Self-avoiding walks and manifolds in random environments

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Self-avoiding walks (SAW's) and manifolds (SAM's) in random environments are studied using a combination of Lifshitz arguments and field-theoretic methods. The number of N-step SAW's starting at the origin, Z, is shown to be a broadly distributed quantity whose typical value, $Z_{\rm typ}$, behaves as $Z_{\rm typ}\sim\langle Z\rangle\exp(-cN^{\alpha})$ below four dimensions. Here $\alpha=2-d\nu$ and $\langle Z\rangle$ is the average number of SAW's at the origin. On the other hand, the integer moments of Z are exponentially larger than the average, i.e., $\langle Z^k\rangle\sim\langle Z\rangle^k\exp[ck^{1/\alpha}(k-1)N]$ for the range $1<k<k_c$. Similar results hold for SAM's. Within the field theory for SAW's the results for $1<k<k_c$ arise from a fluctuation-driven first-order phase transition in the k-replicated theory. Above k_c, Griffiths singularities control the moments of Z.

I. INTRODUCTION

In the present paper we discuss the statistical mechanics of self-avoiding random walks (SAW's) and manifolds (SAM's) in random environments. The subject of SAW's in disordered environments has been intensely studied[1-8] during the past decade but remains controversial. Much of the literature concerns the size of a SAW in a random environment with a variety of qualitatively different answers given for the value of the correlation length exponent both at and above the percolation threshold. The primary emphasis in this paper is on the number of SAW's or SAM's starting from the origin, i.e., the partition function Z. Our object is to determine the statistics of the partition function.

The partition function for SAW's on a pure lattice Z_0 has the leading asymptotic behavior

$$Z_0\simeq\mu^N N^{\gamma-1}\ . \tag{1.1}$$

N is the number of steps in the SAW, μ is the (nonuniversal) connective constant, and γ is a (universal) critical exponent. For a site-diluted lattice with a fraction p of allowed sites, Harris[2] pointed out that

$$\langle Z\rangle=p^N Z_0\ . \tag{1.2}$$

This result follows from the observation that a given chain conformation visits N distinct sites and is therefore allowed with probability p^N. Comparing (1.2) to (1.1) we see that the only effect of the disorder is to modify the nonuniversal connective constant $\mu\to\mu p$. Similar arguments[2] show that the size of the chain averaged over disorder and over starting positions is unmodified by site dilution. These results contributed to the belief entertained by some authors,[2,3] that disorder is irrelevant for SAW's. On the other hand, Derrida[4] conjectured that disorder is relevant and should appear in the behavior of the typical value $Z_{\rm typ}$ as given by $\langle\log(1+Z)\rangle$. Results for the typical value of Z were presented in a recent paper by one of us.[7] These results support Derrida's conjecture and show that the power-law singularity of the pure partition function, Eq. (1.1), is replaced by an essential singularity. Specifically we found that $Z_{\rm typ}\simeq\langle Z\rangle\exp\{-cN^{\alpha}\}$ where α is the "specific-heat" exponent $\alpha=2-d\nu$. Similar essential singularities have been found for directed self-avoiding walks in random environments.[9]

The considerable effort investigated in studying SAW's in random environments is motivated by the following considerations. Firstly, this system models the behavior of polymers in porous media. For example, the mean partition function determines the way in which polymers are divided between a solvent bath and a sample of porous material. The statistics of the partition function determines the distribution of the polymers within the porous material and their large scale diffusive motion.[7]

Self-avoiding walks in random environments are closely connected to the larger subject of phase transitions in random systems. In the most general setting this is manifested by the Szymansik[10] representation of field theories by interacting random walks. Thus we may hope to deepen our understanding of phenomena such as Griffiths singularities and Lifshitz tails by studying the behavior of interacting walks in random environments.

Self-avoiding tethered manifolds[11] generalize the continuum or Edwards version of self-avoiding walks and serve as a simple model for polymeric membranes. The statistical mechanics of SAM's is considerably more complicated than for the special case of SAW's and the behavior of a tethered surface in three dimensions is not well understood. Recent computer simulations[12] suggest that tethered membranes remain flat for all temperatures in contrast to the theoretical predictions of a crumpling transition.[13] One can imagine a manifold of arbitrary dimension D embedded in a space of dimension d. Above the upper critical dimension $d^*=4D/(2-D)$ self-avoidance is irrelevant and the SAM is crumpled, having a Hausdorf dimension $d_H=1/\nu=2D/(2-D)$. Near and below the upper critical dimension it is believed that a crumpled phase exists for sufficiently high temperature and one can compute the exponent ν within an $\epsilon=d^*-d$ expansion. We can now put a SAM in a random environment and study how the environment changes the size

and partition function of the manifold. Muthukumar[6] and Thirumalai[5] have addressed the question of the size of a SAM in a random environment. As in the case of SAW's one finds[6] that the crumpled phase survives and the exponent ν is unchanged by sufficiently weak disorder. In the present paper we study the statistics of the partition function for SAM's in a random environment. The results for SAM's are formally very similar to that for SAW's except that the "specific-heat" exponent $\alpha=2-d\nu$ which determines the relevance of disorder and controls the nonanalyticities in the moments of the partition function is replaced by the more general form $\alpha_D=2D-d\nu$.

In this paper we use two different methods to study SAW's in random environments. The first method is an extension of the "Lifshitz" arguments given in Ref. 7. This approach is based on the observation, Eq. (1.2), that the partition function depends exponentially on the fraction of allowed sites or "porosity" p. The basic assumption is that rare configurations of the environment with unusually high porosity near the origin dominate the positive moments of Z. The calculation consists of identifying the dominant configurations and estimating the partition function of these environments. The approach given here is somewhat more complicated than the traditional Lifshitz method. In the Lifshitz argument for electrons[14] or ordinary random walks[15] in random environments the important fluctuations of the environment are spherical regions. Here the rare events dominating the high moments of the partition function are fluctuations which have nonspherical shapes. Section II A of the paper develops this approach for calculating the positive moments of the partition function for self-avoiding walks on site diluted lattices. Section II B extend the Lifshitz method to self-avoiding manifolds in Gaussian random environments. Section II C presents results for the distribution function for the free energy.

The second approach, presented in Section III, is to map the SAW problem onto an $O(n)$ symmetric field theory with quenched disorder. The disorder is eliminated in favor of interacting replica fields where k replicas are introduced to compute $\langle Z^k\rangle$. The salient feature of this field theory in the appropriate $n\to 0$ limit is that the quartic coupling representing the disorder follows a runaway trajectory under renormalization for $d<4$. For a range of moments $1<k<k_c$ this runaway trajectory leads to a fluctuation driven first order phase transition.[16,17] The first-order transition occurs at a temperature higher than the underlying second order transition, which means, in the SAW language that the connective constant for the kth moment is greater than μp for $k>1$. For $k>k_c$ the bare field theory exhibits a first-order transition. Quantitative predictions for the connective constant are in agreement with the results of Sec. II.

II. "LIFSHITZ" ARGUMENTS FOR Z

A. Self-avoiding walks on site-diluted lattices

1. Preliminaries

We suppose that for large N the moments of the partition function are dominated by rare environments which have many more conformations available than typical environments. These environments support an atypically large number of SAW's because they have a higher than average porosity in a region around the origin. This high porosity region may be the full correlation volume or may be smaller than the correlation volume. Our task is to determine the optimal size, shape and porosity of the regions which dominate the kth moment of the partition function. The larger the local porosity the larger the value of Z in that environment. On the other hand, for a region of a given size the probability of finding such a region diminishes as the local porosity deviates from the average. If the high porosity region is smaller than the correlation volume, the number of chain conformations is suppressed but the possibility of such an environment is enhanced. The balance between these competing effects determines the optimal regions.

Consider a region of the lattice Ω having a volume $|\Omega|$. The local porosity of the region m is defined as the ratio $m=M/|\Omega|$ where M is the number of allowed sites in the region. The probability $\rho_\Omega(m)$ for finding a region Ω with local porosity m is a binomial which, for large $|\Omega|$, becomes a Gaussian,

$$\rho_\Omega(m)=\left[\frac{|\Omega|}{2\pi p(1-p)}\right]^{1/2}\exp\left[-\frac{|\Omega|(m-p)^2}{2p(1-p)}\right]. \tag{2.1}$$

In the limiting case of an undiluted region $m=1$ we have the exact result $\rho_\Omega(m)=p^{|\Omega|}$.

Next suppose a SAW of length N is put in a region Ω having M allowed sites. Let $Z(\Omega)$ be the number of SAW's starting at the origin confined to the region Ω for a given realization of the disorder and let $Z_0(\Omega)$ be the corresponding partition function for a SAW confined to Ω without forbidden sites. Let $\langle\ \rangle_m$ be a constrained average over the configurations of the region for which there are exactly $M=m|\Omega|$ allowed sites. The result for the ratio $\Psi_\Omega(m)\equiv\langle Z(\Omega)\rangle_m/Z_0(\Omega)$ is

$$\Psi_\Omega(m)=\begin{cases}\dfrac{M!(\Omega-N)!}{\Omega!(M-N)!}, & M\geq N\\ 0, & M<N\ .\end{cases} \tag{2.2}$$

To obtain this result, imagine a fixed conformation of the chain in the region and count the number of ways of distributing the M allowed sites in Ω so as to cover the chain. Equation (2.2) is obtained by dividing this number by the total number of ways of distributing M allowed sites in Ω. Since each conformation occupies N sites, this counting is independent of the conformation. In the limit $\Omega\sim M\gg N\gg 1$, the combinatorial result, Eq. (2.2), may be expanded as

$$\ln\Psi_\Omega(m)\simeq N\ln m-\frac{N^2}{2|\Omega|}\frac{1-m}{m}+\cdots\ . \tag{2.3}$$

The first term in this expression is the result, already stated in Eq. (1.2), that the number of conformations available to the chain is suppressed by a factor m^N. The second term represents the difference between fixing the

number of allowed sites as is done here versus fixing the average number of allowed sites as is done in deriving Eq. (1.2). This difference vanishes for $|\Omega| \to \infty$ but is crucial in the arguments which follow. Indeed, we can anticipate one of the central results of the paper by examining Eq. (2.3) in the case that Ω is a correlation volume $|\Omega| \sim N^{d\nu}$ and that the local porosity is just the average porosity $m = p$. This gives an estimate of the typical or most probable value of the partition function Z_{typ} for which we obtain the result

$$Z_{\text{typ}} \simeq p^N Z_0 \exp[-cN^{2-d\nu}(1-p)/2p] . \tag{2.4}$$

The typical value is exponentially smaller than the average, i.e., $Z_{\text{typ}}/\langle Z \rangle \simeq \exp[-cN^{2-d\nu}(1-p)/2p]$. Throughout the paper we use c to denote an arbitrary positive constant that may depend on the dimension. The numerical value of c varies from equation to equation.

In what follows we shall use $\langle Z(\Omega) \rangle_m$ to estimate the partition function for chains starting at the origin in a given environment having region Ω with local porosity m. However, in obtaining $\Psi_\Omega(m)$ and hence $\langle Z(\Omega) \rangle_m$ we have averaged over all those configurations of the disorder with a fixed number of allowed sites. The preaveraging over all configurations with local porosity m washes out some of the fluctuations in Z and is discussed below in Sec. II A 3.

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The next quantity which is needed for the calculation is the number of conformations $Z_0(\Omega)$ for a polymer confined to an undiluted region Ω. The simplest situation is that Ω is at least the correlation length R in all dimensions. In this case, the partition function of the polymer is, to good approximation, the same as in an infinite lattice $Z_0(\Omega) \simeq Z_0$. If Ω is smaller than the correlation length in one or more dimensions, the number of configurations is reduced. Using scaling arguments, Daoud and de Gennes[18,19] obtained the leading behavior of $Z_0(\Omega)$ for a polymer confined in regions of various shapes. For a long tube of diameter L the result is

$$Z_0(\Omega) \simeq Z_0 \exp\{-cNL^{-1/\nu}\} \tag{2.5}$$

where ν is the correlation length exponent. The length of tube occupied by the chain $R_\parallel$ is given by

$$R_\parallel \sim NL^{1-1/\nu} . \tag{2.6}$$

This result can be understood within the "blob" picture[19] in which the polymer is viewed as a string of blobs, each of diameter L. Within each blob the polymer behaves as an unconfined chain so that $L \sim N_{\text{blob}}^{\nu}$ where N_{blob} is the number of monomers in a blob. Equation (2.6) follows from the relation, $R_\parallel \sim (N/N_{\text{blob}})L$. Note that Eqs. (2.5) and (2.6) correctly yield the size and partition function of an unconfined chain in the limit that the tube diameter goes to the correlation length $L \to R \sim N^{\nu}$ in which case $R_\parallel \to R$ and $Z_0(\Omega) \to Z_0$.

More generally, consider a polymer confined in a region Ω that allows the polymer to expand freely in Δ dimensions ($1 \le \Delta < d$) but confines the polymer on a length scale $L \ll R$ in the remaining $d - \Delta$ dimensions. Using the scaling ideas of Ref. 18 we find that the entropic suppression due to confinement is independent of Δ so Eq. (2.5) still holds while the unconfined length scale is given by

$$R_\parallel \sim N^{\nu(\Delta)} L^{1-\nu(\Delta)/\nu(d)} \tag{2.7}$$

where $\nu(\Delta)$ is the correlation length exponent for Δ dimensions.

After identifying a class of optimal shapes for the SAW to reside in we need to estimate the number of distinct shapes in this class. Let g_Ω be this counting factor. If Ω is the correlation volume g_Ω is order unity. If, for example, Ω is a self-avoiding tube of diameter L and length $R_\parallel$ then there will be many equivalent ways of choosing the conformation of the tube. The number of distinct shapes in a given class times the number of SAW's contained in each shape must yield the full partition function thus,

$$g_\Omega \simeq Z_0/Z_0(\Omega) \simeq \exp\{cNL^{-1/\nu}\} . \tag{2.8}$$

We shall use the product $g_\Omega \rho_\Omega(m)$ as an estimate of the probability that a given environment has a *single* region with porosity m in the Ω class of shapes. This estimate is appropriate if $g_\Omega \rho_\Omega(m) \ll 1$ but when $g_\Omega \rho_\Omega(m)$ is order 1 or greater it is likely that each environment has many regions in the class. This violates the original hypothesis that the polymer is confined to a single region.

2. *The moments of Z*

Let us now assemble all of these ingredients to obtain an estimate for the kth moment of the partition function. Our ansatz for $\langle Z^k \rangle$ is

$$\langle Z^k \rangle \simeq \sup_\Omega \left[g_\Omega \int_p^1 dm\, \rho_\Omega(m) [\Psi_\Omega(m) Z_0(\Omega)]^k \right] . \tag{2.9}$$

The idea is that $\langle Z^k \rangle$ is dominated by high porosity environments with a given class of optimal shapes Ω. The integration and variation with respect to the shape is subject to the constraints discussed above-that $g_\Omega \rho_\Omega(m) \ll 1$ and $|\Omega| \le N^{d\nu}$. The integral over porosities m involves only $\rho_\Omega(m)$ and $\Psi_\Omega(m)$. For a given volume $|\Omega|$ this integral can be carried out by steepest descents. The formulas simplify in the weak disorder limit where p approaches unity and we shall quote the results only for this case. The optimal porosity is given by

$$\delta m = \frac{k \delta p N}{|\Omega|} \tag{2.10}$$

where δm is the deviation from the average porosity, $\delta m \equiv m - p$, and δp is the deviation from a pure system $\delta p \equiv 1 - p$. Note that the upper limit of the integral in Eq. (2.9) is at $m = 1$ so that for $kN \ge |\Omega|$ Eq. (2.10) is replaced by $\delta m = \delta p$.

After approximating the integral by the value of the integrand at the optimal porosity δm we must carry out the remaining optimization with respect to the size and shape of the region. This variation takes the form,

$$\langle Z^k\rangle/Z_0^k \simeq \sup_{\Omega}\{\exp[-k\delta pN - c(k-1)L^{-1/\nu}N + k(k-1)\delta pN^2/2|\Omega|]\} .$$

for

$$kN|\Omega| < 1 . \tag{2.11a}$$

For $kN \geq |\Omega|$, m is set to unity and there is no entropic suppression due to forbidden sites ($\Psi \to 1$) so that we have the simpler expression

$$\langle Z^k\rangle/Z_0^k \simeq \sup_{\Omega}\{\exp[-c(k-1)L^{-1/\nu}N - \delta p|\Omega|/2]\}$$

for

$$kN/|\Omega| \geq 1 . \tag{2.11b}$$

We now have a variation with respect to two parameters, the number of unconfined dimensions Δ and the length scale L of confinement in the remaining $d-\Delta$ dimensions. For $k>1$ it is clear from (2.11a) that for a given L one should make the region as small as possible, which is achieved by confining the polymer in a tube ($\Delta=1$) in which case Ω is given by

$$|\Omega| \sim R_{\parallel} L^{d-1} \sim N L^{d-1/\nu} \tag{2.12}$$

and the variation over Ω in Eq. (2.11a) reduces to a variation with respect to the tube diameter

$$\langle Z^k\rangle/Z_0^k \simeq \sup_{L}\{\exp[-kN\delta p + (k-1)N \times(-cL^{-1/\nu} + kL^{1/\nu - d}\delta p/2)]\} . \tag{2.13}$$

The maximum is attained for

$$L_1 \sim (k\delta p)^{-\nu/\alpha} \tag{2.14}$$

with $\alpha = 2 - d\nu$. The corresponding result for the moments $1 \leq k \leq k_c$ is

$$\langle Z^k\rangle \simeq Z_0^k \exp\{-k\delta pN + c(k-1)(k\delta p)^{1/\alpha}N\} \tag{2.15}$$

with

$$k_c \sim \delta p^{\alpha-1} \tag{2.16}$$

defined as the value of k for which the optimal porosity (2.10) reaches unity. For $k=1$ the second term in the exponential vanishes and we recover the exact result, Eq. (1.2), which has the weak disorder form

$$\langle Z\rangle = Z_0 \exp(-\delta pN) . \tag{2.17}$$

For the high order moments $k > k_c$ the optimal regions are undiluted ($m=1$) and the appropriate variational expression is (2.11b). Written in terms of the tube diameter we have

$$\langle Z^k\rangle/Z_0^k \simeq \sup_{L}\{\exp[-c(k-1)NL^{-1/\nu} - NL^{d-1/\nu}\delta p/2]\} . \tag{2.18}$$

The optimal tube diameter L_2 is given by

$$L_2 \sim [(k-1)/\delta p]^{1/d} \tag{2.19}$$

and the moments for $k > k_c$ are given by

$$\langle Z^k\rangle \simeq Z_0^k \exp\{-c\delta p^{1/d\nu}(k-1)^{1-1/d\nu}N\} . \tag{2.20}$$

Finally, consider the low moments $0 < k \leq 1$. The maximum found for the $k>1$ case now becomes a minimum and the quantity in the square brackets in Eq. (2.13) increases as L goes either to zero or infinity. The constraint $g_{\Omega}\rho_{\Omega}(m) \ll 1$ sets a lower bound on L while the upper bound is $L \leq N^{\nu}$. It is straightforward to verify, for any number of free dimensions Δ that the maximum occurs at the upper bound for L. Thus we ignore the term involving L in (2.11a) and set $|\Omega| \sim N^{d\nu}$. The result for $0 < k \leq 1$ is

$$\langle Z^k\rangle \sim Z_0^k \exp[-k\delta pN - ck(1-k)\delta pN^{\alpha}] . \tag{2.21}$$

As k approaches unity from below we again recover the exact result, Eq. (2.17). For $k<1$ the correction to the leading exponential growth of the partition function is an essential singularity in N instead the pure system behavior characterized by a power law. If one defines the typical value $Z_{\rm typ}$ to be $Z_{\rm typ} = \langle Z^k\rangle^{1/k}$ for small k then we recover Eq. (2.4).

A useful way of characterizing the asymptotic large-N behavior of the moments is via the connective constants μ_k defined by

$$\mu_k \equiv \lim_{N\to\infty} \langle Z^k\rangle^{1/kN} . \tag{2.22}$$

The behavior of the kth-order connective constant is found from the results for the moments, Eqs. (2.15), (2.20), and (2.21),

$$\mu_k/\mu = \begin{cases} 1 - c_1[(k-1)/k][\delta p/(k-1)]^{1/d\nu}, & k > k_c \quad (2.23a) \\ 1 - \delta p + c_2[(k-1)/k](k\delta p)^{1/\alpha}, & 1 \leq k \leq k_c \quad (2.23b) \\ 1 - \delta p, & 0 < k \leq 1 . \quad (2.23c) \end{cases}$$

The connective constants are plotted as a function of k in Fig. 1. The high order connective constants approach the pure system value μ reflecting the fact that the high moments probe those rare configurations in which the chain sees a pure region and thus have the same number of conformations as on a pure lattice. On the other hand, for low moments ($0 < k \leq 1$) μ_k is independent of k which reflects the fact that both average and typical chains encounter forbidden sites with probability $\delta p = 1-p$ per step. Note that the connective constants cannot be extra-

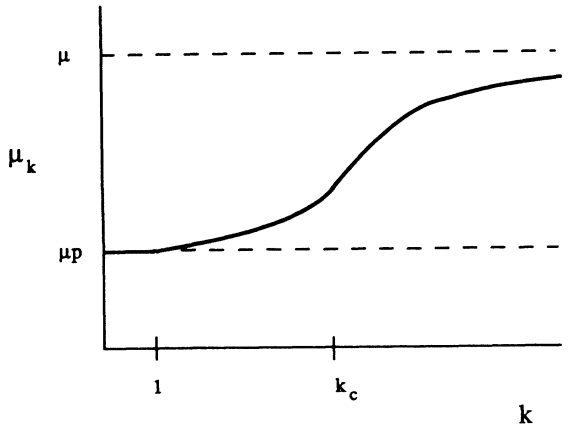

FIG. 1. The connective constants μ_k for the kth moment of the partition function plotted as a function of the order of the moment k. See Eqs. (2.23).

polated from $k>1$ to k near zero since the convexity of $\ln\langle Z^k\rangle$ would be violated.

3. Strong disorder

Since $\langle Z(\Omega)\rangle^k \geq \langle Z(\Omega)^k\rangle$ for $k<1$ and since $Z(\Omega)\sim Z$ when Ω is the correlation volume we see that the results for the low moments, Eq. (2.21), are an upper bound to the correct answer. It is our hypothesis that for weak disorder, porosity fluctuations dominate the fluctuations in Z^k for $k<1$ so that Eq. (2.21) is exact. On the other hand, for strong disorder (δp order unity), Eq. (2.21) is not correct. Since Z is a positive integer valued random variable its probability density $P(Z)$ takes the form

$$P(Z)=P_0\delta(Z)+P_1\delta(Z-1)+\cdots . \tag{2.24}$$

For $p \ll 1$, most configurations have no SAW's at the origin so P_0 is nearly unity. With probability p^N there are configurations with N allowed sites connected to the origin which support a single SAW. These connected sites are themselves a SAW so there are μ^N ways of having configurations with $Z=1$. Configurations with more than one SAW are exponentially rarer so the first two probabilities in Eq. (2.24) are approximately

$$P_0\sim 1-\mu_1^N \tag{2.25a}$$

and

$$P_1\sim\mu_1^N \tag{2.25b}$$

with $\mu_1=\mu p$ the connective constant for $\langle Z\rangle$. Ignoring the rare cases where $Z>1$ in estimating the low moments of Z leads to

$$\langle Z^k\rangle\sim\mu_1^N \tag{2.26}$$

for any k so that the connective constants are given by

$$\mu_k=(\mu p)^{1/k} . \tag{2.27}$$

This result is qualitatively different from the weak disorder results given above, Eq. (2.23c), since here the connective constant decreases toward zero as k approaches zero rather than being independent of k. The transition from weak to strong disorder is not yet understood. The simple argument leading to Eq. (2.27) clearly fails unless $\mu_1<1$ so that perhaps $p=1/\mu$ signals the transition between the strong and weak disorder regimes. The percolation threshold is another intuitively plausible candidate for the transition point. It might also be that the transition occurs at a k-dependent value of p as is seen for the case of directed polymers on disordered hierarchical lattices.[20]

B. Self-avoiding manifolds in a Gaussian random potential

In this section we consider self-avoiding manifolds in an environment with a quenched Gaussian random potential. In addition to generalizing the previous results to manifolds of arbitrary dimension the present calculation illustrates how the Lifshitz method is applied to continuum rather than lattice systems and to a random potential rather than site dilution.

Self-avoiding tethered manifolds are defined by generalizing the Edwards continuum description of self-avoiding walks.[13] In the absence of disorder the partition function is

$$\begin{aligned} Z_0\{w\} &= \int d[\mathbf{r}]\exp\left[-\frac{1}{2}\int_A d^D x|\nabla\mathbf{r}(x)|^2-\frac{w}{2}\int_A d^D x\int_A d^D x'\delta^d(\mathbf{r}(x)-\mathbf{r}(x'))\right] \\ &= \int d[\mathbf{r}]\exp(-\mathcal{H}[\mathbf{r}]) . \end{aligned} \tag{2.28}$$

Here $x=(x_1,x_2,\ldots,x_D)$ is the internal coordinate in the D-dimensional manifold; $\mathbf{r}(x)$ is the position of that point on the manifold in the d-dimensional embedding space; ∇ is the D-dimensional gradient and w is the strength of the self-avoidance. The case $D=1$ is the Edwards continuum version of a SAW. The path integral ranges over all manifolds centered at the origin with a given internal shape and linear size S. The "mass" of the manifold is $|A|=S^D$. To obtain finite results the partition function must be normalized by the ideal manifold partition function $Z_0\{w=0\}$ and regularized, either by the introduction of a short distance cutoff for the interaction s_0 or by dimensional regularization.

For positive w and sufficiently high dimensionality the

manifold is expected to exist in a crumpled phase in which the radius of gyration of the SAM obeys a scaling law

$$R \sim S^{\nu} . \tag{2.29}$$

The Flory value for ν is $(D+2)/(d+2)$ and the ideal manifold (i.e., non-self-avoiding) value is $\nu=(2-D)/2D$.

The partition function for a crumpled manifold in a pure environment behaves exponentially in the mass of the manifold[21]

$$Z_0\{w\}/Z_0\{0\} \sim \exp\left[c_0(w,s_0)\left(\frac{S}{s_0}\right)^{D} + c_1(w,s_0)\left(\frac{S}{s_0}\right)^{D-1} + \cdots \right] . \tag{2.30}$$

As discussed in Ref. 21, the subleading terms depend on the boundary conditions given to the manifold. The leading exponential terms which characterize the "connective constant" for SAM's depend on the cutoff. In dimensional regularization these terms vanish.[21] Except in the case that D is an integer there is no power-law correction to the partition function.

Disorder is added via a δ-correlated Gaussian random potential $V(\mathbf{r})$ with mean zero and variance σ^2:

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \sigma^2\delta^d(\mathbf{r}-\mathbf{r}') . \tag{2.31}$$

For a given realization of the potential, the partition function is

$$Z = \int d[\mathbf{r}] \exp\left[-\mathcal{H}[\mathbf{r}] - \int_A d^D x V(\mathbf{r}(x)) \right] . \tag{2.32}$$

Note that x has the dimensions $[x]=[r]^{2/(2-D)}$ where $[r]$ is a physical length so that $[V]=[r]^{2D/(D-2)}$ and $[\sigma^2]=[w]=[r]^{d-4D/(2-D)}$. The disorder averaged partition function is given by[5,6]

$$\langle Z\rangle = \int d[V]\frac{1}{\mathcal{N}}\int d[\mathbf{r}]\exp\left[-\mathcal{H}[\mathbf{r}] - \int d^d r' \left[\frac{[V(\mathbf{r}')]^2}{2\sigma^2} + \int_A d^D x\, \delta^d(\mathbf{r}(x)-\mathbf{r}')V(\mathbf{r}') \right] \right] . \tag{2.33}$$

where $\mathcal{N}$ is the normalization for the Gaussian disorder. The average over the random potential is easily done by shifting $V(r)$ with the result that

$$\langle Z\rangle = Z_0\{w-\sigma^2\} . \tag{2.34}$$

Thus, the only effect of disorder on the average partition function is to change nonuniversal amplitudes including the connective constant. In contrast to the case of site dilution, disorder increases the partition function since Z_0 is a decreasing function of the interaction strength. If $\sigma^2 > w$ the manifold collapses to a size set by the cutoff and the theory is no longer physically sensible. In what follows we consider the weak disorder limit $\sigma^2 \ll w$. However, we shall see that for sufficiently high moments, the effective disorder becomes large and the manifold collapses so that the theory is restricted to moments $k < k_c$.

In the Lifshitz calculation for site dilution it is high porosity regions which dominate the positive moments of Z. For a Gaussian potential it is regions with large negative values of the local potential which dominate the moments of Z. Let $Z(\Omega)$ be the partition function for SAM's confined to a region Ω and $\langle\ \rangle_m$ represent a constrained average such that the integrated potential in the region is fixed at $-m|\Omega|$. The partially averaged partition function for confined SAM's is

$$\langle Z(\Omega)\rangle_m = \int d[V]\frac{1}{\mathcal{N}_1}\delta\left[\int_\Omega d^d r' V(\mathbf{r}') + |\Omega| m \right] \times \int d[\mathbf{r}]\exp\left[-\mathcal{H}[\mathbf{r}] - \int_\Omega d^d r' \left[\frac{V(\mathbf{r}')^2}{2\sigma^2} + \int_A d^D x \delta^d(\mathbf{r}(x)-\mathbf{r}')V(\mathbf{r}') \right] \right] . \tag{2.35}$$

Here $\mathcal{N}_1$ is the appropriate normalization for the constrained Gaussian average and the path integral ranges over all manifolds confined to Ω. The integrals can be done by shifting the potential and using the Fourier representation of the constraining δ function. The ratio $\Psi_\Omega(m) \equiv \langle Z(\Omega)\rangle_m / \langle Z(\Omega)\rangle$ is given by

$$\ln\Psi_\Omega(m) = mS^D - \frac{S^{2D}}{2|\Omega|}\sigma^2 . \tag{2.36}$$

Note that this result has the same form as for the site-dilution case, Eq. (2.3). The probability density $\rho_\Omega(m)$ for having a shape Ω with local potential m is simply a Gaussian with variance $\sigma^2/|\Omega|$.

The entropy loss due to confinement, the spatial extent of the crumpled manifold in the unconfined dimensions $R_\parallel$, and the number of equivalent conformations of the confining region g_Ω can be obtained by generalizing the scaling or "blob" arguments discussed above. The result

for the confined partition function is

$$\langle Z(\Omega)\rangle \simeq \langle Z\rangle \exp[-cS^D L^{-D/\nu}] \tag{2.37a}$$

and, using Eq. (2.8), it follows that

$$g_\Omega \simeq \exp[cS^D L^{-D/\nu}] . \tag{2.37b}$$

The size of the manifold in the unconfined dimensions is

$$R_\parallel \sim SL^{1-1/\nu} . \tag{2.38}$$

We suppose that the number of free dimensions cannot be less than D if the partition function is to grow exponentially with S^D. Thus the optimum confining shape for moments $k>1$ is a D dimensional "pancake" with extent $R_\parallel$ in d-D dimensions.

The variational ansatz is essentially the same as Eq. (2.9):

$$\langle Z^k\rangle \simeq \sup_\Omega \left[g_\Omega \int_{-\infty}^{+\infty} dm\, \rho_\Omega(m)[\Psi_\Omega(m)\langle Z(\Omega)\rangle]^k \right] . \tag{2.39}$$

The Gaussian integral over the local potential m can be done immediately with the result

$$\langle Z^k\rangle/\langle Z\rangle^k \simeq \sup_\Omega \{\exp[-c(k-1)S^D L^{-D/\nu} + k(k-1)\sigma^2 S^{2D}/2|\Omega|]\} . \tag{2.40}$$

For $k<1$ the optimum region is a correlation volume $|\Omega| = S^D$ and the result is

$$\langle Z^k\rangle/\langle Z\rangle^k \sim \exp[-c\sigma^2 k(1-k)S^{\alpha_D}] \tag{2.41}$$

with the exponent α_D characterizing the relevance of the disorder given by

$$\alpha_D = 2D - d\nu . \tag{2.42}$$

For $k>1$, choosing $|\Omega| \sim S^D L^{d-D/\nu}$ and optimizing with respect to L we obtain

$$\langle Z^k\rangle/\langle Z^k\rangle \sim \exp\{+c(k-1)(k\sigma^2)^{D/\alpha_D} S^D\} . \tag{2.43}$$

α_D is positive below the upper critical dimension $d^* = 4D/(2-D)$. For example, if $d=3$ and $D=2$ the Flory value is $\alpha_D = \frac{8}{5}$.

Because self-avoidance is soft in the continuum description, the manifold can collapse to a region whose size is set by the cutoff with an energy cost which scales as wS^{2D}. This possibility can be introduced into the theory by replacing the confinement factor $(k-1)L^{-D/\nu}$ in Eq. (2.40) by kwS^{2D} and by replacing Ω by a cutoff volume. One sees that the kth moments with $k > k_c \sim (w/\sigma^2)$ are dominated by unphysical, collapsed manifolds.

C. The probability density for lngZ

The moments of Z can be viewed as the characteristic function of the probability distribution for ln$Z$ where ik plays the role of the Fourier variable conjugate to ln$Z$. In particular, let $p(y)$ be the probability density for the random variable y,

$$y = \ln Z/\langle Z\rangle \tag{2.44}$$

then

$$p(y) = \int_{-i\infty}^{+i\infty} dk \frac{1}{2\pi i} e^{-ky}\langle Z^k\rangle/\langle Z\rangle^k . \tag{2.45}$$

Neglecting questions of convergence and analyticity we may evaluate $p(y)$ by a steepest descents calculation assuming the dominant saddle point resides on the real k axis. In carrying out this evaluation it is important to choose the correct regime for k—for small y the small k regime is appropriate and vice versa. For simplicity consider the case of a polymer ($D=1$) in a Gaussian random environment. Using the results of Eqs. (2.41) for the moments of Z and supposing that $\alpha \ll 1$ we obtain the following result:

$$p(y) \sim \exp\left[\frac{-c}{2\sigma^2 S^\alpha} (y + \sigma^2 S^\alpha/2c)^2 \right] \tag{2.46a}$$

for the range

$$-c < \frac{y}{S^\alpha \sigma^2} < c . \tag{2.46b}$$

This result depends on the moments $0<k<1$. Thus for y of order S^α the probability density is a Gaussian with mean $\sigma^2 S^\alpha/2c$ and variance twice the mean. One of the consequences of Eqs. (2.46) is that the scaled variable $(1/S^\alpha)\ln Z/\langle Z\rangle$ goes almost surely to a definite value proportional to $-\sigma^2$ in the limit $S\to\infty$.

For y of order S one obtains results which are best understood by considering two limiting cases. If $\sigma^{2/\alpha} S \ll y < S$ one uses the expression for the moments, Eqs. (2.43), in the limit $k \gg 1$ and obtains

$$p(y) \sim \exp(-cy^{1+\alpha}/\sigma^2 S^\alpha) \tag{2.47}$$

while for $\alpha < y^2/\sigma^{2/\alpha} S < 1$ with $\alpha \ll 1$ we have

$$p(y) \sim \exp\left[-\alpha(\sigma^2/2)^{1/\alpha} S - y - \frac{y^2}{4(\sigma^2/2)^{1/\alpha} S} \right] . \tag{2.48}$$

It is straightforward to check that these expressions reproduce the moments of Z in the appropriate ranges of k; Eq. (2.46a) yields the moments $k<1$, Eq. (2.47) yields the moments $k \gg 1$ while Eq. (2.48) yields the moments near and above $k=1$.

III. FIELD-THEORETIC CALCULATION OF THE MOMENTS OF Z

A. The model

The number of N-step SAW's, $C_N(l)$ starting from a lattice point l on a site-diluted lattice can be obtained from the susceptibility χ_l of an n-vector spin model in the limit $n\to 0$.[19] The susceptibility is defined by

$$\chi_l = \lim_{n\to 0} \sum_{l'} \mathrm{Tr} S_{l,1} S_{l',1} \exp\left[\sum_{(i,j)} K_{ij} p_i p_j S_i \cdot S_j \right] . \tag{3.1}$$

Here $S_l=(S_{l1},\ldots,S_{ln})$ is an n-component spin vector at site l, normalized so that $\sum_{\alpha=1}^{n} S_{l\alpha}S_{l\alpha}\equiv S_l\cdot S_l=n$, $K_{ij}=KL_{ij}$ with $L_{ij}=1$ if i and j are nearest neighbors and $L_{ij}=0$ otherwise. The random lattice is defined by setting $p_i=1$ if i is an allowed site and $p_i=0$ otherwise.

The susceptibility is the generating function for the number of SAW's,[18]

$$\chi_l=\sum_{N=0}^{\infty} K^N C_N(l)\ . \tag{3.2a}$$

The inverse of Eq. (3.2a) is,

$$C_N(l)=\int_c dK\frac{1}{2\pi i}\frac{\chi_l}{K^{N+1}}\ , \tag{3.2b}$$

with c a contour that encircles the origin without enclosing any of the singularities of χ_l. Note that K in Eqs. (3.2) is the inverse temperature and (3.2a) is the high temperature expansion of the susceptibility. The representation, Eqs. (3.2), relating the SAW problem and the n-vector spin model is valid only in the high temperature or disordered phase. For $N\to\infty$ the integral in Eq. (3.2b) is dominated by the singularity in χ_l nearest to the origin, i.e., by the phase transition in the n-vector ($n\to 0$) model.

Below we argue that the N dependence of the moments of Z,

$$\langle Z^k\rangle=\langle C_N(l)^k\rangle \tag{3.3a}$$

can be determined from the moments of χ.

$$\langle\chi_l^k\rangle=\left\langle\left[\sum_{N=0}^{\infty} K^N C_N(l)\right]^k\right\rangle\ . \tag{3.3b}$$

Here the angular brackets denote an average over the disorder in which each site variable is chosen independently with

$$p_i=\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\ .\end{cases} \tag{3.4}$$

To compute $\langle\chi_l^k\rangle$ we introduce k replicas $\alpha=1,2,\ldots,k$, and use Eq. (3.1) to write

$$\langle\chi_l^k\rangle=\left\langle \mathrm{Tr}\left[\sum_{\alpha=1}^{k}\sum_{l'} S_{l1}^{\alpha}S_{l'1}^{\alpha}\right]\exp\left[\sum_{\alpha=1}^{k}\sum_{(i,j)} K_{ij}p_ip_jS_i^{\alpha}\cdot S_j^{\alpha}\right]\right\rangle\ , \tag{3.5}$$

with S_l^{α} an n-component spin vector in the αth replica at site l. Using the Gaussian transformation

$$\exp\left[\sum_{\alpha=1}^{k}\sum_{(i,j)} K_{ij}p_ip_jS_i^{\alpha}\cdot S_j^{\alpha}\right]=\text{const}\times\prod_{l\alpha\beta}\int d\phi_{l\beta}^{\alpha}\exp\left\{-\sum_{\alpha=1}^{k}\sum_{(i,j)} K_{ij}^{-1}\phi_i^{\alpha}\cdot\phi_j^{\alpha}+\sum_{\alpha=1}^{k}\sum_{i} p_iS_i^{\alpha}\cdot\phi_i^{\alpha}\right\} \tag{3.6}$$

the trace and disorder average in Eq. (3.5) can be performed. The result is a field theory in terms of the ϕ fields that is given by

$$\langle\chi_l^k\rangle=\text{const}\times\int[d\phi]\left[\prod_{\alpha=1}^{k}\phi_{l1}^{\alpha}\sum_{l'}\phi_{l'l}^{\alpha}\right]\exp(-\mathcal{H}_k)\ . \tag{3.7a}$$

Here $\mathcal{H}_k$ is the k replica Hamiltonian,

$$\mathcal{H}_k=\sum_{\substack{i,j\\ \alpha,\beta}}\phi_{i\beta}^{\alpha}K_{ij}^{-1}\phi_{j\beta}^{\alpha}-\sum_i\ln\left[(1-p)+p\prod_{\alpha}\left[1+\frac{1}{2}\sum_{\beta}\phi_{i\beta}^{\alpha 2}\right]\right]\ . \tag{3.7b}$$

In the continuum limit, the field theory for $\langle\chi^k\rangle$ is

$$\langle\chi_l^k\rangle\to\langle\chi^k(\mathbf{x})\rangle=\text{const}\times\int[D\phi]\left[\prod_{\alpha=1}^{k}\phi_l^{\alpha}(\mathbf{x})\int d\mathbf{x}'\phi_l^{\alpha}(\mathbf{x}')\right]\times\exp[-\mathcal{H}_k] \tag{3.8a}$$

with

$$\mathcal{H}_k=\frac{1}{2}\int d\mathbf{x}\sum_{\alpha}[A\phi^{\alpha}(\mathbf{x})\cdot\phi^{\alpha}(\mathbf{x})+\nabla\phi^{\alpha}(\mathbf{x})\cdot\nabla\phi^{\alpha}(\mathbf{x})]-\frac{1}{a^d}\int d\mathbf{x}\ln\left[(1-p)+p\prod_{\alpha}\left[1+u\sum_{\beta}\phi_{\beta}^{\alpha}(\mathbf{x})^2\right]\right] \tag{3.8b}$$

where a is the lattice spacing, z is the coordination number, $A=z/a^2$, and $u=z^2K/2a^{2-d}$. If the phase transition in this magnetic model is continuous, or nearly continuous, and if the spatial dimension is greater than or equal to three, then the critical behavior will be determined by the ϕ^4 approximation to Eq. (3.8b). To $O(\phi^4)$, Eq. (3.8b) gives

$$\mathcal{H}_k=\frac{1}{2}\sum_{\alpha=1}^{k}\int d\mathbf{x}[t\phi^{\alpha}(\mathbf{x})\cdot\phi^{\alpha}(\mathbf{x})+\nabla\phi^{\alpha}(\mathbf{x})\cdot\nabla\phi^{\alpha}(\mathbf{x})]+\bar{u}\sum_{\alpha=1}^{k}\int d\mathbf{x}[\phi^{\alpha}(\mathbf{x})\cdot\phi^{\alpha}(\mathbf{x})]^2-\frac{\Delta}{8}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mathbf{x}[\phi^{\alpha}(\mathbf{x})\cdot\phi^{\alpha}(\mathbf{x})][\phi^{\beta}(\mathbf{x})\cdot\phi^{\beta}(\mathbf{x})] \tag{3.9a}$$

with

$$t=\frac{z}{a^2}(1-pzK)\ ,$$
$$\bar{u}=\frac{z^2(pzK)^2}{8a^{4-d}}\ , \tag{3.9b}$$
$$\frac{\Delta}{8}=\frac{\bar{u}(1-p)}{p}\ .$$

Note that the coupling between replicas is $\Delta \sim(1-p)\sim\delta p$.

The approximate Hamiltonian given by Eqs. (3.9) is stable only if $\bar{u}>(k-1)\Delta/4$. Otherwise the quartic approximation cannot be used and the full Hamiltonian given by Eq. (3.8b) is needed. Using that $\Delta\sim\delta p \ll 1$ this yields a critical $k(=k_c)$ given by $k_c=\delta p^{-1}$ which agrees with the Lifshitz result, Eq. (2.16), if fluctuation effects are neglected [i.e., set $\alpha=0$ in Eq. (2.16)]. The quartic field theory is equivalent to the Edwards walk with Gaussian disorder discussed above in Sec. II B and the instability of the field theory at k_c corresponds to the collapse transition in the Edwards theory.

In Sec. III B we discuss the behavior of $\langle\chi^k(\mathbf{x})\rangle$ for $1\le k\le k_c$ using the quartic field theory. In Sec. III C we consider the $k\gg k_c$ behavior of $\langle\chi^k(\mathbf{x})\rangle$.

B. The moments of Z for $1\le k\le k_c$

To begin we note that for the $k=1$ case the last term in Eq. (3.9a) is absent and the Hamiltonian $\mathcal{H}_1$ is the pure system Hamiltonian with K replaced by pK. The leading singularity in $\langle\chi(\mathbf{x})\rangle$ near the ferromagnetic transition for $\mathcal{H}_1$ is then given by

$$\langle\chi(\mathbf{x})\rangle=\frac{c}{(1-pK\mu)^\gamma}\ , \tag{3.10a}$$

with γ the pure system susceptibility exponent and μ the connective constant or effective coordination number. For $N\to\infty$, Eqs. (3.2) and (3.3) give

$$\langle Z\rangle=c(p\mu)^N N^{\gamma-1} \tag{3.10b}$$

in agreement with Eq. (1.2).

For $k\neq1$, the replicas are coupled by the last term in Eq. (3.9a) and the problem is qualitatively different than in the pure or $k=1$ cases. To find the consequences of this we construct renormalization-group (RG) flow equations for the parameters t, $\bar{u}$, and Δ that appear in Eq. (3.9a). We use standard field theoretic techniques[22] and work to one-loop order which will enable use to construct an $\epsilon=4-d$ expansion. Denoting the renormalized quantities by t_R, $\bar{u}_R$, and Δ_R, the flow equations are

$$b\frac{dt_R}{db}=2t_R-4t_R\bar{u}_R\ , \tag{3.11a}$$
$$b\frac{d\bar{u}_R}{db}=\epsilon\bar{u}_R-32\bar{u}_R^{\,2}\ , \tag{3.11b}$$
$$b\frac{d\Delta_R}{db}=\epsilon\Delta_R+2\Delta_R^2-16\Delta_R\bar{u}_R \tag{3.11c}$$

with b the RG length scale. The coupling constants in Eqs. (3.11) have been scaled by $(2\pi)^d/S_d$ with S_d the area of a d-dimensional unit sphere. It is interesting to note that Eqs. (3.11a) and (3.11b) are determined by two- and four-point vertex functions, respectively, within a single replica. This implies the quantity Δ (or Δ_R) in Eq. (3.9a) does not couple to these flow equations to any order in perturbation theory because for these vertex functions the last term in Eq. (3.9a) is effectively of $O(n)$ and vanishes in the SAW limit. This implies that Eqs. (3.11a) and (3.11b) are identical to those in the pure system, and, by themselves, have the same fixed point (FP) as the pure system. This fact has been used[3] to conclude that disorder is irrelevant for SAW's. We shall see that this is not a correct conclusion because Eq. (3.11c) leads to a runaway trajectory to Δ_R and this has dramatic consequences for the number of SAW's at the origin. We also note that the RG flow equations (3.11) are independent of k, the number of replicas. Again, this is true only because of the $n\to0$, SAW limit.

For $d>4$ ($\epsilon<0$) Eqs. (3.11) predict that small disorder ($\sim\Delta_R$) is irrelevant in a RG sense. For $d<4$ ($\epsilon>0$), Eq. (3.11b) implies a FP value of $\bar{u}_R$ given by

$$\bar{u}_R^{\,*}=\frac{\epsilon}{32}+O(\epsilon^2)\ . \tag{3.12}$$

Equation (3.12) inserted in Eq. (3.11c) shows that even small disorder is relevant for $d<4$. Furthermore, Eq. (3.11c) has no FP and instead scales or runs away to large values of Δ_R. As is the case for the closely related problem of cubic anisotropies,[16,22] here a runaway RG trajectory implies a fluctuation driven first-order phase transition. To demonstrate this we construct a free energy for a presumed ordered phase and compare it to the free energy of the high temperature phase.[17,22]

To proceed we make the assumption that the symmetry breaking in the ordered phase is the same in each replica and in the $\beta=1$ spin direction

$$\langle\phi_\beta^\alpha\rangle=\delta_{\beta1}\bar{\phi}\ . \tag{3.13}$$

With this ansatz the renormalized free energy relative to the disordered phase to one-loop order $F_R(\bar{\phi})$ is given by

$$\begin{aligned}F^R(\bar{\phi})=&\frac{kt_R\bar{\phi}^2}{2}+k\bar{\phi}^4\left[\bar{u}_R-\frac{k-1}{8}\Delta_R\right]\\&-kf\left[t_R+4\bar{u}_R\bar{\phi}^2-\frac{k-1}{2}\Delta_R\bar{\phi}^2\right]\\&+(k-1)f\left[t_R+12\bar{u}_R\bar{\phi}^2+\Delta_R\bar{\phi}^2-\frac{k-1}{2}\Delta_R\bar{\phi}^2\right]\\&+f[t_R+12\bar{u}_R\bar{\phi}^2-\tfrac{3}{2}(k-1)\Delta_R\bar{\phi}^2]\end{aligned} \tag{3.14a}$$

with

$$f(x)=\frac{x^2}{8}(\log x+\tfrac{1}{2})\ . \tag{3.14b}$$

The first-order phase transition is manifest in the free energy when the RG length scale reaches b^* such that

$$\bar{u}_R(b^*)=\frac{(k-1)}{8}\Delta_R(b^*)\ . \tag{3.15}$$

The correlation length ξ at the transition is identified with this length scale $\xi=b^*$. In the weak disorder limit $\delta p \ll \epsilon$, b^* must be large to satisfy Eq. (3.15) since $\Delta_R(b=1)\sim\delta p \ll \bar{u}_R(b\to\infty)=\epsilon/32$. Thus, in solving Eq. (3.15), we can replace $\bar{u}_R(b)$ in Eq. (3.11c) by its fixed-point value yielding

$$b\frac{d\Delta_R(b)}{db}=\frac{\epsilon}{2}\Delta_R(b)+2\Delta_R^2(b)\ . \tag{3.16}$$

Solving Eq. (3.17a) for b yields

$$b=\left[\frac{\Delta_R(b)}{\Delta_R(1)}\frac{[\epsilon+4\Delta_R(1)]}{[\epsilon+4\Delta_R(b)]}\right]^{2/\epsilon}\ . \tag{3.17}$$

At b^*, Eqs. (3.15) and (3.12) yield

$$\Delta_R(b^*)=\frac{\epsilon}{4(k-1)} \tag{3.18a}$$

and Eqs. (3.17) and (3.18a) give

$$b^*=\xi\sim\left[\frac{\epsilon}{k\delta p}\right]^{2/\epsilon}\ . \tag{3.18b}$$

To leading order in ϵ, $\alpha=\epsilon/4$ and $\nu=\frac{1}{2}$, so that the correlation length given by Eq. (3.18b) is identical to the confinement length given by Eq. (2.14).

To determine the connective constant for $k>1$ we need to determine the transition temperature for the k-replica field theory $t_c^{(k)}$. We note that there is an underlying continuous transition at $t_R(1)=0$ but that the actual first-order transition temperature is at a higher temperature $t_R(1)=t_c^{(k)}>0$. At the first-order phase transition

$$\begin{aligned} &F_R(\bar{\phi})=0\ , \\ &\frac{\partial F_R(\bar{\phi})}{\partial\bar{\phi}}=0\ . \end{aligned} \tag{3.19}$$

These two equations yield[22]

$$t_R(b^*)=a\epsilon\ , \tag{3.20a}$$

with $a=e^{-3/2}/16$. Equations (3.11a), (3.11b), and (3.20) give

$$t_c^{(k)}=\frac{a\epsilon}{(b^*)^2}\sim(k\delta p)^{4/\epsilon}\ . \tag{3.20b}$$

What does the first-order phase transition in the magnetic model mean in the polymer representation? We first note that in general to obtain $\langle Z^k\rangle=\langle C_N^k(l)\rangle$ one needs to introduce k distinct temperatures $K_1,\ldots,K_k$ [cf. Eq. (3.2b)]. However, it can be shown[23] that to obtain the leading N dependence of $\langle Z^k\rangle$ it is sufficient to use only one K. If we introduce the critical temperature $t=1-p\mu K$, and write

$$Z=C_N(l)=(p\mu)^N g_N(l)\ . \tag{3.21a}$$

then for small t

$$\langle\chi^k(\mathbf{x})\rangle\sim\int_0^\infty dN\exp(-tkN)\langle g_N^k(l)\rangle\ . \tag{3.21b}$$

The first-order phase transition in the magnetic model implies that $\langle\chi^k(\mathbf{x})\rangle$ is discontinuous at $t=t_c^{(k)}$. From Eq. (3.21b) this in turn implies

$$\langle g_N^k(l)\rangle=\exp(t_c^{(k)}kN)\ . \tag{3.22a}$$

From Eqs. (3.21a), (3.22a), and (2.22) the kth-order connective constant for $1<k<k_c$ is

$$\mu_k=p\mu\exp(t_c^{(k)})\sim p\mu[1+c(k\delta p)^{4/\epsilon}]\ . \tag{3.22b}$$

To leading order in an ϵ expansion, Eq. (3.22b) is in agreement with Eq. (2.23b).

C. The moments of Z for $k\gg k_c$

The effective Hamiltonian given by Eq. (3.8b) can be written

$$\mathcal{H}=\frac{1}{2}\int dx[\nabla\phi^\alpha(\mathbf{x})]\cdot[\nabla\phi^\alpha(\mathbf{x})]+\int dx\ U(\phi^2)\ , \tag{3.23a}$$

with the potential $U(\phi^2)$ given by

$$\begin{aligned} U(\phi^2)=&\frac{A}{2}\sum_\alpha\phi^\alpha\cdot\phi^\alpha-\frac{1}{a^d}\sum_\alpha\ln(1+u\phi^\alpha\cdot\phi^\alpha) \\ &-\frac{1}{a^d}\ln\left[p+\delta p\exp\left[-\sum_\alpha\ln(1+u\phi^\alpha\cdot\phi^\alpha)\right]\right]\ . \end{aligned} \tag{3.23b}$$

As already noted, if $U(\phi^2)$ is expanded to $O(\phi^4)$ then the theory is unstable for large enough k, with the least stable direction being along the replica diagonal [cf. Eq. (3.13)]. For $k\to\infty$ the first two terms in Eq. (3.23b) are of $O(k)$, while the last term is of $O(1)$. To the leading and subleading order as $k\to\infty$ we have

$$\begin{aligned} U(\phi^2)\sim&\frac{A}{2}\sum_{\alpha=1}^k\phi^\alpha\cdot\phi^\alpha \\ &-\frac{1}{a^d}\sum_{\alpha=1}^k\ln[1+u\phi^\alpha\cdot\phi^\alpha]-\frac{1}{a^d}\log p\ . \end{aligned} \tag{3.24}$$

The two leading terms in Eq. (3.24) are the sum of k uncoupled pure system potentials. We conclude that in the $k\to\infty$ limit the phase transition in the magnetic model is just the pure system phase transition. This implies that the connective constant approaches the pure system connective constant μ as $k\to\infty$. This is in accord with Eq. (2.23a).

For large but finite k, Eq. (3.24) is a faithful representation of Eq. (3.23b) as long as $k\phi^2\gg1$. In fact, if fluctuation effects are ignored then Eqs. (3.23) and (3.24) imply a first-order phase transition with, using $\log p\simeq-\delta p$,

$$\begin{aligned} t_c^{(k)}&\sim-\left[\frac{\delta p}{k}\right]^{1/2}\ , \\ \bar{\phi}^2&\sim-\left[\frac{\delta p}{k}\right]^{1/2}\ . \end{aligned} \tag{3.25}$$

Here $t_c^{(k)}$ is the phase transition temperature measured relative to the pure system transition at $t=0$ and $\bar{\phi}$ is the amplitude of the magnetization at $t_c^{(k)}$.

To take into account fluctuation effects we use a scaling argument. The first two terms in Eq. (3.24) lead to a

pure system free energy f that scales like

$$f \sim k|t|^{2-\alpha} . \tag{3.26}$$

The first-order phase transition takes place when this term is comparable to the last term in Eq. (3.24), $-\delta p$. This implies

$$t_c^{(k)} \sim -\left[\frac{\delta p}{p}\right]^{1/(2-\alpha)} = -\left[\frac{\delta p}{k}\right]^{1/d\nu} . \tag{3.27}$$

where the last equality follows if hyperscaling is valid. The arguments from the end of the previous subsection relating the magnetic problem to the polymer problem can be repeated here except that the transition temperature is measured relative to the pure system critical temperature, i.e., $t=1-\mu K$. The kth-order connective constant for $k \gg k_c$ is then given by

$$\mu_k = \mu\left[1 - c\left[\frac{\delta p}{k}\right]^{1/d\nu}\right] . \tag{3.28}$$

Equation (3.28) is in agreement with Eq. (2.23a).

We conclude this section with two remarks. First, the connective constant μ_k for $k<1$ cannot be determined with the methods of Sec. III B. This follows because $\Delta_R(b^*)$ given by Eq. (3.18a) is negative which corresponds to an unphysical initial condition for the RG. Second, Eq. (3.22b) is identical in structure to Eq. (2.23b) except for a factor $(k-1)/k$. The methods of Sec. III B requires that $k-1 > O(\epsilon)$ since otherwise the Ginsburg criterion is violated and one must consider additional fluctuations. Thus, in the range where the field theory is applicable it is in agreement with the Lifshitz arguments up to factors of order unity.

IV. DISCUSSION

We have treated the statistical mechanics of self-avoiding walks in randomly diluted environments using two distinct methods. For the positive integer moments of the partition function both methods are in agreement and show that the partition function Z is a broadly distributed random variable. More precisely, the connective constants, which describe the exponential growth rate of the moments $\langle Z^k \rangle$, increase with k. On the other hand, for the low moments $(0<k<1)$ and for sufficiently weak disorder we find that $\langle Z^k \rangle / \langle Z \rangle^k \sim \exp\{-ck(1-k)\delta p N^{\alpha}\}$ with $\alpha = 2-d\nu < 1$. We have only been able to obtain results for the low moments using the heuristic "Lifshitz" method of Sec. II. The field theory solution, valid for $k > 1+O(\epsilon)$, cannot be extrapolated to $k<1$ suggesting that a replica symmetry breaking solution might be needed to obtain the low moments of Z.

The Lifshitz calculation for SAW's was generalized to the case of self-avoiding D-dimensional crumpled manifolds with qualitatively similar results. The main change that occurs is that the "specific-heat" exponent α which characterizes the essential singularities induced by the disorder is replaced by $\alpha_D = 2D - d\nu$.

In a recent publication, Meir and Harris[8] study SAW's in diluted environments from a different perspective. Using field-theoretic methods similar to those developed for the diluted Ising model they construct a 6-ϵ expansion to investigate the region near the percolation threshold. They find that SAW's on percolation clusters are described by a fixed point with new values of the critical exponents while above the percolation threshold disorder is irrelevant. These results are consistent with ours since above four dimensions we also predict that weak disorder is irrelevant. The existence of a percolation fixed point near six dimensions lends weight to the idea that the transition from strong to weak disorder below four dimensions occurs at the percolation threshold though this is not accessible via a 4-ϵ expansion. There is, however, an argument which runs counter to this hypothesis and depends on a subtle difference between the class of quantities studied in Ref. 8 and those studied in this work. Let $C_N(i,j)$ be the number of N-step SAW's from i to j in a given environment. The quantities which we study in Sec. III are defined by

$$\chi^{(k)} = \left\langle \left[\sum_j \sum_N K^N C_N(0,j) \right]^k \right\rangle \tag{4.1a}$$

while the quantities studied in Ref. 8 are defined by

$$G^{(k)} = \left\langle \sum_j \left[\sum_N K^N C_N(0,j) \right]^k \right\rangle . \tag{4.1b}$$

Consider the simple case of $k=0$, then $\chi^{(0)}$ is the probability of having some walk at the origin so that $\chi^{(0)}=p$ while $G^{(0)}$ is the expected number of sites connected to the origin which diverges as $p \to p_c$. It may well be that for arbitrary k one finds that $\chi^{(k)}$ is smooth at p_c while $G^{(k)}$ is singular there.

Ordinary random walks and directed random walks in random environments are closely related to SAW's in random environments. In both cases disorder leads to a broad distribution for Z due to rare configurations which dominate the moments of Z. The average partition function $\langle Z_{\mathrm{RW}} \rangle$ for ordinary random walks on diluted lattices was obtained rigorously by Donsker and Varadhan[24] and later field-theoretic investigations[23,25] extended the result to all of the positive integer moments of Z_{RW} yielding

$$\langle Z_{\mathrm{RW}}^k \rangle = z^{kN} \exp[-c \ln(1/p)^{2/(d+2)} (kN)^{d/(d+2)}] \tag{4.2}$$

where z is the coordination number and c is a constant depending on the dimension. Lifshitz arguments have been constructed[15,26] for the first two moments of Z_{RW}. An interesting feature of these arguments is that the optimal regions are confining in all d dimensions and have no scatterers ($\Delta=0$ and $m=1$ in the notation of Sec. II). The size of the confining region scales as $L \sim N^{2/(d+2)}$. Confinement in all dimensions is permissible for RW's but not SAW's. In contrast to the SAW result, the connective constant for RW's is that of the pure system for all moments. The stretched exponential correction arises within the field theory from an instanton calculation and, as pointed out by Lubensky,[23] is a kind of Griffiths singularity. The instanton calculation could be applied to computing corrections to the leading exponential behavior for the high moments of the SAW partition function.

Directed walks are free to move in any direction in d transverse dimensions but must move forward in one "time" dimension. The positive integer moments of the partition function, Z_{DW}, for $1+1$ dimensional continuous directed walks in Gaussian random environments have been obtained by Kardar[27] with the result

$$\langle Z_{DW}^k \rangle \sim \exp[c_1 kt + c_2 k(k^2-1)t] \quad (4.3)$$

where t is the walk length in the time direction. This result resembles the results for $k>1$ for a SAW in a random environment in that the connective constants increase as a power of k larger than unity and the first moment is trivially affected by the disorder. In contrast to the SAW case, it has been argued[27] that (4.3) holds for all positive k. The results presented here for SAW's cast doubt on the extrapolation of Eq. (4.3) from integer k to $k<1$.

Note added. Two works that are not yet published relevant to the present paper have come to our attention. In one paper, S. P. Obukhov independently derives the recursion relations obtained in Sec. III and from them draws the conclusion that the free energy is controlled by a strong disorder fixed point. In the other, J. D. Honeycutt and D. Thirumalai develop a Lifshitz theory for SAW's based upon tube-shaped regions.

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