\beta)L}\sinh(\beta x)$$
$$=(2/\pi)^{1/2}Le^{L-x}[(L-x)^{q-1/2}K_{1/2-q}(L-x)-(L+x)^{q-1/2}K_{1/2-q}(L+x)]\ . \tag{31}$$

When L is large we can use the asymptotic estimate[14]

$$K_{1/2-q}(L\pm x)\simeq(\pi/2)^{1/2}(L\pm x)^{-1/2}e^{-(L\pm x)}\ . \tag{32}$$

Finally

$$\langle t^q\rangle\simeq\langle t^q\rangle_{(-)}\simeq L^q(1-e^{-2x})\ . \tag{33}$$

In this case $\langle t^q\rangle$ grows as L^q but exhibits no critical behavior as a function of q.

B. Dichotomous noise: Linear potential

Let $X(t)$ be a random process driven by dichotomous noise in a constant force field; its evolution equation is

$$\dot X(t)=v+F(t)\ , \tag{34}$$

where $-1<v<1$ and $F(t)=\pm1$ is dichotomous noise with an exponential switching time distribution $\psi(t)=e^{-t}$. The Laplace transform of the first-passage-time probability density function is[15]

$$\hat f(s/x)=e^{-\gamma x}\frac{e^{\gamma L}[(\alpha+\gamma)\sinh(\beta x)+\beta\cosh(\beta x)]+(1+v)^{-1}\sinh[\beta(L-x)]}{(\alpha+\gamma)\sinh(\beta L)+\beta\cosh(\beta L)} \tag{35}$$

where

$$\alpha=\frac{1+s}{1+v},\quad \beta=\frac{[v^2+s(s+2)]^{1/2}}{1-v^2},\quad \gamma=\frac{v(1+s)}{1-v^2}\ . \tag{36}$$

We note that if $-1<v<1$ then $\beta>\gamma$. Now for large L and x held fixed we have from Eq. (35) the following approximation:

$$\hat f(s/x)\simeq\frac{e^{-(\beta+\gamma)x}}{(1+v)(\alpha+\beta+\gamma)}+e^{(\gamma-\beta)L}\left[e^{-(\gamma-\beta)x}+e^{-(\gamma+\beta)x}\frac{\beta-\alpha-\gamma}{\beta+\alpha+\gamma}\right]\ . \tag{37}$$

The substitution of Eq. (37) into Eq. (6) yields

$$\langle t^q\rangle\simeq\frac{-1}{\Gamma(1-q)}[I_0(q,x)+I_1(q,x,L)]\ , \tag{38}$$

where

$$I_0(q,x)=(1+v)^{-1}\int_0^\infty ds\,s^{-q}\frac{\partial}{\partial s}\left[\frac{e^{-(\beta+\gamma)x}}{\alpha+\beta+\gamma}\right]\ , \tag{39}$$

and

$$I_1(q,x,L)=\int_0^\infty ds\,s^{-q}\frac{\partial}{\partial s}\left[e^{(\gamma-\beta)L}\left[e^{-(\gamma-\beta)x}+e^{-(\gamma+\beta)x}\frac{\beta-\alpha-\gamma}{\beta+\alpha+\gamma}\right]\right]\ . \tag{40}$$

It is easily seen from Eq. (39) that $I_0(q,x)$ converges when $0<q<1$ and is independent of L. Let us now evaluate $I_1(q,x,L)$ for $L\gg1$ and $L\gg x$. We make the change of variable

$$s=v\tau+(1+\tau^2)^{1/2}-1\equiv s(\tau)\ . \tag{41}$$

When $1>v>0$ the lower limit of the integral in Eq. (40), $s=0$, corresponds to $\tau=0$ and we find after some algebra

$$I_1(q,x,L)=-\int_0^\infty d\tau[s(\tau)]^{-q}e^{-\tau L}\varphi(\tau,L,x)\ , \tag{42}$$

where

$$\varphi(\tau,L,x)=(L-x)e^{\tau x}+\frac{1-v}{1+v}\left\{\left[L-\left[\frac{1+v^2}{1-v^2}+\frac{2v\tau}{(1-v^2)(1+\tau^2)^{1/2}}\right]x\right]\frac{\tau-(1+v^2)^{1/2}}{\tau+(1+\tau^2)^{1/2}}-\frac{2}{[\tau+(1+\tau^2)^{1/2}](1+\tau^2)^{1/2}}\right\}$$
$$\times\exp\frac{-x}{(1+\tau^2)^{1/2}}[(1+v^2)\tau+v(1+\tau^2)^{1/2}]\ . \tag{43}$$

Since $L \gg 1$ then the Laplace method allows an asymptotic evaluation of Eq. (42); the final result reads

$$I_1(q,x,L) \simeq -\left[\frac{L}{v}\right]^q \left[1-\left[\frac{1-v}{1+v}\right] e^{(2vx)/(1-v^2)}\right] \Gamma(1-q) \ . \tag{44}$$

Introducing Eqs. (44) and (39) into Eq. (38) we finally obtain the relation

$$\langle t^q \rangle \simeq \left[\frac{L}{v}\right]^q \left[1-\left[\frac{1-v}{1+v}\right] e^{-(2vx/(1-v^2))}\right] , \tag{45}$$

where $0<q<1$ and $0<v<1$.

When $-1<v<0$ the bound $s=0$ corresponds to

$$\bar{\tau} \equiv \frac{2|v|}{1-v^2} \tag{46}$$

[cf. Eq. (41)] and instead of Eq. (42) we now have

$$I_1(q,x,L) = -\int_{\bar{\tau}}^{\infty} d\tau [s(\tau)]^{-q} e^{-\tau L} \varphi(\tau,L,x) \ , \tag{47}$$

from which it follows that

$$|I_1(q,x,L)| \le e^{-\bar{\tau}L} \int_{\bar{\tau}}^{\infty} d\tau [s(\tau)]^{-q} |\varphi(\tau,L,x)| \ . \tag{48}$$

Equation (48) shows that, when $L \gg 1$, $I_1(q,x,L)$ is transcendentally small, hence [cf. Eqs. (38) and (39)]

$$\langle t^q \rangle \simeq -\frac{(1+v)^{-1}}{\Gamma(1-q)} \int_{\bar{\tau}}^{\infty} d\tau [s(\tau)]^{-q} \frac{\partial}{\partial \tau} \frac{e^{-(\beta+\gamma)x}}{\alpha+\beta+\gamma} \ . \tag{49}$$

For $0<q<1$ this integral converges and is independent of L. Therefore $\langle t^q \rangle$ is finite and has no phase transition as a function of q.

IV. CONCLUSIONS

We have considered the problem of first-passage-time noninteger moments for some diffusion and dichotomous processes. The critical behavior of $\langle t^q \rangle$ found for the Wiener process also appears in the free process driven by dichotomous noise in exactly the same way and with the same critical exponents. This critical behavior means an absence of uniform scaling in the underlying physical system and might indicate a fractal nature of the dichotomous noise which is not obvious at first glance since the trajectories of free processes driven by the random telegraph signal are much smoother than the Wiener process trajectories. Finally the addition of a linear potential breaks down any kind of critical behavior. This fact is not obvious at first glance either because diffusion in a constant field of force can be reduced to the Wiener process by a simple change of variable and for the dichotomous case the trajectories have similar analytical properties to those of the free process.[15]

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APPENDIX: DERIVATION OF EQS. (17) AND (19)

From the definition of $I(\theta)$ given by Eq. (16) we have

$$I(\theta) = I'(0)\theta + O(\theta^2) \ , \tag{50}$$

where

$$I'(0) = \int_0^{\infty} d\sigma\, \sigma^{-2q} \frac{\partial}{\partial \sigma} \frac{\sigma(1-\cosh\sigma)}{\sinh\sigma} \ . \tag{51}$$

Integrating by parts and taking into account that $1 > q > \frac{1}{2}$ we get

$$I'(0) = -2q \int_0^{\infty} d\sigma\, \sigma^{-2q} \frac{\cosh\sigma - 1}{\sinh\sigma} \equiv -K' \ , \tag{52}$$

where $0<K'<\infty$. Substitution of Eq. (52) into Eq. (50) yields Eq. (17).

For $q=\frac{1}{2}$ we write Eq. (15) in the form

$$\langle t^{1/2} \rangle \simeq (2/\pi) L I(\theta) \ , \tag{53}$$

where

$$I(\theta) = \tfrac{1}{2} \int_0^{\infty} d\sigma\, \sigma^{-1} \frac{\partial}{\partial \sigma} \frac{\cosh[\sigma(1-2\theta)]}{\cosh\sigma} \tag{54}$$

and

$$I(0) = 0 \ . \tag{55}$$

From Eq. (54) we have

$$\begin{aligned} I'(\theta) &= -\int_0^{\infty} d\sigma \frac{\partial}{\partial\sigma} \frac{\sinh[\sigma(1-2\theta)]}{\cosh\sigma} - \int_0^{\infty} d\sigma\, \sigma^{-1} \frac{\sinh[\sigma(1-2\theta)]}{\cosh\sigma} \\ &= -\int_0^{\infty} d\sigma\, \sigma^{-1} \frac{\sinh[\sigma(1-2\theta)]}{\cosh\sigma} \end{aligned} \tag{56}$$

and[14]

$$I'(\theta) = 2 \ln \tan\left[\frac{\pi\theta}{2}\right] \simeq 2 \ln \left[\frac{\pi\theta}{2}\right] \tag{57}$$

hence [cf. Eq. (55)]

$$I(\theta) \simeq 2\theta \left[\ln\left[\frac{\pi\theta}{2}\right] - 1\right] \ . \tag{58}$$

Introducing Eq. (58) into Eq. (53) we find Eq. (19).

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