Statistical-physical theory of multivariate temporal fluctuations: Global characterization and temporal correlation

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(Received 10 July 1989)

The fluctuation-spectrum theory and the generalized time-correlation function method established for the statistical characterization of univariate temporal fluctuations are extended to multivariate fluctuations by introducing the concept of the complete set of temporal fluctuations for the overall description of dynamical behaviors of the physical system. We develop the eigenfunction expansion for relevant quantities and further construct the continued-fraction expansion. The present approach enables us to study the explicit cross correlation among several time sequences as well as a detailed description of the dynamical behaviors of the system.

I. INTRODUCTION

Recent studies on fluctuation dynamics, especially on chaotic dynamics, brought about several new concepts in statistical physics. One of the important contributions is the multifractal theory of strange objects,[1–5] established first on the basis of the Renyi exponent D_q.[6–8] A similar approach to the study of fluctuations around the metric entropy of chaotic dynamical systems has been formulated in Ref. 9 by utilizing generalized entropy. Another approach to fluctuation dynamics, especially on the statistical characterization of univariate sequential fluctuations, was developed as the fluctuation-spectrum theory,[10,11] by introducing the characteristic function λ_q.[12,13]

The common features of the multifractal theory and the fluctuation-spectrum theory are mainly threefold. The first is that both of them deal with the rates describing the enhancement or the reduction of relevant quantities as the scale over which they are defined is changed. Second, the various aspects of fluctuations of the rates can be singled out by introducing the so-called filtering parameter q.[13,14] Third, they have the thermodynamics formalism relevant to the global characterization of fluctuations, giving no explicit information about the generation law of fluctuations. On the above third point Feigenbaum, Jensen, and Procaccia[15] tried to study the correlation in strange objects in connection with the global characterization. On the other hand, the approach to temporal correlation has been developed by the present authors as the generalized time-correlation function theory.[16–18] These are the correlation problem embedded in an observed strange object or in an observed univariate time series.

The fluctuation-spectrum theory and the generalized time-correlation function theory seem to be sufficient in order to statistically analyze a univariate steady time series. However, from the viewpoint of an overall description of the dynamical behaviors of the physical system, the observation of univariate time series is not necessarily sufficient. The fundamental aim of the present paper is to develop a statistical-physical approach to multivariate temporal fluctuations.

This paper is organized as follows. In Sec. II we briefly describe the reason for the necessity of the study of multivariate temporal fluctuations. The statistical-physical theory for them is developed in Sec. III, especially concerning their global (long-time) statistical characteristics. This is reconsidered in Sec. IV from the ensemble-processing viewpoint. Such a global approach turns out to have a formalism similar to equilibrium thermodynamics (Sec. V). In Sec. VI, the relevant quantities for Markov processes are derived from first principles. It is shown that the global aspect is determined by the largest eigenvalue of the extended master operator $H(\mathbf{q})$. Others contribute to explicit temporal correlations. A more practical and workable approach to the overall analyses of multivariate temporal fluctuations is proposed in Sec. VII as the continued-fraction expansion for the characteristic function $M_{\mathbf{q}}(n)$. Two examples are illustrated in Secs. VIII and IX. The concluding remarks are given in Sec. X.

II. NECESSITY OF THE OBSERVATION OF MULTIVARIATE TEMPORAL FLUCTUATIONS

In previous papers[10,12–14] we dealt with a univariate time series $\{u(j)\}=\{u(1),u(2),u(3),\dots\}$, observed from the physical system, discussing how a scale-dependent time average,

$$\alpha_n=\frac{1}{n}\sum_{j=1}^{n}u(j) \tag{2.1}$$

converges to the long-time average α_∞ as n is increased. As far as we are concerned, with a univariate time series, the statistical characterization with the fluctuation-spectrum theory and the generalized time-correlation function theory seems to be sufficient. However, from the viewpoint of the characterization of the overall dynamical behaviors of the system, we meet another problem.

As a simple example, let us take the three-valued Bernoulli map,

$$x_{n+1}=f(x_n)$$
$$\equiv\begin{cases} x_n/p_1, & x_n\in I_1\equiv[0,p_1) \\ (x_n-p_1)/p_2, & x_n\in I_2\equiv[p_1,p_1+p_2) \\ [x_n-(p_1+p_2)]/p_3, & x_n\in I_3\equiv[p_1+p_2,1] \end{cases} \tag{2.2}$$

$(p_1+p_2+p_3=1)$. The fluctuation dynamics of local expansion rates is discussed with

$$u(n)\equiv\ln|f'(x_n)|=\begin{cases} \ln p_1^{-1}, & x_n\in I_1 \\ \ln p_2^{-1}, & x_n\in I_2 \\ \ln p_3^{-1}, & x_n\in I_3\ . \end{cases} \tag{2.3}$$

Take a subsequence with the span n in $\{u(j)\}$, where the state point is n_1 times in I_1, n_2 times in I_2, and $n_3(=n-n_1-n_2)$ times in I_3. Let α' be the average value of $\{u(j)\}$ in this region, i.e.,

$$\alpha_n=\frac{1}{n}[n_1\ln p_1^{-1}+n_2\ln p_2^{-1}+(n-n_1-n_2)\ln p_3^{-1}]$$
$$=\alpha'\ . \tag{2.4}$$

We cannot uniquely determine the numbers n_1 and n_2. So in order to determine n_1 and n_2 for a precise characterization of the subsequence, we need the observation of another scale-dependent average $\beta_n=n^{-1}\sum_{j=1}^{n}v(j)$, where $v(n)$ is, e.g., the coarse-graining position,

$$v(n)=\begin{cases} p_1/2, & x_n\in I_1 \\ p_1+p_2/2, & x_n\in I_2 \\ p_1+p_2+p_3/2, & x_n\in I_3\ . \end{cases} \tag{2.5}$$

Now the numbers n_1 and n_2 are able to be uniquely determined with (2.4) and

$$\beta_n=\frac{1}{n}\left[n_1\frac{p_1}{2}+n_2\left[p_1+\frac{p_2}{2}\right]+(n-n_1-n_2)\left[p_1+p_2+\frac{p_3}{2}\right]\right]\ . \tag{2.6}$$

The above example hints at the importance of the study of multivariate temporal fluctuations for a more detailed description of the system. We may set up the problem on the multivariate fluctuations in a more general way as follows. Let $\Psi_{\rm I}(n)$ be the internal state of the system giving the subsequence, with the span n, where α_n takes a value between α' and $\alpha'+d\alpha'$. Let us take another subsequence, whose α_n is the same value as in $\Psi_{\rm I}(n)$. This state is called $\Psi_{\rm II}(n)$. The fact that the α_n's for the two subsequences are the same does not necessarily imply that $\Psi_{\rm I}(n)$ and $\Psi_{\rm II}(n)$ are the same. Since we observe the system through the quantity α_n, the above situation simply means that we cannot distinguish them. In this sense the states $\Psi_{\rm I}(n)$ and $\Psi_{\rm II}(n)$ are degenerate for the observation of α_n. This degeneracy can be dissolved by measuring the quantity

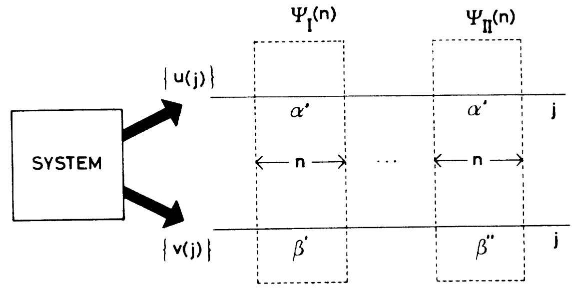

FIG. 1. Schematic figures showing the degeneracy for the observation only for α_n. A univariate time series may not distinguish different internal states of the system.

$$\beta_n=\frac{1}{n}\sum_{j=1}^{n}v(j)\ , \tag{2.7}$$

the time series $\{v(j)\}$ being different from $\{u(j)\}$. Namely, even α_n takes the same value in two subsequences, the observation of β_n in these subsequences may give different values β' and β'' for $\Psi_{\rm I}(n)$ and $\Psi_{\rm II}(n)$, respectively (Fig. 1). If the degeneracy cannot be dissolved by observing α_n and β_n, one should increase the number of other time series to be observed. This is the fundamental reason for the importance of the survey of multivariate temporal fluctuations.

The above consideration suggests the introduction of a set of temporal fluctuations (time series), which sufficiently describe the overall dynamical characteristics of the system. This set will be termed the complete set of temporal fluctuations. In the above three-valued Bernoulli map, the complete set is composed of two different time series, provided that we are concerned with the dynamics for discretized states I_1, I_2, and I_3. As will be discussed in Sec. V, this set can be compared with the set of thermodynamics variables relevant for the complete description of the thermodynamic state of the system under consideration.

III. STATISTICAL-PHYSICAL CHARACTERIZATION OF MULTIVARIATE TEMPORAL FLUCTUATIONS

Let us consider real m-variable *steady* time series experimentally observed,

$$\{\mathbf{u}(j)\}_{j=1}^{nN}=\{\mathbf{u}(1),\mathbf{u}(2),\mathbf{u}(3),\ldots,\mathbf{u}(nN)\}\ . \tag{3.1}$$

They are supposed to be composed of m *different* time series,[19] i.e., $\mathbf{u}(n)=\mathrm{col}[u_1(n),u_2(n),\ldots,u_m(n)]$. The m will be hereafter called the dimension of $\mathbf{u}(n)$. For the overall description of dynamical behaviors of the system, it is desirable that these m time series constitute the complete set of temporal fluctuations. However, the following arguments hold even if $\mathbf{u}(n)$ is not necessarily chosen as the complete set.

We divide the above sequence into N subsequences, each of which has the span n:

$$\{\mathbf{u}(j)\}_{j=1}^{n},\{\mathbf{u}(j)\}_{j=n+1}^{2n},\ldots,\{\mathbf{u}(j)\}_{j=kn+1}^{(k+1)n},\ldots,\{\mathbf{u}(j)\}_{j=(N-1)n+1}^{nN}.$$

The set of these subsequences constitutes the ensemble S_n. The quantity

$$\boldsymbol{\alpha}_n(k)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{u}((k-1)n+j) \quad (3.2)$$

($k=1,2,3,\ldots,N$) is the average of $\{\mathbf{u}(j)\}$ in the kth subsequence (ensemble member). The number density of subsequences for which $\boldsymbol{\alpha}_n$ takes a value in between $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}'+d\boldsymbol{\alpha}'$ is given by

$$\nu_n(\boldsymbol{\alpha}')=\sum_{k=1}^{N}\delta(\boldsymbol{\alpha}_n(k)-\boldsymbol{\alpha}') , \quad (3.3)$$

$\delta(\boldsymbol{\alpha}_n-\boldsymbol{\alpha}')$ $[\equiv\prod_{\mu=1}^{m}\delta((\alpha_n)_\mu-\alpha'_\mu)]$ being the δ function. The probability density $\rho_n(\boldsymbol{\alpha}')$ that $\boldsymbol{\alpha}_n$ takes a value $\boldsymbol{\alpha}'$ is thus calculated by

$$\rho_n(\boldsymbol{\alpha}')=\lim_{N\to\infty}\frac{\nu_n(\boldsymbol{\alpha}')}{N}=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\delta(\boldsymbol{\alpha}_n(k)-\boldsymbol{\alpha}') . \quad (3.4)$$

The average of $F(\boldsymbol{\alpha}_n)$ in the ensemble S_n is given by

$$\langle F(\boldsymbol{\alpha}_n)\rangle\equiv\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}F(\boldsymbol{\alpha}_n(k)) =\int_{-\infty}^{\infty}F(\boldsymbol{\alpha}')\rho_n(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' , \quad (3.5)$$

where $\int_{-\infty}^{\infty}A d\boldsymbol{\alpha}'=\int_{-\infty}^{\infty}d\alpha'_1\int_{-\infty}^{\infty}d\alpha'_2\cdots\int_{-\infty}^{\infty}d\alpha'_m A$. Especially, the order-$\mathbf{q}$ characteristic function is defined by

$$M_{\mathbf{q}}(n)\equiv\langle\exp(n\mathbf{q}\cdot\boldsymbol{\alpha}_n)\rangle =\int_{-\infty}^{\infty}\rho_n(\boldsymbol{\alpha}')e^{n\mathbf{q}\cdot\boldsymbol{\alpha}'}d\boldsymbol{\alpha}' , \quad (3.6)$$

where $\mathbf{q}=\mathrm{col}(q_1,q_2,\ldots,q_m)$ with real numbers q_μ ($\mu=1,2,\ldots,m$) and the scalar product $\mathbf{a}\cdot\mathbf{b}\equiv\sum_{\mu=1}^{m}a_\mu b_\mu$. The extensivity of $n\boldsymbol{\alpha}_n$ suggests the introduction of the similarity structure function $\phi(\mathbf{q})$ via[20]

$$\phi(\mathbf{q})=\lim_{n\to\infty}\frac{1}{n}\ln M_{\mathbf{q}}(n) . \quad (3.7)$$

Assume that $\rho_n(\boldsymbol{\alpha}')$ asymptotically obeys

$$\rho_n(\boldsymbol{\alpha}')\sim n^{m/2}\exp[-\sigma(\boldsymbol{\alpha}')n] \quad (3.8)$$

for a large n. The scalar function $\sigma(\boldsymbol{\alpha}')$ defined by

$$\sigma(\boldsymbol{\alpha}')=-\lim_{n\to\infty}\frac{1}{n}\ln\rho_n(\boldsymbol{\alpha}') \quad (3.9)$$

is called the *fluctuation spectrum*,[10] and evaluates the asymptotic generation probability of the value $\boldsymbol{\alpha}'$ for a large n.

Inserting the asymptotic form (3.8) into (3.6), and applying the steepest descent method, one finds

$$\phi(\mathbf{q})=-\min_{\boldsymbol{\alpha}'}[\sigma(\boldsymbol{\alpha}')-\mathbf{q}\cdot\boldsymbol{\alpha}'] . \quad (3.10)$$

Namely, if we set[21]

$$\boldsymbol{\alpha}\equiv\frac{\partial\phi(\mathbf{q})}{\partial\mathbf{q}} , \quad (3.11)$$

where $(\partial/\partial\mathbf{q})_\mu=\partial/\partial q_\mu$, the fluctuation spectrum is given by the Legendre transform

$$\sigma(\boldsymbol{\alpha})=\mathbf{q}\cdot\boldsymbol{\alpha}-\phi(\mathbf{q}) . \quad (3.12)$$

By inserting (3.11) into (3.12), σ is alternatively written as

$$\sigma(\boldsymbol{\alpha})=|\mathbf{q}|\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}}\left[\frac{\phi(\mathbf{q})}{|\mathbf{q}|}\right] , \quad (3.13)$$

where $|\mathbf{q}|=\sqrt{\mathbf{q}\cdot\mathbf{q}}$. Combining (3.11) and (3.12) yields

$$\mathbf{q}=\frac{\partial\sigma(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}} . \quad (3.14)$$

The uniqueness of the natural probability density implies

$$\phi(\mathbf{q}=\mathbf{0})=0 , \quad (3.15)$$

$$\sigma(\boldsymbol{\alpha}(\mathbf{q}=\mathbf{0}))=0 . \quad (3.16)$$

The fluctuation spectrum $\sigma(\boldsymbol{\alpha}')$ vanishes only for $\boldsymbol{\alpha}'=\boldsymbol{\alpha}(\mathbf{q}=\mathbf{0})$.

The steepest-descent method is applicable when the inequality

$$\sum_\mu\sum_\nu[\chi(\mathbf{q})^{-1}]_{\mu\nu}z_\mu z_\nu\geq 0 \quad (3.17)$$

holds for an arbitrary real vector $\mathbf{z}$, z_μ being its μth component. $\chi(\mathbf{q})^{-1}$ is the inverse matrix of $\chi(\mathbf{q})$. The $\mu\nu$ element of $\chi(\mathbf{q})$ is defined by

$$\chi_{\mu\nu}(\mathbf{q})\equiv\frac{\partial\alpha_\mu}{\partial q_\nu}=\frac{\partial^2\phi(\mathbf{q})}{\partial q_\mu\partial q_\nu}=\chi_{\nu\mu}(\mathbf{q}) , \quad (3.18)$$

and its inverse matrix has elements

$$[\chi(\mathbf{q})^{-1}]_{\mu\nu}=\frac{\partial q_\mu}{\partial\alpha_\nu}=\frac{\partial^2\sigma(\boldsymbol{\alpha})}{\partial\alpha_\mu\partial\alpha_\nu}=[\chi(\mathbf{q})^{-1}]_{\nu\mu} . \quad (3.19)$$

Furthermore, if we introduce the matrix $\chi\{\boldsymbol{\alpha}'\}$ by

$$[\chi\{\boldsymbol{\alpha}'\}^{-1}]_{\mu\nu}\equiv\frac{\partial^2\sigma(\boldsymbol{\alpha}')}{\partial\alpha'_\mu\partial\alpha'_\nu} , \quad (3.20)$$

it follows that

$$\chi(\mathbf{q})=\chi\{\boldsymbol{\alpha}\} . \quad (3.21)$$

The condition (3.17) implies that all eigenvalues of $\chi(\mathbf{q})$ or $\chi\{\boldsymbol{\alpha}'\}$ should be positive.

The above global characterization gives no explicit information about the temporal correlation in $\{\mathbf{u}(n)\}$. This can be described with the *order-*$\mathbf{q}$ *(generalized) time-correlation function* $Q_{\mathbf{q}}(n)$ (Refs. 16–18) defined through

$$M_{\mathbf{q}}(n)=Q_{\mathbf{q}}(n)\exp[\phi(\mathbf{q})n] , \quad (3.22)$$

where

$$\lim_{n\to\infty}\frac{1}{n}\ln Q_{\mathbf{q}}(n)=0 . \quad (3.23)$$

Namely, the poles of

$$\Xi_{\mathbf{q}}(\omega)=\sum_{n=0}^{\infty} Q_{\mathbf{q}}(n)\cos(\omega n) \tag{3.24}$$

($\omega\neq 0$) are relevant to the characteristic frequencies $\{\omega_{\mathbf{q}}^{(l)}\}$ and the damping rates $\{\gamma_{\mathbf{q}}^{(l)}\}$ for characteristic motions in $\{\mathbf{u}(n)\}$. As will be shown in Secs. VI and VII, $Q_{\mathbf{q}}(n)$ is generically expanded as[16–18]

$$Q_{\mathbf{q}}(n)=J_{\mathbf{q}}^{(0)}+\sum_{l}{}' J_{\mathbf{q}}^{(l)}\exp[-(i\omega_{\mathbf{q}}^{(l)}+\gamma_{\mathbf{q}}^{(l)})n] \ . \tag{3.25}$$

IV. ENSEMBLE PROCESSING AND PROBABILITY DENSITIES

We turn to the ensemble processing first introduced in Ref. 14 for a univariate time series. The majority of the ensemble members in S_n are in the peak region of $\rho_n(\boldsymbol{\alpha}')$ (the central limit theorem). In order to magnify the fluctuation characteristics of the minority in tail regions, a new ensemble $S_n(\mathbf{q})$ is constructed so that the number density of ensemble members is changed according to

$$\nu_n(\boldsymbol{\alpha}';\mathbf{q})\propto \nu_n(\boldsymbol{\alpha}')e^{n\mathbf{q}\cdot\boldsymbol{\alpha}'} \ . \tag{4.1}$$

The probability density $\rho_n(\boldsymbol{\alpha}';\mathbf{q})$ that $\boldsymbol{\alpha}_n$ takes a value $\boldsymbol{\alpha}'$ in the *processed-ensemble* $S_n(\mathbf{q})$ is obtained as

$$\begin{aligned}\rho_n(\boldsymbol{\alpha}';\mathbf{q})&=\lim_{N\to\infty}\frac{\nu_n(\boldsymbol{\alpha}';\mathbf{q})}{N(\mathbf{q})}\\ &=\lim_{N\to\infty}\frac{1}{N(\mathbf{q})}\sum_{\bar{k}=1}^{N(\mathbf{q})}\delta(\bar{\boldsymbol{\alpha}}_n(\bar{k})-\boldsymbol{\alpha}')\\ &=\rho_n(\boldsymbol{\alpha}')g_n(\boldsymbol{\alpha}';\mathbf{q}) \ .\end{aligned} \tag{4.2}$$

Here

$$N(\mathbf{q})\equiv\int_{-\infty}^{\infty}\nu_n(\boldsymbol{\alpha}';\mathbf{q})d\boldsymbol{\alpha}' \tag{4.3}$$

is the total number of ensemble members in $S_n(\mathbf{q})$, and $\bar{\boldsymbol{\alpha}}_n(\bar{k})$ is the value of $\boldsymbol{\alpha}_n$ in the $\bar{k}$th ensemble member of $S_n(\mathbf{q})$. g_n is the processing factor,

$$g_n(\boldsymbol{\alpha}';\mathbf{q})=\frac{e^{n\mathbf{q}\cdot\boldsymbol{\alpha}'}}{M_{\mathbf{q}}(n)} \tag{4.4}$$

[$g_n(\boldsymbol{\alpha}';0)=1$]. The average of $F(\boldsymbol{\alpha}_n)$ in the ensemble $S_n(\mathbf{q})$ is calculated by

$$\begin{aligned}\langle F(\boldsymbol{\alpha}_n);\mathbf{q}\rangle&=\lim_{N\to\infty}\frac{1}{N(\mathbf{q})}\sum_{\bar{k}=1}^{N(\mathbf{q})}F(\bar{\boldsymbol{\alpha}}_n(\bar{k}))\\ &\equiv\int_{-\infty}^{\infty}\rho_n(\boldsymbol{\alpha}';\mathbf{q})F(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'\end{aligned} \tag{4.5}$$

[$\langle F(\boldsymbol{\alpha}_n);0\rangle=\langle F(\boldsymbol{\alpha}_n)\rangle$]. Especially,

$$\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle=\frac{\partial}{\partial\mathbf{q}}\left[\frac{1}{n}\ln M_{\mathbf{q}}(n)\right] \ . \tag{4.6}$$

The quantity defined by

$$\begin{aligned}\chi_{\mu\nu}(\mathbf{q},n)&\equiv\frac{\partial(\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle)_\mu}{\partial q_\nu}=\frac{\partial^2}{\partial q_\mu\partial q_\nu}\left[\frac{1}{n}\ln M_{\mathbf{q}}(n)\right]\\ &=\chi_{\nu\mu}(\mathbf{q},n)\end{aligned} \tag{4.7}$$

has the meaning of the *susceptibility* evaluating the change of the average $\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle$ under the infinitesimal change of the degree of processing, indicated by the variable $\mathbf{q}$. The introduction of the correlation function

$$v_{\mu\nu}(\mathbf{q},n)\equiv\langle(\boldsymbol{\alpha}_n-\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle)_\mu(\boldsymbol{\alpha}_n-\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle)_\nu;\mathbf{q}\rangle \tag{4.8}$$

[$v_{\mu\nu}(\mathbf{q},n)=v_{\nu\mu}(\mathbf{q},n)$] leads to the relation

$$\chi_{\mu\nu}(\mathbf{q},n)=nv_{\mu\nu}(\mathbf{q},n) \ . \tag{4.9}$$

This is the direct interrelation between the response function $\chi(\mathbf{q},n)$ and the variance $v(\mathbf{q},n)$. Since $\boldsymbol{\alpha}=\lim_{n\to\infty}\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle$, one finds

$$\chi_{\mu\nu}(\mathbf{q})=\lim_{n\to\infty}\chi_{\mu\nu}(\mathbf{q};n)=\lim_{n\to\infty}[nv_{\mu\nu}(\mathbf{q};n)] \ , \tag{4.10}$$

which is equal to the quantity defined in (3.18). We obtain

$$\langle\boldsymbol{\alpha}_n;\mathbf{q}\rangle=\boldsymbol{\alpha}+\frac{\partial}{\partial\mathbf{q}}\left[\frac{1}{n}\ln Q_{\mathbf{q}}(n)\right] \ , \tag{4.11}$$

$$\chi_{\mu\nu}(\mathbf{q},n)=\chi_{\mu\nu}(\mathbf{q})+\frac{\partial^2}{\partial q_\mu\partial q_\nu}\left[\frac{1}{n}\ln Q_{\mathbf{q}}(n)\right] \ , \tag{4.12}$$

where $Q_{\mathbf{q}}(n)$ is the order-$\mathbf{q}$ time-correlation function defined in (3.22).

The insertion of the asymptotic forms of (4.4) and $\rho_n(\boldsymbol{\alpha}')$ into (4.2) yields the asymptotic law[14]

$$\rho_n(\boldsymbol{\alpha}';\mathbf{q})\sim n^{m/2}\exp[-\sigma(\boldsymbol{\alpha}';\mathbf{q})n] \ , \tag{4.13}$$

where

$$\begin{aligned}\sigma(\boldsymbol{\alpha}';\mathbf{q})&\equiv\sigma(\boldsymbol{\alpha}')-\mathbf{q}\cdot\boldsymbol{\alpha}'+\phi(\mathbf{q})\\ &=\sigma(\boldsymbol{\alpha}')-\sigma(\boldsymbol{\alpha})-\frac{\partial\sigma(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\cdot(\boldsymbol{\alpha}'-\boldsymbol{\alpha}) \ ,\end{aligned} \tag{4.14}$$

will be called the order-q fluctuation spectrum[14] [$\sigma(\boldsymbol{\alpha}';0)=\sigma(\boldsymbol{\alpha}')$]. Expanding (4.14) around $\boldsymbol{\alpha}'=\boldsymbol{\alpha}$ and retaining the lowest-order term, one gets

$$\sigma(\boldsymbol{\alpha}';\mathbf{q})=\tfrac{1}{2}\sum_\mu\sum_\nu[\chi(\mathbf{q})^{-1}]_{\mu\nu}(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_\mu(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_\nu \ , \tag{4.15}$$

which is non-negative [Eq. (3.17)]. Therefore $\rho_n(\boldsymbol{\alpha}';\mathbf{q})$ asymptotically takes the Gaussian form

$$\rho_n(\boldsymbol{\alpha}';\mathbf{q})\sim n^{m/2}\exp\left[-\frac{n}{2}\sum_\mu\sum_\nu[\chi(\mathbf{q})^{-1}]_{\mu\nu}\times(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_\mu(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_\nu\right] \tag{4.16}$$

near its peak position ($\boldsymbol{\alpha}'=\boldsymbol{\alpha}$), i.e., for $|\boldsymbol{\alpha}'-\boldsymbol{\alpha}|<O(1/\sqrt{n})$. Hence $\boldsymbol{\alpha}$ is found to be the average of $\boldsymbol{\alpha}_n$, taken over the processed ensemble $S_n(\mathbf{q})$ for large n.

The conventional central limit theorem result is written, for large n, as

$$\rho_n(\boldsymbol{\alpha}') \sim n^{m/2}\exp\left[-\frac{n}{4}\sum_{\mu}\sum_{\nu}(D^{-1})_{\mu\nu}(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_{\mu} \times(\boldsymbol{\alpha}'-\boldsymbol{\alpha})_{\nu}\right], \quad (4.17)$$

where $\boldsymbol{\alpha}=\boldsymbol{\alpha}(0)=\langle \mathbf{u}(n)\rangle$, and D is a certain constant matrix. Equation (4.16) is namely the direct generalization of (4.17) to the processed-ensemble $S_n(\mathbf{q})$. The quantity D is thus given by $D=\chi(0)/2$.[22] Furthermore, the asymptotic form of the generalized time-correlation function $Q_{\mathbf{q}}(n)$ [Eq. (3.22)] for $\mathbf{q}\to 0$ has an explicit interrelation with the conventional double time-correlation function of $\{\mathbf{u}(j)\}$. See Appendix A.

V. ANALOGS TO THERMODYNAMICS

In Ref. 10 we have pointed out an analog on the mathematical structures in the fluctuation-spectrum theory for univariate temporal fluctuations and in conventional equilibrium thermodynamics. The thermodynamics formalism for univariate temporal fluctuations corresponds to a hydrostatic system contained in a rigid wall, where no external work is done and whose thermodynamic state is completely determined only with temperature. When the wall is movable, we should add the system volume as another thermodynamic variable.[23] Furthermore, when particle (constituent) numbers are changeable, the particle numbers and the corresponding chemical potentials are added as thermodynamic variables. In this way as a natural extension of the analog for a univariate time-series case, we can expect a further analog for a multivariate time-series case. In this section, we first give several characteristic functions obtained by the Legendre transforms of the similarity structure function ϕ as well as their physical meanings. Then we compare the present formalism with the mathematical structure in the equilibrium thermodynamics.

Consider first, for simplicity, the bivariate time series. Let us construct a new ensemble from the original ensemble S_n. It consists of ensemble members of S_n, whose $(\alpha_n)_2$ are in between α_2 and $\alpha_2+d\alpha_2$. This is called $S_n^{(1)}\{\alpha_2\}$. Namely, all members in $S_n^{(1)}\{\alpha_2\}$ take the value $(\alpha_n)_2=\alpha_2$. Let $M_{q_1}^{(1)}\{\alpha_2|n\}$ be the average of $\exp[nq_1(\alpha_n)_1]$ taken over this ensemble:

$$M_{q_1}^{(1)}\{\alpha_2|n\}=\langle \exp[nq_1(\alpha_n)_1]\delta((\alpha_n)_2-\alpha_2)\rangle = \int_{-\infty}^{\infty}\rho_n(\alpha_1',\alpha_2)\exp(nq_1\alpha_1')d\alpha_1' , \quad (5.1)$$

where $\rho_n(\alpha_1',\alpha_2')$ $[=\rho_n(\boldsymbol{\alpha}')]$ is defined in (3.4).

We introduce a new function

$$A_1(q_1,\alpha_2)=\lim_{n\to\infty}\frac{1}{n}\ln M_{q_1}^{(1)}\{\alpha_2|n\} . \quad (5.2)$$

Inserting the asymptotic form (3.8) into (5.1) and applying the steepest-descent method, we get

$$A_1(q_1,\alpha_2)=-\min_{\alpha_1'}[\sigma(\alpha_1',\alpha_2)-q_1\alpha_1'] , \quad (5.3)$$

where $\sigma(\alpha_1',\alpha_2')$ $[=\sigma(\boldsymbol{\alpha}')]$ is the fluctuation spectrum. By introducing the quantity α_1, a function of q_1 and α_2, through

TABLE I. The characteristic (thermodynamic) functions obtained from the similarity structure function ϕ, and their Pfaffian differential forms. Each of them is a function of independent variables given in the middle column. $\sum_j'$ stands for the summation except $j=1,2$, and the curly brackets $\{q_j\}'$ and $\{\alpha_j\}'$ mean the sets of the variables q_j and α_j except $j=1$ and 2, respectively. For the meanings of the functions A_1 and A_2, see the text.

Characteristic functions	Independent variables	Pfaffian differential form
ϕ	$q_1,q_2,\{q_j\}'$	$d\phi=\alpha_1dq_1+\alpha_2dq_2+\sum_j'\alpha_jdq_j$
$\bar\phi=\phi-\Sigma_j'q_j\alpha_j$	$q_1,q_2,\{\alpha_j\}'$	$d\bar\phi=\alpha_1dq_1+\alpha_2dq_2-\sum_j'q_jd\alpha_j$
$A_1=\bar\phi-q_2\alpha_2$	$q_1,\alpha_2,\{\alpha_j\}'$	$dA_1=\alpha_1dq_1-q_2d\alpha_2-\sum_j'q_jd\alpha_j$
$A_2=\bar\phi-q_1\alpha_1$	$\alpha_1,q_2,\{\alpha_j\}'$	$dA_2=-q_1d\alpha_1+\alpha_2dq_2-\sum_j'q_jd\alpha_j$
$A_3=A_1-q_1\alpha_1$ $=A_2-q_2\alpha_2$ $=\bar\phi-q_1\alpha_1-q_2\alpha_2$ $(=-\sigma)$	$\alpha_1,\alpha_2,\{\alpha_j\}'$	$dA_3=-q_1d\alpha_1-q_2d\alpha_2-\sum_j'q_jd\alpha_j$
$\psi_1=-\dfrac{A_1}{q_1}$	$\dfrac{1}{q_1},\alpha_2,\{\alpha_j\}'$	$d\psi_1=-A_3d\left[\dfrac{1}{q_1}\right]+\dfrac{q_2}{q_1}d\alpha_2+\sum_j'\dfrac{q_j}{q_1}d\alpha_j$
$\psi_2=-\dfrac{\bar\phi}{q_1}$	$\dfrac{1}{q_1},q_2,\{\alpha_j\}'$	$d\psi_2=-A_2d\left[\dfrac{1}{q_1}\right]-\dfrac{\alpha_2}{q_1}dq_2+\sum_j'\dfrac{q_j}{q_1}d\alpha_j$

$$q_1=\frac{\partial\sigma(\alpha_1,\alpha_2)}{\partial\alpha_1}\ , \tag{5.4}$$

the function A_1 is rewritten as

$$A_1(q_1,\alpha_2)=-\sigma(\alpha_1,\alpha_2)+q_1\alpha_1$$
$$=\phi(q_1,q_2)-q_2\alpha_2\ . \tag{5.5}$$

The variable q_2 has been defined by $q_2=\partial\sigma(\alpha_1,\alpha_2)/\partial\alpha_2$ [Eq. (3.14)], and the similarity structure function is given by

$$\phi(\mathbf{q})\equiv\phi(q_1,q_2)=q_1\alpha_1+q_2\alpha_2-\sigma(\alpha_1,\alpha_2)$$

[Eq. (3.12)].

The function A_1 is therefore given by the Legendre transform of ϕ, and is relevant to the fluctuation characteristics of the "univariate" fluctuations $(\boldsymbol{\alpha}_n)_1$ in the ensemble $S_n^{(1)}\{\alpha_2\}$ [Eqs. (5.1) and (5.2)]. If the fluctuations $(\boldsymbol{\alpha}_n)_1$ and $(\boldsymbol{\alpha}_n)_2$ are statistically independent of each other, then the function A_1 is obtained as

$$A_1(q_1,\alpha_2)=A_1'(q_1)+A_1''(\alpha_2)\ ,$$

where A_1' and A_1'' are functions of q_1 and α_2, respectively. In other words, if there exists a cross term of functions of q_1 and α_2 in A_1, it evaluates the *cross correlation* between $(\boldsymbol{\alpha}_n)_1$ and $(\boldsymbol{\alpha}_n)_2$. In a similar way to the above we can define another ensemble $S_n^{(2)}\{\alpha_1\}$, and a new characteristic function $A_2(\alpha_1,q_2)$, which describes the fluctuation characteristics of $(\boldsymbol{\alpha}_n)_2$ under the constraint that $(\boldsymbol{\alpha}_n)_1$ take the value α_1.

The above discussion can be straightforwardly extended to trivariate or quadrivariate time series and so on. As was seen above, relevant characteristic (thermodynamic) functions are obtained by the Legendre transforms of the similarity structure function ϕ. In Table I, thermodynamic functions, their independent variables, and their Pfaffian differential forms are summarized. The function A_1 defined as in (5.2) is relevant to the univariate fluctuation $(\boldsymbol{\alpha}_n)_1$ in the subensemble of S_n, whose other members take values $(\boldsymbol{\alpha}_n)_j=\alpha_j$ $(j=2,3,\ldots,m)$. Namely, A_1 describes the fluctuation characteristics of $(\boldsymbol{\alpha}_n)_1$ in S_n on the condition that other members take values $(\boldsymbol{\alpha}_n)_2=\alpha_2, (\boldsymbol{\alpha}_n)_3=\alpha_3,\ldots,(\boldsymbol{\alpha}_n)_m=\alpha_m$. The function A_2 is obtained in a way similar to A_1. The function A_3 is namely the fluctuation spectrum $[A_3=-\sigma(\boldsymbol{\alpha})]$. In Table II, two possible analogs of the present approach to the equilibrium thermodynamics are summarized.[25] One easily finds that, as expected, $q_1,q_2,\ldots,q_m$ are *intensive* variables and $\alpha_1,\alpha_2,\ldots,\alpha_m$ are *extensive* ones.

TABLE II. Two possible analogs of the present formalism to the thermodynamics (Ref. 24). The parentheses (,) in the first column imply that, e.g., the entropy S is a function of U and V, and the Massieu function Ψ is a function of $1/T$ and p, etc. N_k and μ_k are the number and the chemical potential of the kth constituent, respectively.

Thermodynamic variables	Analog I	Analog II
Temperature T	$\dfrac{1}{q_1}$	q_1
Entropy S (U,V)	$-A_3(=\sigma)$	$-\alpha_1$
Pressure p (magnetic field $-h$)	$\dfrac{q_2}{q_1}$	q_2
Volume V (magnetization M)	α_2	α_2
Internal energy U (S,V)	α_1	$A_3(=-\sigma)$
Enthalpy H (S,p)	$\alpha_1+\dfrac{q_2}{q_1}\alpha_2$	A_2
Helmholtz free energy F (T,V)	$\dfrac{A_1}{q_1}(=-\psi_1)$	A_1
Gibbs free energy G (T,p)	$\dfrac{\bar{\phi}}{q_1}(=-\psi_2)$	$\bar{\phi}$
Massieu function $\Psi=-F/T$ $(1/T,V)$	$-A_1$	$-\dfrac{A_1}{q_1}(=\psi_1)$
Planck function $\Phi=-G/T$ $(1/T,p)$	$-\bar{\phi}$	$-\dfrac{\bar{\phi}}{q_1}(=\psi_2)$
Particle numbers $\{N_k\}$	$\{\alpha_j\}'$	$\{\alpha_j\}'$
Chemical potentials $\{\mu_k\}$	$\left\{-\dfrac{q_j}{q_1}\right\}'$	$\{-q_j\}'$

VI. EIGENVALUE PROBLEM FOR MARKOV PROCESSES

Let $\{\mathbf{x}_n\}$ obey a stochastic process described by the master equation

$$P_{n+1}(\mathbf{x})=\int_{-\infty}^{\infty}T(\mathbf{x},\mathbf{x}')P_n(\mathbf{x}')d\mathbf{x}'$$
$$\equiv HP_n(\mathbf{x})\ , \tag{6.1}$$

where $P_n(\mathbf{x})$ is the probability density that $\mathbf{x}_n$ takes a value in between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$. $T(\mathbf{x},\mathbf{x}')$ is the transition probability density from the $\mathbf{x}'$ state to the $\mathbf{x}$ state in unit step with the normalization condition

$$\int_{-\infty}^{\infty}T(\mathbf{x},\mathbf{x}')d\mathbf{x}=1\ . \tag{6.2}$$

H is the master operator. We consider the case where $\mathbf{u}(n)$ evolves through the dynamical variable $\mathbf{x}_n$, i.e., $\mathbf{u}(n)$ is a unique function of $\mathbf{x}_n$ as

$$\mathbf{u}(n)=\mathbf{u}\{\mathbf{x}_n\}\ . \tag{6.3}$$

With the steady density $P_*(\mathbf{x})$ satisfying

$$P_*(\mathbf{x})=HP_*(\mathbf{x}) , \tag{6.4}$$

the function $M_{\mathbf{q}}(n)$ can be written as

$$M_{\mathbf{q}}(n)=\int_{-\infty}^{\infty}[H(\mathbf{q})]^n P_*(\mathbf{x})d\mathbf{x} , \tag{6.5}$$

where $H(\mathbf{q})$ is the linear operator defined by

$$H(\mathbf{q})F(\mathbf{x})\equiv H[e^{\mathbf{q}\cdot\mathbf{u}\{\mathbf{x}\}}F(\mathbf{x})] = \int_{-\infty}^{\infty} T(\mathbf{x},\mathbf{x}')e^{\mathbf{q}\cdot\mathbf{u}\{\mathbf{x}'\}}F(\mathbf{x}')d\mathbf{x}' \tag{6.6}$$

$[H=H(0)]$.[26–28]

Let us consider the eigenvalue problem of $H(\mathbf{q})$,[16]

$$H(\mathbf{q})\psi_{\mathbf{q}}^{(l)}(\mathbf{x})=\exp(\phi_{\mathbf{q}}^{(l)})\psi_{\mathbf{q}}^{(l)}(\mathbf{x}) , \tag{6.7}$$

where $\exp[\phi_{\mathbf{q}}^{(l)}]$ and $\psi_{\mathbf{q}}^{(l)}(\mathbf{x})$ are the lth eigenvalue and the corresponding eigenfunction, respectively $(l=0,1,2,\ldots)$. By assuming the completeness of the set $\{\psi_{\mathbf{q}}^{(l)}(\mathbf{x})\}$, the invariant probability density $P_*(\mathbf{x})$ is uniquely expanded as

$$P_*(\mathbf{x})=\sum_l k_{\mathbf{q}}^{(l)}\psi_{\mathbf{q}}^{(l)}(\mathbf{x}) , \tag{6.8}$$

where $\{k_{\mathbf{q}}^{(l)}\}$ are expansion coefficients. The insertion of (6.8) into (6.5) leads to

$$M_{\mathbf{q}}(n)=\sum_l J_{\mathbf{q}}^{(l)}\exp(\phi_{\mathbf{q}}^{(l)}n) , \tag{6.9}$$

where

$$J_{\mathbf{q}}^{(l)}=k_{\mathbf{q}}^{(l)}\int_{-\infty}^{\infty}\psi_{\mathbf{q}}^{(l)}(\mathbf{x})d\mathbf{x} . \tag{6.10}$$

By noting that the quantity $\max_l[|\exp(\phi_{\mathbf{q}}^{(l)})|]$ $\{=\exp[\max_l(\mathrm{Re}\phi_{\mathbf{q}}^{(l)})]\}$ is not degenerate,[16,29] the similarity structure function can be set as

$$\phi(\mathbf{q})=\max_l(\mathrm{Re}\phi_{\mathbf{q}}^{(l)})\equiv\phi_{\mathbf{q}}^{(0)} . \tag{6.11}$$

Equation (3.15) implies $\phi_0^{(0)}=0$, which corresponds to the "eigenvalue equation" (6.4).

The generalized time-correlation function $Q_{\mathbf{q}}(n)$ defined in (3.22) thus turns out to be expanded as (3.25), where

$$\omega_{\mathbf{q}}^{(l)}=-\mathrm{Im}(\phi_{\mathbf{q}}^{(l)}) , \tag{6.12}$$

$$\gamma_{\mathbf{q}}^{(l)}=\phi(\mathbf{q})=\mathrm{Re}(\phi_{\mathbf{q}}^{(l)}) \quad (>0) . \tag{6.13}$$

The symbol $\sum_l'$ in (3.25) means the summation, except $l=0$. The Fourier transform $\Xi_{\mathbf{q}}(\omega)$ [Eq. (3.24)] is written as

$$\Xi_{\mathbf{q}}(\omega)=\mathrm{Re}\sum_l{}' \frac{J_{\mathbf{q}}^{(l)}}{1-\exp[i(\omega-\omega_{\mathbf{q}}^{(l)})-\gamma_{\mathbf{q}}^{(l)}]} . \tag{6.14}$$

The poles of $\Xi_{\mathbf{q}}(\omega)$ describe the characteristic frequencies $\{\omega_{\mathbf{q}}^{(l)}\}$ and the damping rates $\{\gamma_{\mathbf{q}}^{(l)}\}$ of motions embedded in $\{\mathbf{u}(n)\}$.[16]

When $\{\mathbf{u}(n)\}$ consists of discrete values, more detailed analyses are given in Appendix B, and simple examples are illustrated in Secs. VIII and IX.

VII. CONTINUED-FRACTION EXPANSION OF $M_{\mathbf{q}}(n)$

The eigenfunction expansion in the preceding sections is possible only when the generation law of $\{\mathbf{u}(n)\}$ is known. Even when the generation law is known, however, usually it is difficult to solve the eigenvalue problem. Furthermore, when only the numerical data are available, such an approach is meaningless. To dissolve these embarrassments, we can employ the continued-fraction expansion for $M_{\mathbf{q}}(n)$.

Let the characteristic function

$$(n)_0\equiv M_{\mathbf{q}}(n) \tag{7.1}$$

obey the equation of motion

$$(n+1)_0=(\hat{1})_0(n)_0+\sum_{j=0}^{n-1}(n-1-j)_1(j)_0 , \tag{7.2}$$

where $(\hat{1})_0\equiv(1)_0/(0)_0$ and $(n)_1$ is the memory kernel.[30] Equation (7.2) can be regarded as the definition of the memory kernel $(n)_1$. Assume that $(n)_1$ obeys the equation of motion

$$(n+1)_1=(\hat{1})_1(n)_1+\sum_{j=0}^{n-1}(n-1-j)_2(j)_1 , \tag{7.3}$$

where $(\hat{1})_1=(1)_1/(0)_1$. $(n)_2$ is a new memory kernel.

Repeating this procedure successively, we get the set of functions $\{(n)_k,\ k=0,1,2,\ldots\}$, where the kth memory function $(n)_k$ obeys

$$(n+1)_k=(\hat{1})_k(n)_k+\sum_{j=0}^{n-1}(n-1-j)_{k+1}(j)_k , \tag{7.4}$$

with $(\hat{1})_k=(1)_k/(0)_k$. This generates a new memory kernel $(j)_{k+1}$. By introducing the Laplace transform

$$[\mu]_k\equiv\sum_{n=0}^{\infty}(n)_k\mu^n , \tag{7.5}$$

Eq. (7.4) is solved to yield

$$[\mu]_k=\frac{(0)_k}{1-\mu(\hat{1})_k-\mu^2[\mu]_{k+1}} . \tag{7.6}$$

Therefore the Laplace transform

$$M_{\mathbf{q}}[\mu]\equiv\sum_{n=0}^{\infty}M_{\mathbf{q}}(n)\mu^n=[\mu]_0 \tag{7.7}$$

is expanded as[17]

$$M_{\mathbf{q}}[\mu]=\cfrac{1}{1-\mu(\hat{1})_0-\cfrac{\mu^2(0)_1}{1-\mu(\hat{1})_1-\cfrac{\mu^2(0)_2}{1-\mu(\hat{1})_2-\cfrac{\mu^2(0)_3}{1-\mu(\hat{1})_3-\ddots}}}} . \tag{7.8}$$

The expansion coefficients are given by the quantities $\{(0)_k,(1)_k,\ k=0,1,2,\ldots\}$. Since the kth step characteristic function $(n)_k$ is successively determined by one-step lower ones as

$$(n)_{k+1}=\frac{1}{(0)_k}\left[(n+2)_k-(\hat{1})_k(n+1)_k-\sum_{j=0}^{n-1}(j)_{k+1}(n-j)_k\right], \tag{7.9}$$

$(n)_k$ can be expressed in terms of quantities $(1)_0,(2)_0,\ldots,(n+2k)_0$. Hence the coefficients $(0)_k$ and $(1)_k$ are determined by the sets of quantities $\{(j)_0, j=1,2,3,\ldots,2k\}$ $\{(j)_0, j=1,2,3,\ldots,2k+1\}$, respectively. They are easily numerically obtained.

Let $\{\mu_{\mathbf{q}}^{(l)}\}$ be the set of poles of the expansion (7.8). On the assumption that $M_{\mathbf{q}}[\mu]$ is expanded as

$$M_{\mathbf{q}}[\mu]=\sum_l \frac{J_q^{(l)}}{1-\mu/\mu_{\mathbf{q}}^{(l)}}, \tag{7.10}$$

$\{J_{\mathbf{q}}^{(l)}\}$ being the expansion coefficients, its inverse Laplace transform yields[17]

$$M_{\mathbf{q}}(n)=\sum_l J_{\mathbf{q}}^{(l)}(\mu_{\mathbf{q}}^{(l)})^{-n}. \tag{7.11}$$

Since $M_{\mathbf{q}}(n)$ is positive, the closest pole to the origin is not degenerate and is positive. So, by setting

$$\mu_{\mathbf{q}}^{(0)}=\min_l(|\mu_{\mathbf{q}}^{(l)}|), \tag{7.12}$$

the similarity structure function is obtained as

$$\phi(\mathbf{q})=\ln(\mu_{\mathbf{q}}^{(0)})^{-1} \tag{7.13}$$

($\mu_0^{(0)}=1$). Since we are dealing with the same problem as in Sec. IV provided that the process is Markovian, the result (7.11) should coincide with (6.9). Arranging the order in $[\exp(\phi_{\mathbf{q}}^{(l)})]$ and $\{\mu_{\mathbf{q}}^{(l)}\}$, we obtain

$$(\mu_{\mathbf{q}}^{(l)})^{-1}=\exp(\phi_{\mathbf{q}}^{(l)}). \tag{7.14}$$

For the evaluation of the continued-fraction expansion from a practical standpoint, it should be discarded at a finite order. Simple approximation methods are mentioned in Appendix C.

VIII. EXAMPLE 1: THREE-VALUED BERNOULLI PROCESS

The first example is the bivariate discrete-value time series in Sec. I, generated by the three-valued Bernoulli process. The matrix $H(\mathbf{q})$ (Appendix B) has the element

$$H_{lk}(\mathbf{q})=p_l\exp[\mathbf{q}\cdot\mathbf{r}^{(k)}], \tag{8.1}$$

where

$$\mathbf{r}^{(1)}=\begin{bmatrix}\ln p_1^{-1}\\ p_1/2\end{bmatrix},\quad \mathbf{r}^{(2)}=\begin{bmatrix}\ln p_2^{-1}\\ p_1+p_2/2\end{bmatrix},\quad \mathbf{r}^{(3)}=\begin{bmatrix}\ln p_3^{-1}\\ p_1+p_2+p_3/2\end{bmatrix}. \tag{8.2}$$

The matrix $\mathbf{B}$ [Eq. (B24)] is given by

$$B=\begin{bmatrix}\ln(p_3/p_1) & -(1+p_2)/2\\ \ln(p_3/p_2) & -(1-p_1)/2\end{bmatrix}. \tag{8.3}$$

The eigenvalues of the generalized evolution matrix $H(\mathbf{q})$ are two zeros and one positive value. The positive eigenvalue determines the similarity structure function $\phi(q_1,q_2)$ $[\equiv\phi(\mathbf{q})]$ as

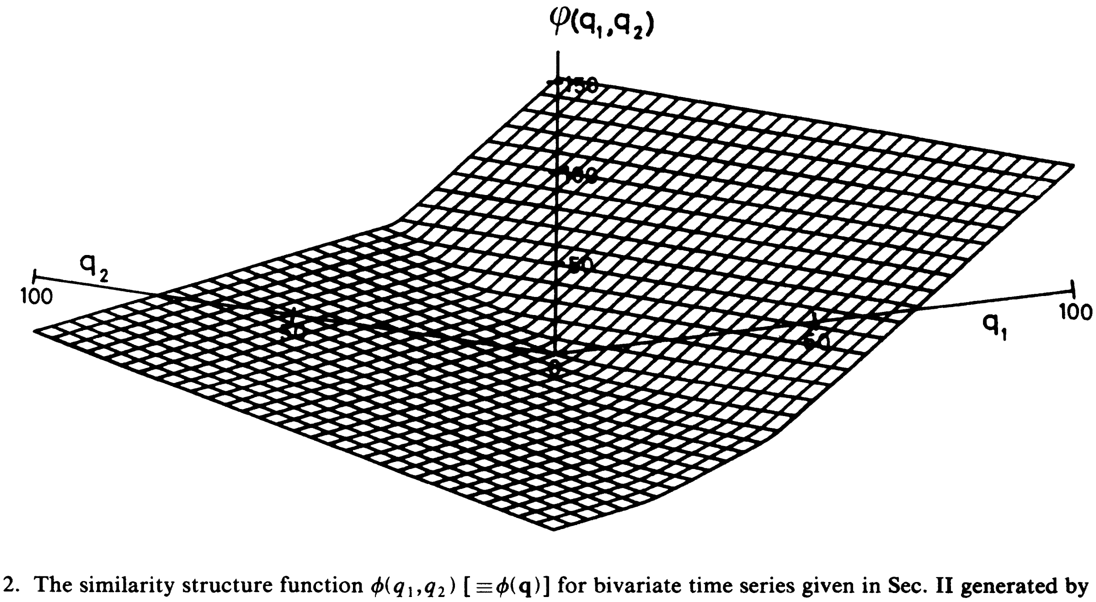

FIG. 2. The similarity structure function $\phi(q_1,q_2)$ $[\equiv\phi(\mathbf{q})]$ for bivariate time series given in Sec. II generated by the three-valued Bernoulli process. Analytic expression is given in (8.4). The parameter values are $p_1=0.1$, $p_2=0.3$, and $p_3=1-p_1-p_2=0.6$.

$$\phi(q_1,q_2)=\ln\left\{p_1^{1-q_1}\exp\left[\frac{q_2p_1}{2}\right]+p_2^{1-q_1}\exp\left[q_2\left[p_1+\frac{p_2}{2}\right]\right]+p_3^{1-q_1}\exp\left[q_2\left[p_1+p_2+\frac{p_3}{2}\right]\right]\right\}. \tag{8.4}$$

One can immediately derive the thermodynamic variables $\boldsymbol{\alpha}$, A_1, σ, etc., from (8.4). Especially the long-time average of $\mathbf{u}(n)$ is given by

$$\begin{bmatrix}\alpha_1(0)\\ \alpha_2(0)\end{bmatrix}=\begin{bmatrix}p_1\ln p_1^{-1}+p_2\ln p_2^{-1}+p_3\ln p_3^{-1}\\ \frac{1}{2}\end{bmatrix}. \tag{8.5}$$

The thermodynamic variables derived from (8.4) are drawn in Figs. 2–5.

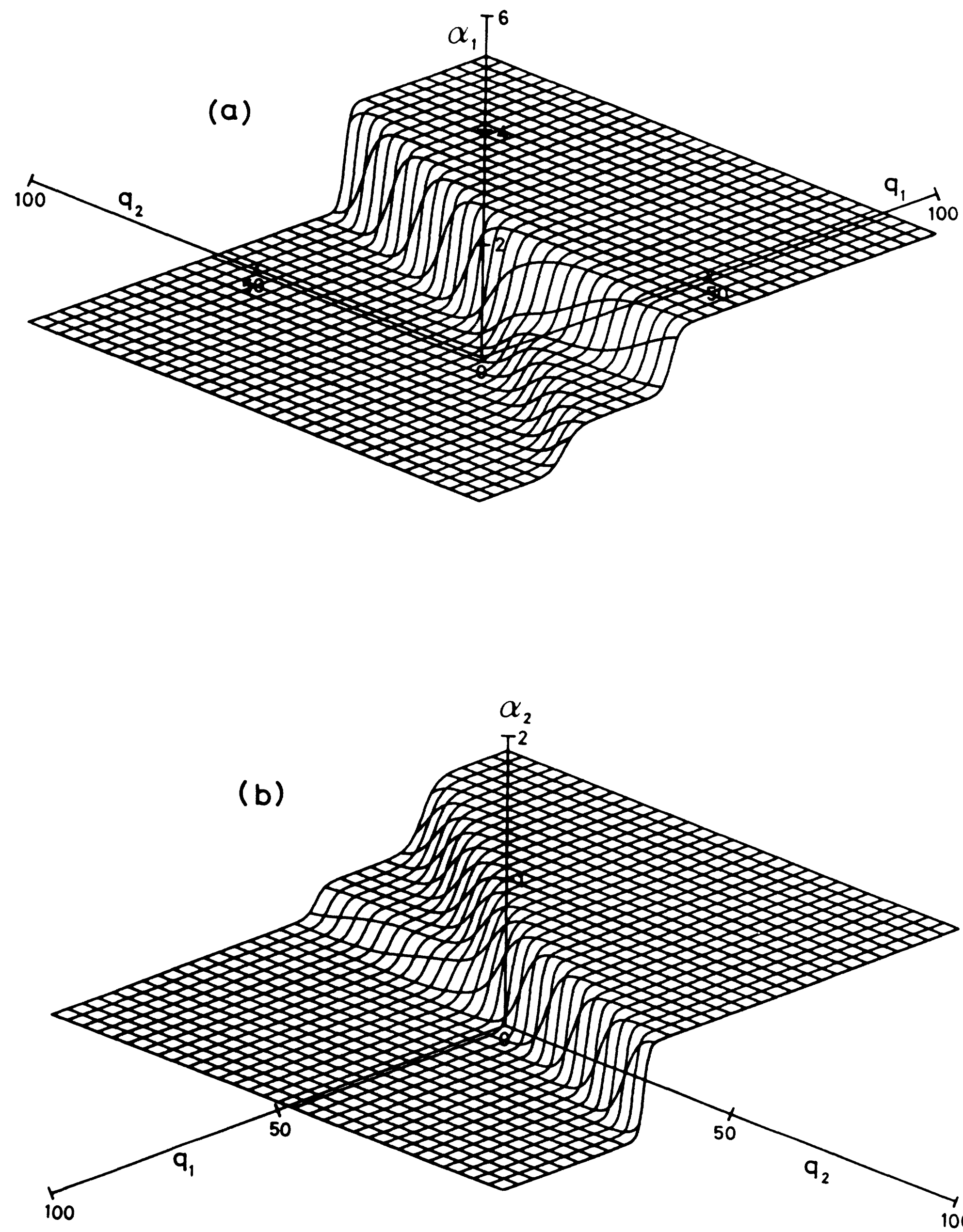

FIG. 3. The averages α_1 and α_2 of the fluctuations $\{u_1(n)\}$ and $\{u_2(n)\}$ over the processed ensemble $S_n(\mathbf{q})$. These are derived from $\phi(q_1,q_2)$ in Fig. 2 with the definition (3.11).

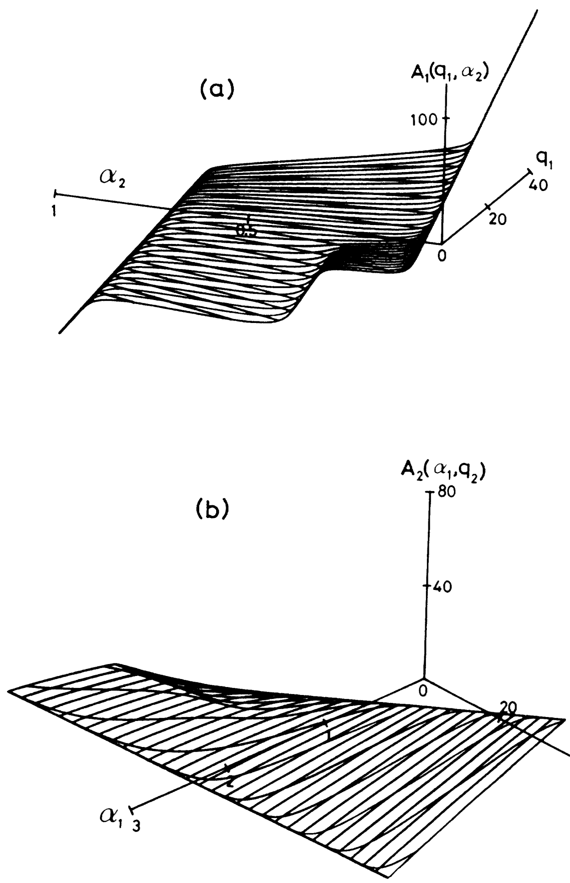

FIG. 4. The characteristic functions $A_1(q_1,\alpha_2)$ and $A_2(\alpha_1,q_2)$ derived from $\phi(q_1,q_2)$ in Fig. 2 through the Legendre transforms [Eq. (5.5) and Table I].

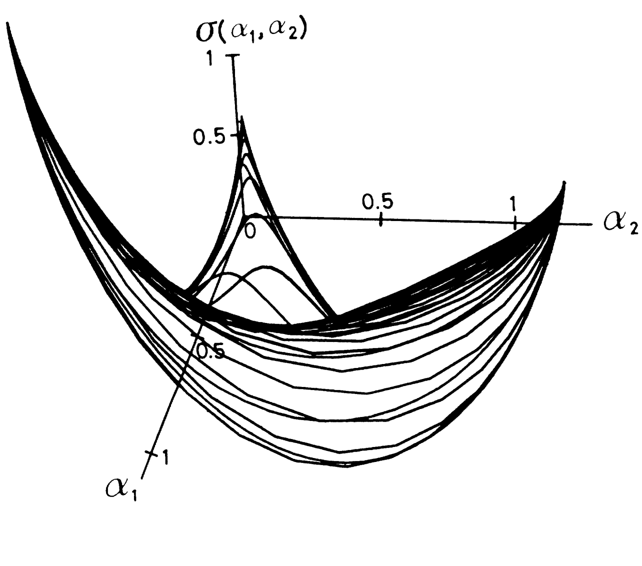

FIG. 5. The fluctuation spectrum σ corresponding to the similarity structure function in Fig. 2.

IX. EXAMPLE 2: A STOCHASTIC MODEL

The next concrete system is the three-state Markovian discrete process (B1) with the evolution matrix

$$H=\begin{bmatrix} h' & r & 0 \\ h & r' & h \\ 0 & r & h' \end{bmatrix} \tag{9.1}$$

($h'=1-h$, $r'=1-2r$, $0<h<1$, $0<r<\frac{1}{2}$). The invariant probability distribution is immediately obtained as

$$\mathbf{P}_*=\frac{1}{2r+h}\begin{bmatrix} r \\ h \\ r \end{bmatrix} . \tag{9.2}$$

We consider the bivariate time series $u_1(n),u_2(n)$, generated by the above process.

Noting that this process has a symmetry between the first and the third states, let the first time series $u_1(n)$ consist of three values as

$$u_1(n)=\begin{cases} -1 \\ 0 \\ 1 \end{cases} . \tag{9.3}$$

Namely, $u_1(n)$ takes -1, 0, and $+1$ if the state at the step n is in the first, second, and third states, respectively.[31] Recalling that the time series $\{u_1(n)\}$ is symmetric under the inversion of the sign, we define the new time series $\{u_2(n)\}$ by

$$u_2(n)=\begin{cases} 0 \\ 0 \\ 1 \end{cases} . \tag{9.4}$$

This is the "symmetry-breaking" time series, and is independent of $\{u_1(n)\}$. So the combined time series $\{\mathbf{u}(n)\}$ take three values as

$$\mathbf{r}^{(1)}=\begin{bmatrix} -1 \\ 0 \end{bmatrix},\quad \mathbf{r}^{(2)}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\quad \mathbf{r}^{(3)}=\begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{9.5}$$

For the above bivariate time series, the extended evolution matrix is written as

$$H(\mathbf{q})=\begin{bmatrix} h'e^{-q_1} & r & 0 \\ he^{-q_1} & r' & he^{q_1+q_2} \\ 0 & r & h'e^{q_1+q_2} \end{bmatrix} . \tag{9.6}$$

The eigenvalues of $H(\mathbf{q})$ are solutions of

$$\zeta^3-a_{\mathbf{q}}\zeta^2+b_{\mathbf{q}}\zeta-c_{\mathbf{q}}=0 , \tag{9.7}$$

where

$$a_{\mathbf{q}}=r'+h'(e^{q_1+q_2}+e^{-q_1}) , \tag{9.8a}$$

$$b_{\mathbf{q}}=h'^2e^{q_2}+(r'h'-rh)(e^{q_1+q_2}+e^{-q_1}) , \tag{9.8b}$$

$$c_{\mathbf{q}}=h'(r'-h)e^{q_2} . \tag{9.8c}$$

In comparison with the chaos-induced diffusion,[32] h is

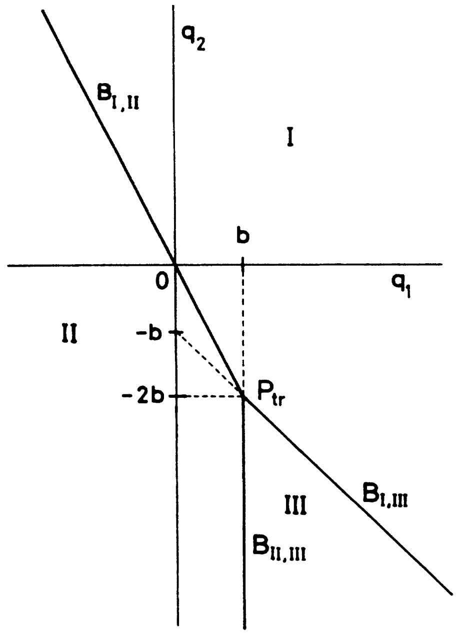

FIG. 6. Phase diagram for bivariate time series generated by the stochastic model with (9.1) for $h \to 0$. There exist three thermodynamic phases I, II, and III, which are clearly separated by boundaries $B_{I,II}(2q_1+q_2=0)$, $B_{I,III}(q_1+q_2=-b)$ and $B_{II,III}(q_1=b)$, where $b=-\ln r'$. These phases merge together at the tricritical point $P_{tr}(b,-2b)$. The thermodynamic quantities in each phase are summarized in Table III. The susceptibilities $\{\chi_{\mu\nu}(\mathbf{q})\}$ except on three boundaries all vanish. The χ_{11}, χ_{12}, and χ_{22} diverge on the boundaries $B_{I,II}$ and $B_{I,III}$. On $B_{II,III}$, χ_{11} diverges, and χ_{12} and χ_{22} vanish.

hereafter supposed to be appropriately small and to satisfy $r'>h$ ($c_{\mathbf{q}}>0$). The lth eigenvector of $H(\mathbf{q})$ is obtained as

$$\boldsymbol{\psi}_{\mathbf{q}}^{(l)}=\begin{pmatrix} r(\zeta_{\mathbf{q}}^{(l)}-h'e^{q_1+q_2}) \\ (\zeta_{\mathbf{q}}^{(l)}-h'e^{q_1+q_2})(\zeta_{\mathbf{q}}^{(l)}-h'e^{-q_1}) \\ r(\zeta_{\mathbf{q}}^{(l)}-h'e^{-q_1}) \end{pmatrix}, \tag{9.9}$$

TABLE III. Relevant quantities for the bivariate time series for the stochastic model with the evolution matrix (9.1) ($h \to 0$). The $\gamma_{\mathbf{q}}^{(1)}$ and $\gamma_{\mathbf{q}}^{(2)}$ are damping rates appearing in $Q_{\mathbf{q}}(n)$ [Eq. (9.10)]. These exhibit the critical slowing-down ($\gamma_{\mathbf{q}} \to 0$) near the phase boundaries. The notation $2q_1+q_2(B_{I,II})$ in I implies that the damping rate $2q_1+q_2$ in phase I undergoes critical slowing down near the boundary $B_{I,II}$. Near the tricitical point $P_{tr}(b,-2b)$, both $\gamma_{\mathbf{q}}^{(1)}$ and $\gamma_{\mathbf{q}}^{(2)}$ exhibit critical slowing down.

	$\phi(q_1,q_2)$	α_1	α_2	$\gamma_{\mathbf{q}}^{(1)},\gamma_{\mathbf{q}}^{(2)}$
I	q_1+q_2	1	1	$q_1+q_2+b(B_{I,III})$ $2q_1+q_2(B_{I,II})$
II	$-q_1$	-1	0	$-2q_1-q_2(B_{I,II})$ $-q_1+b(B_{II,III})$
III	$-b$	0	0	$q_1-b(B_{II,III})$ $-q_1-q_2-b(B_{I,III})$

$\zeta_{\mathbf{q}}^{(l)}$ [$=\exp(\phi_{\mathbf{q}}^{(l)})$] being the corresponding eigenvalue ($l=0,1,2$).

Since we are interested mainly in the small-h case, we first take the limit $h \to 0$. Equation (9.7) is immediately solved to yield e^{-q_1}, $e^{q_1+q_2}$, and r'. Comparing these eigenvalues with each other, one can determine the similarity structure function $\phi(\mathbf{q})$ and the generalized time-correlation function $Q_{\mathbf{q}}(n)$. On the q_1-q_2 plane there appear thermodynamic phases I, II, and III, as is shown in Fig. 6. The quantities characteristic of these phases are summarized in Table III.[33]

For a finite but sufficiently small h, all the eigenvalues are real when $\mathbf{q}$ is in the region of same order as in Fig. 6. One should note that when (9.7) has three real solutions, all of them are positive. Let $\zeta_{\mathbf{q}}^{(0)}$ be the largest eigenvalue. One gets $\phi(\mathbf{q})=\ln\zeta_{\mathbf{q}}^{(0)}$ and

$$Q_{\mathbf{q}}(n)=J_{\mathbf{q}}^{(0)}+J_{\mathbf{q}}^{(1)}\exp(-\gamma_{\mathbf{q}}^{(1)}n)+J_{\mathbf{q}}^{(2)}\exp(-\gamma_{\mathbf{q}}^{(2)}n), \tag{9.10}$$

where

$$\gamma_{\mathbf{q}}^{(l)}=\ln(\zeta_{\mathbf{q}}^{(0)}/\zeta_{\mathbf{q}}^{(l)})(>0) \quad (l=1,2). \tag{9.11}$$

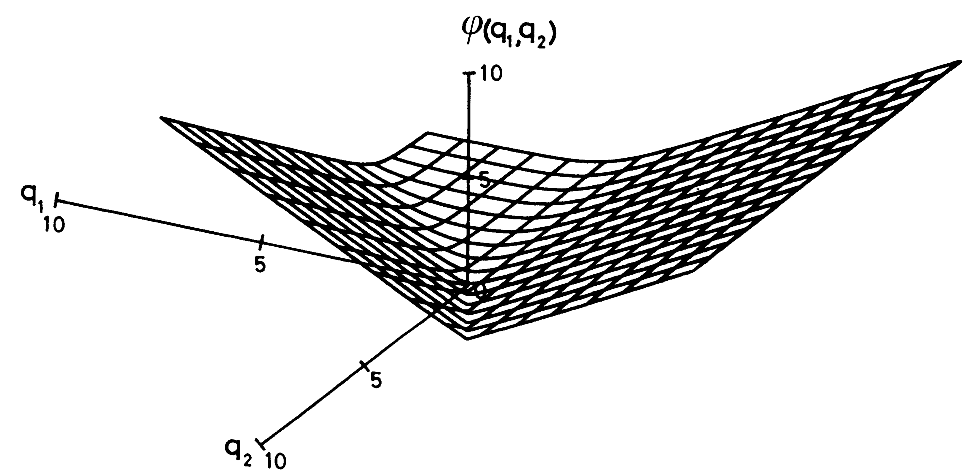

FIG. 7. The similarity structure function $\phi(q_1,q_2)$ [$\equiv\phi(\mathbf{q})$] for bivariate time series generated by the stochastic model in Sec. IX for a finite but small h. The $\phi(\mathbf{q})$ is determined by the largest eigenvalue of the cubic equation (9.7). Since h is finite, the boundaries among three characteristic phases are unclear. We set $h=0.25$ and $r=0.3$ ($r'=1-2r=0.4$ and $b\equiv-\ln r'\simeq0.916$).

Figure 7 is the similarity structure function $\phi(q_1,q_2)$ $[\equiv\phi(\mathbf{q})]$ for $h=0.25$ and $r=0.3$. The thermodynamic quantities derived from $\phi(q_1,q_2)$ are drawn in Figs. 7–10. Furthermore, the damping rates $\gamma_{\mathbf{q}}^{(1)}$ and $\gamma_{\mathbf{q}}^{(2)}$ are illustrated in Fig. 11.

X. CONCLUDING REMARKS

In this paper we proposed the fluctuation-spectrum theory and the generalized time-correlation function theory extended to multivariate temporal fluctuations. This is based on the introduction of the concept, the *complete set of temporal fluctuations (time series)* for the

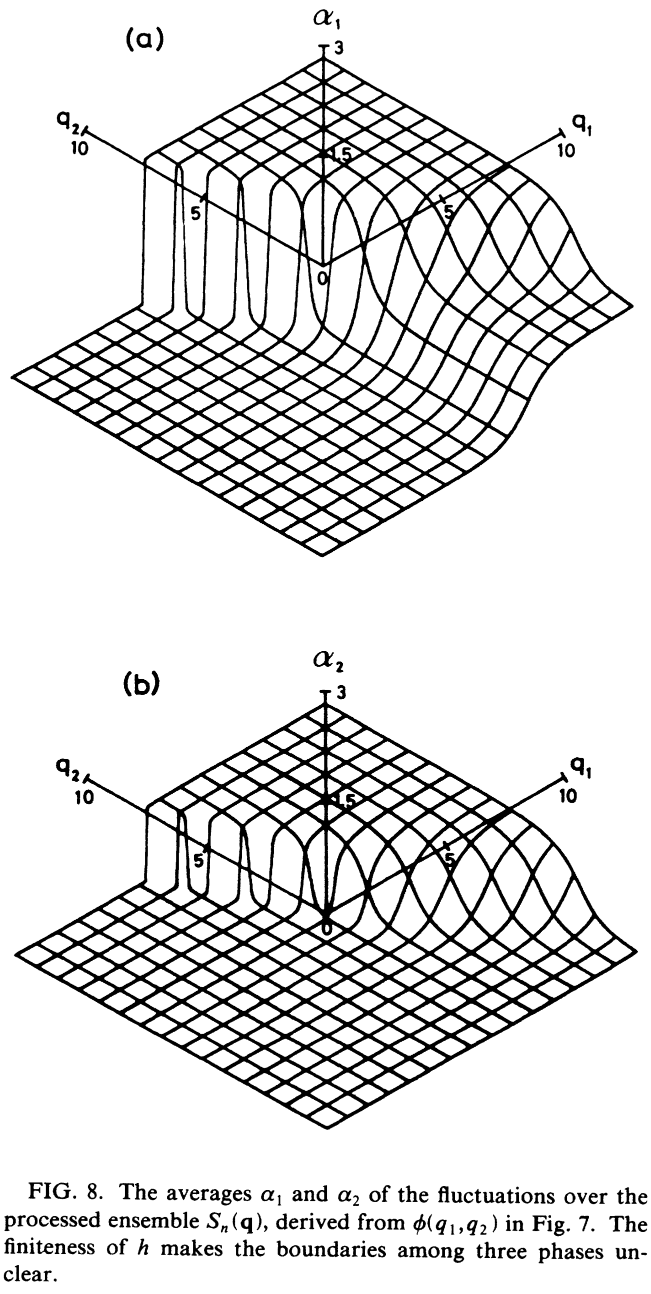

FIG. 8. The averages α_1 and α_2 of the fluctuations over the processed ensemble $S_n(\mathbf{q})$, derived from $\phi(q_1,q_2)$ in Fig. 7. The finiteness of h makes the boundaries among three phases unclear.

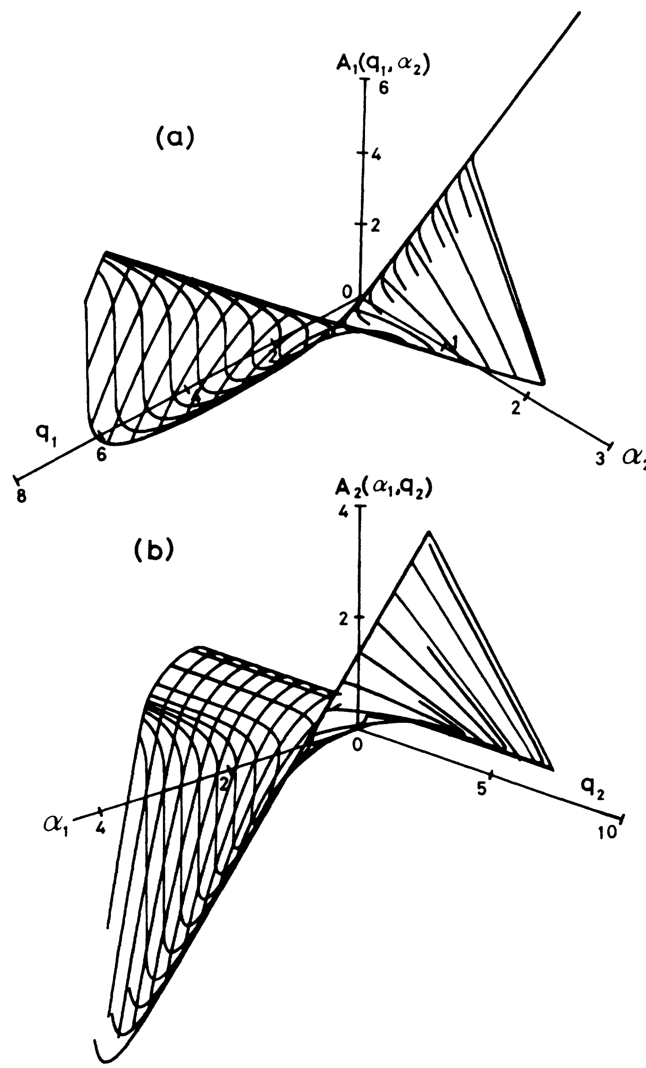

FIG. 9. The characteristic functions $A_1(q_1,\alpha_2)$ and $A_2(\alpha_1,q_2)$ derived from $\phi(q_1,q_2)$ in Fig. 7 through the Legendre transforms.

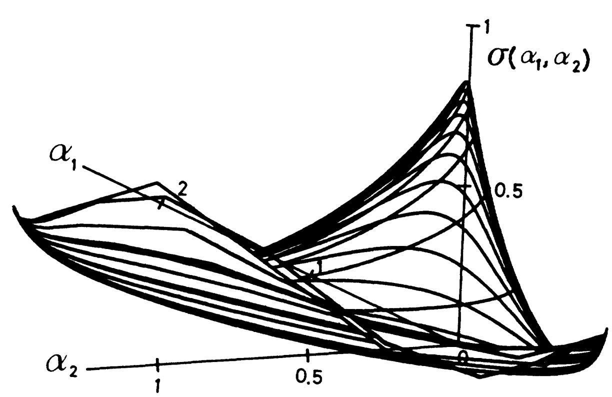

FIG. 10. The fluctuation spectrum σ derived from the similarity structure function ϕ in Fig. 7.

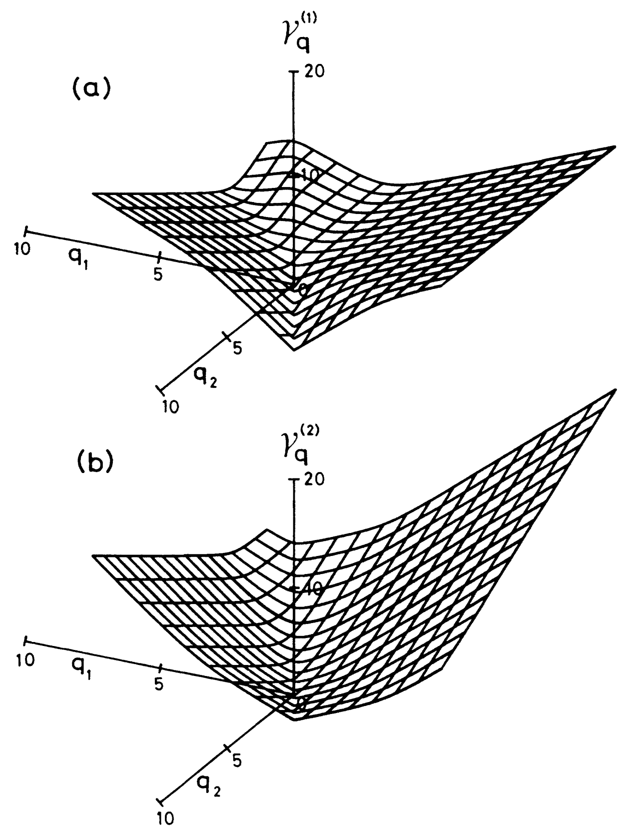

FIG. 11. The q dependences of the damping rates $\gamma_{\mathbf{q}}^{(1)}$ and $\gamma_{\mathbf{q}}^{(2)}$ for the model (9.1) [Eq. (9.11)], where the parameter values are the same as in Fig. 7. The damping rates in the conventional double-time-correlation function are given by $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$.

overall description of the dynamical behaviors of the system under consideration. This new concept is clearly the extension of the set of thermodynamic variables in the equilibrium thermodynamics. One should remark that to consider the complete set of temporal fluctuations and to study temporal correlation in a univariate time series are quite different subjects (Sec. II). The main part of the present approach can be straightforwardly extended to spatially homogeneous fluctuations.

A global characterization partially similar to that in Sec. III for the fluctuations of local expansion rates in two-dimensional chaotic maps has been discussed in Ref. 34. Furthermore, very recently Honda and Matsushita[35] proposed a theory to obtain the complete spectra of singularities by introducing a new variable to distinguish strange sets. Their study implies the necessity of an additional variable for a more precise description of strange sets. In this way we stand at the stage where we can expect the new development of the multifractal theory and the fluctuation-spectrum theory by observing multivariate fluctuations.

The most remarkable aspect of the present approach in comparing with conventional analyses of multivariate fluctuations is that the variable **q**, which is not contained in the time series $\{\mathbf{u}(n)\}$, plays the central role. The variable **q** singles out various statistical characteristics embedded in multivariate temporal fluctuations. This is the extension of the filtering parameter concept found for a univariate time series. The statistical dynamics for $|\mathbf{q}| \ll \kappa$ and $|\mathbf{q}| \gg \kappa$, κ being the convergence radius of the cumulant expansion of $\phi(\mathbf{q})$, are usually quite different from each other. The study of the overall dynamical behaviors of the physical system turns out to be possible by observing the statistical characteristics in the whole regimes of **q**. This immediately leads to the phase diagram for temporal fluctuations. Simple examples are given in Secs. VIII and IX.

The present approach also enables us to evaluate the explicit cross correlation among several time series. A brief discussion for the cross correlation has been given in Sec. V. Here we give an alternative method for studying it. Let us define the similarity structure function for the μth time series by

$$\phi_\mu(q) \equiv \lim_{n\to\infty} \frac{1}{n} \ln\left\langle \exp\left[q \sum_{j=1}^{n} u_\mu(j) \right] \right\rangle . \tag{10.1}$$

This is determined by the similarity structure function ϕ in (3.7) as

$$\phi_\mu(q) = \phi(q\hat{\mathbf{e}}^\mu) , \tag{10.2}$$

where $\hat{\mathbf{e}}^\mu$ is the unit vector with the νth component $(\hat{\mathbf{e}}^\mu)_\nu = \delta_{\mu\nu}$. We define the function $\phi^{\rm corr}(\mathbf{q})$ by

$$M_{\mathbf{q}}(n) \sim \left[\prod_{\mu=1}^{m} \left\langle \exp\left[q_\mu \sum_{j=1}^{n} u_\mu(j) \right] \right\rangle \right] \exp[\phi^{\rm corr}(\mathbf{q}) n] , \tag{10.3}$$

i.e.,

$$\phi^{\rm corr}(\mathbf{q}) = \phi(\mathbf{q}) - \sum_{\mu=1}^{m} \phi(q_\mu \hat{\mathbf{e}}^\mu) . \tag{10.4}$$

If components $(\boldsymbol{\alpha}_n)_1, (\boldsymbol{\alpha}_n)_2, \ldots, (\alpha_n)_m$ are statistically independent of each other, $\phi^{\rm corr}(\mathbf{q})$ vanishes. So $\phi^{\rm corr}(\mathbf{q})$ turns out to measure the explicit cross correlation, more precisely speaking, the global cross correlation, among multivariate time series.

As shown in Refs. 36 and 37, the thermodynamics variables and the generalized time-correlation function obey the static and dynamic scaling laws for univariate time series for the relevant variables near several chaotic transition points. This suggests that in the case of multivariate time series we may also expect the static and dynamic scaling laws for $\phi(\mathbf{q})$, $\Xi_{\mathbf{q}}(\omega)$, etc., near the transition points. Such studies are planned to be reported elsewhere.

The present formalism has a close relation with other statistical-mechanical formalisms for the global characterization of relevant fluctuations.[4] Recently, Crutchfield and Young[38] proposed another similar formalism for information processing complexity of nonlinear dynamical systems by introducing a measure of complexity different from the information-theoretic entropies and dimensions. Thus it seems that the statistical-thermodynamics approach is able to be the powerful tool for the global characterization of temporal fluctuations as in nonlinear-dynamical systems.

ACKNOWLEDGMENTS

The authors are grateful to Professor H. Mori and Professor T. Yamada for continuous encouragement and valuable discussions. They also thank Professor K. Honda for sending us Ref. 35 prior to publication.

APPENDIX A: INTERRELATION BETWEEN THE CONVENTIONAL DOUBLE-TIME-CORRELATION FUNCTION AND THE $\mathbf{q}\to 0$ STATISTICS OF $M_{\mathbf{q}}(n)$

The symmetrized double-time-correlation function defined by

$$C_{\mu\nu}(j-l)\equiv\tfrac{1}{2}[\langle \delta u_\mu(j)\delta u_\nu(l)\rangle+\langle \delta u_\nu(j)\delta u_\mu(l)\rangle] \qquad (A1)$$

$[\delta u_\mu \equiv u_\mu - \langle u_\mu\rangle = u_\mu - \alpha_\mu(0)]$, is an even function of the time difference $j-l$, because of the stationarity of the time series.

It is an easy task to see the relations, for $n\geq 1$,

$$\begin{aligned}C_{\mu\nu}(n)&=\Delta_n \lim_{\mathbf{q}\to 0}\frac{\partial^2 \ln M_{\mathbf{q}}(n)}{\partial q_\mu \partial q_\nu}\\&=\Delta_n \lim_{\mathbf{q}\to 0}\frac{\partial^2 \ln Q_{\mathbf{q}}(n)}{\partial q_\mu \partial q_\nu}\\&=\Delta_n[n\chi_{\mu\nu}(\mathbf{q}\to \mathbf{0},n)]\\&=\Delta_n[n^2 v_{\mu\nu}(\mathbf{q}\to\mathbf{0},n)]\ ,\end{aligned} \qquad (A2)$$

With the linear operator Δ_n defined through

$$\Delta_n G(n)\equiv[G(n+1)-2G(n)+G(n-1)]/2\ ,$$

and

$$\begin{aligned}\frac{1}{n}\lim_{\mathbf{q}\to 0}\frac{\partial^2 \ln M_{\mathbf{q}}(n)}{\partial q_\mu \partial q_\nu}&=\chi_{\mu\nu}(\mathbf{q}\to\mathbf{0},n)\\&=n v_{\mu\nu}(\mathbf{q}\to\mathbf{0},n)\\&=\frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{n}C_{\mu\nu}(j-l)\ .\end{aligned} \qquad (A3)$$

In this sense $C_{\mu\nu}(n)$ and the $\mathbf{q}\to\mathbf{0}$ statistics of $M_{\mathbf{q}}(n)$ are equivalent to each other.[18]

We first note that on the assumption of the expansion (3.25) the quantity

$$j_{\mu\nu}^{(l)}\equiv\tfrac{1}{2}\lim_{\mathbf{q}\to 0}\frac{\partial^2 J_{\mathbf{q}}^{(l)}}{\partial q_\mu \partial q_\nu} \qquad (A4)$$

remains finite and that $\lim_{\mathbf{q}\to 0}J_{\mathbf{q}}^{(l)}=\lim_{\mathbf{q}\to 0}\partial J_{\mathbf{q}}^{(l)}/\partial q_\mu=0$ for $l\neq 0$. This can be easily seen with the cumulant expansion for $M_{\mathbf{q}}(n)$. The insertion of the expansion (3.25) into (A2) yields

$$C_{\mu\nu}(n)=\sum_l{}' K_{\mu\nu}^{(l)}\exp[-(i\omega_0^{(l)}+\gamma_0^{(l)})n] \qquad (A5)$$

with the expansion coefficient[16,18]

$$K_{\mu\nu}^{(l)}=4j_{\mu\nu}^{(l)}\sinh^2\left[\frac{i\omega_0^{(l)}+\gamma_0^{(l)}}{2}\right]\ . \qquad (A6)$$

The power spectrum is therefore given by

$$\begin{aligned}P_{\mu\nu}(\omega)&\equiv\sum_{n=-\infty}^{\infty}C_{\mu\nu}(n)\cos(\omega n)\\&=\tfrac{1}{2}\sum_l{}' K_{\mu\nu}^{(l)}\frac{\sinh(i\omega_0^{(l)}+\gamma_0^{(l)})}{\sinh^2[(i\omega_0^{(l)}+\gamma_0^{(l)})/2]+\sin^2(\omega/2)}\ .\end{aligned} \qquad (A7)$$

One finds that characteristic frequencies $\{\omega_0^{(l)}\}$ and damping rates $\{\gamma_0^{(l)}\}$ in the conventional double-time-correlation function are completely determined by poles of $\Xi_{\mathbf{q}}(\omega)$ for the limit $\mathbf{q}\to\mathbf{0}$. Especially, (A7) for $\omega\to 0$ is written as

$$\begin{aligned}P_{\mu\nu}(\omega\to 0)&=\chi_{\mu\nu}(\mathbf{q}\to\mathbf{0})\\&=2\sum_l{}' j_{\mu\nu}^{(l)}\sinh(i\omega_0^{(l)}+\gamma_0^{(l)})\\&=\sum_l{}' K_{\mu\nu}^{(l)}\coth\left[\frac{i\omega_0^{(l)}+\gamma_0^{(l)}}{2}\right]\ .\end{aligned} \qquad (A8)$$

The susceptibility for $\mathbf{q}=\mathbf{0}$ is expanded as

$$\begin{aligned}\chi_{\mu\nu}(\mathbf{0},n)&=\chi_{\mu\nu}(\mathbf{0})\\&\quad-\frac{2}{n}\sum_l{}' j_{\mu\nu}^{(l)}\{1-\exp[-(i\omega_0^{(l)}+\gamma_0^{(l)})n]\}\ .\end{aligned} \qquad (A9)$$

APPENDIX B: MARKOVIAN DISCRETE-VALUE PROCESSES

In this appendix we consider an L-state Markov process. When the state at step n is in the lth, $\mathbf{u}(n)$ takes the value $\mathbf{r}^{(l)}$ $[=\mathrm{col}(r_1^{(l)},r_2^{(l)},\ldots,r_m^{(l)})]$ $(l=1,2,\ldots,L)$. Let H_{lk} be the transition probability from the kth state to the lth state in unit step $(\sum_{l=1}^{L}H_{lk}=1)$. The master equation for $P_n^{(l)}$, the probability that the state at step n is in the lth state, is written as

$$P_{n+1}^{(l)}=\sum_{k=1}^{L}H_{lk}P_n^{(k)}\ , \qquad (B1)$$

$(\sum_{l=1}^{L}P_n^{(l)}=1)$. The characteristic function $M_{\mathbf{q}}(n)$ is obtained as

$$M_{\mathbf{q}}(n)=\sum_{l=1}^{L}\{[H(\mathbf{q})]^n\mathbf{P}_*\}_l\ , \qquad (B2)$$

where

$$H_{lk}(\mathbf{q})\equiv H_{lk}f_{\mathbf{q}}^{(k)} \qquad (B3)$$

with

$$f_{\mathbf{q}}^{(l)}\equiv\exp(\mathbf{q}\cdot\mathbf{r}^{(l)})\ , \qquad (B4)$$

and $\mathbf{P}_*=\mathrm{col}(P_*^{(1)},P_*^{(2)},\ldots,P_*^{(L)})$ is the steady probability distribution satisfying

$$P_*^{(l)}=\sum_{k=1}^{L}H_{lk}P_*^{(k)}\ . \qquad (B5)$$

Let $E(f_{\mathbf{q}}^{(1)},f_{\mathbf{q}}^{(2)},\ldots,f_{\mathbf{q}}^{(L)})$ be an eigenvalue of the matrix $H(\mathbf{q})$. One easily observes that E satisfies

$$E(f_{\mathbf{q}}^{(1)},f_{\mathbf{q}}^{(2)},\ldots,f_{\mathbf{q}}^{(L)}) =\xi E(\xi^{-1}f_{\mathbf{q}}^{(1)},\xi^{-1}f_{\mathbf{q}}^{(2)},\ldots,\xi^{-1}f_{\mathbf{q}}^{(L)}) \quad (B6)$$

with an arbitrary ξ. The similarity structure function $\phi(\mathbf{q})$ is therefore given in the form

$$\phi(\mathbf{q})=\ln[e^{\mathbf{q}\cdot\mathbf{r}^{(L)}}s(\mathbf{w}_{\mathbf{q}})] , \quad (B7)$$

with $\mathbf{w}_{\mathbf{q}}=\mathrm{col}(w_{\mathbf{q}}^{(1)},w_{\mathbf{q}}^{(2)},\ldots,w_{\mathbf{q}}^{(L-1)})$, where

$$w_{\mathbf{q}}^{(l)}=\exp[\mathbf{q}\cdot(\mathbf{r}^{(l)}-\mathbf{r}^{(L)})] \quad (l=1,2,\ldots,L-1) \quad (B8)$$

and we have set $\xi=f_{\mathbf{q}}^{(L)}$ in (B6). The quantity $\exp(\mathbf{q}\cdot\mathbf{r}^{(L)})s(\mathbf{w}_q)$ is the largest eigenvalue of $H(\mathbf{q})$. So we obtain

$$\alpha_\mu=r_\mu^{(L)}+\sum_{l=1}^{L-1}(r_\mu^{(l)}-r_\mu^{(L)})\frac{\partial \ln s(\mathbf{w}_{\mathbf{q}})}{\partial \ln w_{\mathbf{q}}^{(l)}} \quad (B9)$$

($\mu=1,2,\ldots,m$). The susceptibility matrix is determined as

$$\chi_{\mu\nu}(\mathbf{q})=\sum_{l=1}^{L-1}\sum_{k=1}^{L-1}(r_\mu^{(l)}-r_\mu^{(L)}) \times(r_\nu^{(k)}-r_\nu^{(L)})\frac{\partial^2 \ln s(\mathbf{w}_{\mathbf{q}})}{\partial \ln w_{\mathbf{q}}^{(l)}\partial \ln w_{\mathbf{q}}^{(k)}} . \quad (B10)$$

In order to carry out Legendre transforms of ϕ, one should solve $q_1,q_2,\ldots,q_m$ as functions of $\alpha_1,\alpha_2,\ldots,\alpha_m$ by inversely solving (B9). Since there are m independent equations connecting $\mathbf{q}$ with $\boldsymbol{\alpha}$, one can uniquely determine $\mathbf{q}$ in terms of $\boldsymbol{\alpha}$ for any m. However, repeating the discussion similar to that in Sec. II, one easily sees that it is sufficient to observe $(L-1)$ independent time series for the complete description of the L-state discrete process. Especially, if $m>L-1$, then $[m-(L-1)]$ time series do not give independent information. Hereafter in this appendix the dimension of $\mathbf{u}(n)$ is therefore chosen as

$$m=L-1 . \quad (B11)$$

One should note that this is equivalent to the condition that the quantities $w_{\mathbf{q}}^{(1)},w_{\mathbf{q}}^{(2)},\ldots,w_{\mathbf{q}}^{(L-1)}$ are uniquely solved as functions of $\alpha_1,\alpha_2,\ldots,\alpha_{L-1}$ in (B9).

Now we turn to the correlation problem. In the present process the eigenvalue equation (6.7) is written as

$$H(\mathbf{q})\boldsymbol{\psi}_{\mathbf{q}}^{(l)}=\exp(\phi_{\mathbf{q}}^{(l)})\boldsymbol{\psi}_{\mathbf{q}}^{(l)} \quad (B12)$$

with the lth eigenvector $\boldsymbol{\psi}_{\mathbf{q}}^{(l)}$ ($l=0,1,2,\ldots,L-1$). The zeroth eigenvalue is chosen as the largest one, i.e., $\exp(\phi_{\mathbf{q}}^{(0)})=e^{\mathbf{q}\cdot\mathbf{r}^{(L)}}s(\mathbf{w}_{\mathbf{q}})$. Inserting the expansion

$$\mathbf{P}_*=\sum_{l=0}^{L-1}k_{\mathbf{q}}^{(l)}\boldsymbol{\psi}_{\mathbf{q}}^{(l)} , \quad (B13)$$

into (B2) gives (6.9) with

$$J_{\mathbf{q}}^{(l)}=k_{\mathbf{q}}^{(l)}\sum_{j=1}^{L}(\boldsymbol{\psi}_{\mathbf{q}}^{(l)})_j . \quad (B14)$$

The expansion coefficients $\{k_{\mathbf{q}}^{(l)}\}$ are determined by

$$k_{\mathbf{q}}^{(l-1)}=\sum_{j=1}^{L}[R(\mathbf{q})^{-1}]_{lj}P_*^{(j)} , \quad (B15)$$

where $R(\mathbf{q})$ is the matrix with the jl element $(\boldsymbol{\psi}_{\mathbf{q}}^{(l-1)})_j$.

Let us hereafter consider a purely random process, where the evolution matrix has the element

$$H_{lk}=p_l . \quad (B16)$$

The invariant probability is immediately obtained as

$$P_*^{(l)}=p_l . \quad (B17)$$

There is only one nonvanishing, positive eigenvalue of $H(\mathbf{q})$. So $Q_{\mathbf{q}}(n)=1$. The logarithm of the nonzero eigenvalue determines $\phi(\mathbf{q})$ as

$$\phi(\mathbf{q})=\ln\left[\sum_{l=1}^{L}p_l f_{\mathbf{q}}^{(l)}\right] , \quad (B18)$$

and therefore

$$\boldsymbol{\alpha}=\frac{\sum_{l=1}^{L}p_l\mathbf{r}^{(l)}f_{\mathbf{q}}^{(l)}}{\sum_{l=1}^{L}p_l f_{\mathbf{q}}^{(l)}} =\frac{\sum_{l=1}^{L-1}p_l\mathbf{r}^{(l)}w_{\mathbf{q}}^{(l)}+p_L\mathbf{r}^{(L)}}{\sum_{l=1}^{L-1}p_l w_{\mathbf{q}}^{(l)}+p_L} . \quad (B19)$$

Especially the long-time average of $\mathbf{u}(n)$ is given by

$$\boldsymbol{\alpha}(\mathbf{q}=\mathbf{0})=\sum_{l=1}^{L}p_l\mathbf{r}^{(l)} . \quad (B20)$$

Equation (B19) is solved to yield

$$w_{\mathbf{q}}^{(l)}=\frac{p_L}{p_l}[U(\boldsymbol{\alpha})^{-1}(\mathbf{r}^{(L)}-\boldsymbol{\alpha})]_l\equiv\frac{p_L}{p_l}\xi_l(\boldsymbol{\alpha}) \quad (>0) . \quad (B21)$$

where $U(\boldsymbol{\alpha})$ is the $(L-1)\times(L-1)$ matrix with the element

$$U_{lk}(\boldsymbol{\alpha})=\alpha_l-r_l^{(k)} . \quad (B22)$$

Combining (B8) with (B21) gives

$$q_l=\sum_{k=1}^{L-1}(B^{-1})_{lk}\ln\left[\frac{p_L}{p_k}\xi_k(\boldsymbol{\alpha})\right] , \quad (B23)$$

where the matrix B has the element

$$B_{lk}=r_k^{(l)}-r_k^{(L)} . \quad (B24)$$

A slight calculation after the insertion of (B23) into (3.12) with (B18) yields the fluctuation spectrum[39]

$$\sigma(\boldsymbol{\alpha}')=\sum_{l=1}^{L}\pi_l(\boldsymbol{\alpha}')\ln\left[\frac{\pi_l(\boldsymbol{\alpha}')}{p_l}\right] , \quad (B25)$$

where

$$\pi_l(\boldsymbol{\alpha}')\equiv\sum_{k=1}^{L-1}(\alpha_k'-r_k^{(L)})(B^{-1})_{kl} \quad (B26)$$

for $l=1,2,\ldots,L-1$ and

$$\pi_L(\boldsymbol{\alpha}')\equiv 1-\sum_{l=1}^{L-1}\pi_l(\boldsymbol{\alpha}') \ . \tag{B27}$$

One easily finds

$$\boldsymbol{\alpha}'=\sum_{l=1}^{L}\pi_l(\boldsymbol{\alpha}')\mathbf{r}^{(l)} \ . \tag{B28}$$

Employing the inequality $x\ln x \geq x-1$, we can prove

$$\sigma(\boldsymbol{\alpha}')\geq 0 \ . \tag{B29}$$

The equality holds only when $\boldsymbol{\alpha}'$ satisfies

$$\pi_l(\boldsymbol{\alpha}')=p_l \tag{B30}$$

($l=1,2,\ldots,L$). The $\boldsymbol{\alpha}'$ giving (B30) is identical to $\boldsymbol{\alpha}(\mathbf{0})$ in (B20), the long-time average of $\mathbf{u}(n)$. For $\boldsymbol{\alpha}'=\boldsymbol{\alpha}(\mathbf{0})$, (B28) reduces to (B20). One therefore obtains

$$\pi_l(\boldsymbol{\alpha}')=p_l+\sum_{k=1}^{L-1}[\alpha'_k-\alpha_k(\mathbf{0})](B^{-1})_{kl} \tag{B31}$$

for $l=1,2,\ldots,L-1$.

Finally the susceptibilities are expanded as

$$\chi_{\mu\nu}(\mathbf{q})=\frac{e^{-2\phi(\mathbf{q})}}{2}\sum_{l=1}^{L}\sum_{k=1}^{L}p_lp_k(r_\mu^{(l)}-r_\mu^{(k)})\times(r_\nu^{(l)}-r_\nu^{(k)})f_{\mathbf{q}}^{(l)}f_{\mathbf{q}}^{(k)} \ , \tag{B32}$$

$$\chi_{\mu\nu}\{\boldsymbol{\alpha}'\}=\tfrac{1}{2}\sum_{l=1}^{L}\sum_{k=1}^{L}(r_\mu^{(l)}-r_\mu^{(k)})\times(r_\nu^{(l)}-r_\nu^{(k)})\pi_l(\boldsymbol{\alpha}')\pi_k(\boldsymbol{\alpha}') \ . \tag{B33}$$

Since $\pi_l(\boldsymbol{\alpha}')$ is a linear function of $\boldsymbol{\alpha}'$, $\chi_{\mu\nu}\{\boldsymbol{\alpha}'\}$ is the quadratic function of $\boldsymbol{\alpha}'$.

APPENDIX C: FINITE-POLE APPROXIMATIONS OF THE CONTINUED-FRACTION EXPANSION

From a practical viewpoint, the infinite series of the continued-fraction expansion should be discarded at a finite order. Hereafter we present a few low-order discard approximations.[17]

The simplest approximation is to retain only one pole by assuming $(0)_1=0$, which leads to $\phi(\mathbf{q})=\ln(\hat{1})_0=\ln M_{\mathbf{q}}(1)$ and $Q_{\mathbf{q}}(n)=1$. This corresponds to the complete neglect of temporal correlation.

The lowest-order approximation containing temporal correlation is the two-pole approximation, where

$$(0)_2=0 \ . \tag{C1}$$

This immediately leads to

$$\phi(\mathbf{q})=\ln\left[\frac{a_{\mathbf{q}}+(a_{\mathbf{q}}^2-4b_{\mathbf{q}})^{1/2}}{2}\right] , \tag{C2}$$

$$Q_{\mathbf{q}}(n)=\begin{cases}J_{\mathbf{q}}^{(+)}+J_{\mathbf{q}}^{(-)}\exp(-\gamma_{\mathbf{q}}n) & (b_{\mathbf{q}}>0)\\ J_{\mathbf{q}}^{(+)}+J_{\mathbf{q}}^{(-)}(-1)^n\exp(-\gamma_{\mathbf{q}}n) & (b_{\mathbf{q}}<0)\end{cases} \tag{C3}$$

where

$$a_{\mathbf{q}}=(\hat{1})_0+(\hat{1})_1 \ , \tag{C4a}$$

$$b_{\mathbf{q}}=(\hat{1})_0(\hat{1})_1-(0)_1 \ , \tag{C4b}$$

$$J_{\mathbf{q}}^{(+)}=1-J_{\mathbf{q}}^{(-)}=\frac{1}{2}-\frac{(\hat{1})_1-a_{\mathbf{q}}/2}{(a_{\mathbf{q}}^2-4b_{\mathbf{q}})^{1/2}} \quad (>0) \ , \tag{C4c}$$

$$\gamma_{\mathbf{q}}=\ln\left|\frac{a_{\mathbf{q}}+(a_{\mathbf{q}}^2-4b_{\mathbf{q}})^{1/2}}{a_{\mathbf{q}}-(a_{\mathbf{q}}^2-4b_{\mathbf{q}})^{1/2}}\right| \quad (>0) \ . \tag{C4d}$$

The next is the three-pole approximation obtained by setting

$$(0)_3=0 \ . \tag{C5}$$

The cubic equation $M_{\mathbf{q}}[\mu]^{-1}=0$ has two types of solutions: (i) three real solutions, and (ii) one real and a pair of complex-conjugate solutions. When three real solutions exist, the one that is the closest to the origin determines $\phi(\mathbf{q})$. The generalized time-correlation function is given as

$$Q_{\mathbf{q}}(n)=\begin{cases}J_{\mathbf{q}}^{(0)}+J_{\mathbf{q}}^{(1)}\exp(-\gamma_{\mathbf{q}}^{(1)}n)+J_{\mathbf{q}}^{(2)}\exp(-\gamma_{\mathbf{q}}^{(2)}n) \ , & \text{(C6a)}\\ J_{\mathbf{q}}^{(0)}+J_{\mathbf{q}}^{(1)}(-1)^n\exp(-\gamma_{\mathbf{q}}^{(1)}n)+J_{\mathbf{q}}^{(2)}\exp(-\gamma_{\mathbf{q}}^{(2)}n) \ , & \text{(C6b)}\\ J_{\mathbf{q}}^{(0)}+(-1)^n[J_{\mathbf{q}}^{(1)}\exp(-\gamma_{\mathbf{q}}^{(1)}n)+J_{\mathbf{q}}^{(2)}\exp(-\gamma_{\mathbf{q}}^{(2)}n)] \ , & \text{(C6c)}\end{cases}$$

$\gamma_{\mathbf{q}}^{(1)}$ and $\gamma_{\mathbf{q}}^{(2)}$ being positive, where (C6a)–(C6c) correspond to three-positive, one-negative and two-positive, and two-negative and one-positive solutions, respectively. On the other hand, when there exist a pair of complex-conjugate solutions, the real solution, which should be positive, is larger than the absolute value of the complex solution, and determines $\phi(\mathbf{q})$. The correlation function $Q_{\mathbf{q}}(n)$ is expanded as

$$Q_{\mathbf{q}}(n)=J_{\mathbf{q}}^{(0)}+2\,\mathrm{Re}\{J_{\mathbf{q}}^{(1)}\exp[-(i\omega_{\mathbf{q}}+\gamma_{\mathbf{q}})n]\} \ , \tag{C7}$$

where $\omega_{\mathbf{q}}$ is the characteristic frequency and $\gamma_{\mathbf{q}}$ (>0) is the inverse lifetime.

In general the N-pole approximation is given by setting

$$(0)_N=0 \ . \tag{C8}$$

Increasing the number of poles, we can successively take into account the higher-order contributions. This is especially efficient when finite poles are dominant in the system dynamics. When an infinite number of poles contribute to the continued-fraction expansion, as near the intermittency chaos transitions, the above perturbative expansion may not converge.

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[19]Observed components of time series are called to be *different* when a component $(\boldsymbol{\alpha}_n)_\mu$ of the local average $\boldsymbol{\alpha}_n$ [Eq. (3.2)] is not determined by other components of $\boldsymbol{\alpha}_n$ for any n.

[20]If we put $A_{n+1}^{(\mathbf{q})}=\exp[\mathbf{q}\cdot\sum_{j=1}^{n}\mathbf{u}(j)]=\exp(n\mathbf{q}\cdot\boldsymbol{\alpha}_n)$, it is self-similar in the sense that the ratio $A_{n+1}^{(\mathbf{q})}/A_n^{(\mathbf{q})}$ $(=\exp[\mathbf{q}\cdot\mathbf{u}(n)])$ is statistically independent of n because of the steadiness of $\{\mathbf{u}(n)\}$. The terminology *similarity structure function* is due to the fact that $\phi(\mathbf{q})$ is characteristic of statistical self-similarity of $A_n^{(\mathbf{q})}$ (Ref. 10). By introducing the cumulant-generating function $\Phi_{\mathbf{q}}(n)\equiv\langle\exp(n\mathbf{q}\cdot\boldsymbol{\alpha}_n)-1\rangle_c$ $=\ln M_{\mathbf{q}}(n)$, $\langle\cdots\rangle_c$ being the cumulant average, $\phi(\mathbf{q})$ can be expanded as

$$\phi(\mathbf{q})\equiv\lim_{n\to\infty}\frac{\Phi_{\mathbf{q}}(n)}{n}$$

$$=\sum_{l=1}^{\infty}\sum_{\mu_1}\cdots\sum_{\mu_l}c(\mu_1,\mu_2,\ldots,\mu_l)q_{\mu_1}\cdots q_{\mu_l}$$

with the expansion coefficients $[c(\mu_1,\mu_2,\ldots,\mu_l)]$. This is meaningful only when $|\mathbf{q}|$ is less than a convergence radius κ.[10]

[21]Hereafter the letter $\boldsymbol{\alpha}$ $[=\boldsymbol{\alpha}(\mathbf{q})]$ is used for the special value defined in (3.11), which is a function of the variable $\mathbf{q}$. For a value of a local average $\boldsymbol{\alpha}_n$, we use the letter $\boldsymbol{\alpha}'$.

[22]For $|\mathbf{q}|\ll\kappa$, κ being the convergence radius of the cumulant expansion (Ref. 10), $\phi(\mathbf{q})$ is usually expanded as

$$\phi(\mathbf{q})=\sum_{\mu}\alpha_\mu(0)q_\mu+\sum_{\mu,\nu}D_{\mu\nu}q_\mu q_\nu\quad(D_{\mu\nu}=D_{\nu\mu})\ .$$

One obtains $\alpha_\mu=\alpha_\mu(0)+2\sum_\nu D_{\mu\nu}q_\nu$ and $\chi_{\mu\nu}(\mathbf{q})=2D_{\mu\nu}$. The fluctuation spectrum is given as $\sigma(\boldsymbol{\alpha}')$ $=\frac{1}{4}\sum_\mu\sum_\nu(D^{-1})_{\mu\nu}[\alpha'_\mu-\alpha_\mu(0)][\alpha'_\nu-\alpha_\nu(0)]$. This agrees with (4.17). For $|\mathbf{q}|\gg\kappa$, the asymptotic $\mathbf{q}$ dependences of ϕ, α_μ, and $\chi_{\mu\nu}$ are highly different from the above. This means that $\rho_n(\boldsymbol{\alpha}')$ is far from the Gaussian (4.17) for a large $|\boldsymbol{\alpha}'-\boldsymbol{\alpha}(0)|$.

[23]When the system is a magnetic substance under an external magnetic field, the work given by the product of the total magnetization and the field should be taken into account.

[24]Analog II in univariate time series (Ref. 10), where the interrelations among relevant quantities are written as $\alpha=d(q\lambda_q)/dq$, $\sigma(\alpha)=q^2d\lambda_q/dq$, $q=d\sigma(\alpha)/d\alpha$, and $\lambda_q=\alpha-\sigma(\alpha)/q$, corresponds to the replacement $q\to T$, $\alpha\to-S$, $q\lambda_q\to F$, and $-\sigma(\alpha)\to U$. Furthermore, one should remark that in the conventional thermodynamics the interrelations among thermodynamics variables T, p, S, U, V, etc., hold under the transformation $T\to1/T$, $S\to-U$, $p\to p/T$, $V\to V$, $U\to-S$, $\mu_i\to\mu_i/T$, and $N_i\to N_i$, and therefore $F\to-\Psi$, $G\to-\Phi$, $\Psi\to-F$, and $\Phi\to-G$. This corresponds to analog II in Table II.

[25]The condition (3.17) corresponds to the stability condition of the equilibrium state in the conventional thermodynamics, and the positivity of eigenvalues of $\chi(\mathbf{q})$ leads to the thermodynamic inequalities. It seems that in the present case there is no relation corresponding to the Gibbs-Duhem relation $SdT-Vdp+\sum_k N_k d\mu_k=0$.

[26]For a chaotic one-dimensional map $x_{n+1}=f(x_n)$, the transition probability density is given by

$$T(x,x')=\delta(f(x')-x)$$

$$=\sum_j\delta(x'-y_j)|f'(y_j)|^{-1}\ ,$$

where $\sum_j$ is the summation over $\{y_j\}$ satisfying $x=f(y_j)$. The master operator H is identical to the Frobenius-Perron operator, and $H(\mathbf{q})$ is written as[16]

$$H(\mathbf{q})F(x)=H[e^{\mathbf{q}\cdot\mathbf{u}\{x\}}F(x)]$$

$$=\sum_j e^{\mathbf{q}\cdot\mathbf{u}\{y_j\}}F(y_j)|f'(y_j)|^{-1}\ .$$

For a univariate time series $H(\mathbf{q})$ reduces to the H_q previously introduced (Ref. 16). Furthermore, for fluctuations of local expansion rates we can put $u\{x\}=\ln|f'(x)|$. Then

$$H_qF(x)=\sum_j F(y_j)|f'(y_j)|^{q-1}\ .$$

The eigenvalue problem of the above H_q has been discussed in the context both of chaotic repellers (Refs. 9 and 27) and of the fluctuation dynamics of local expansion rates (Ref. 28).

[27]P. Szépfalusy and T. Tél, Phys. Rev. A **34**, 2520 (1986); Z. Kövács and T. Tél (unpublished).

[28]N. Mori, T. Kobayashi, H. Hata, T. Morita, T. Horita, and H. Mori, Prog. Theor. Phys. **81**, 60 (1989); T. Kobayashi, N. Mori, H. Hata, T. Horita, T. Yoshida, and H. Mori, *ibid.* **82**, 1 (1989).

[29]If there is only one largest eigenvalue, it should be positive, since $M_{\mathbf{q}}(n)$ is always positive. If we assume that there are a pair of complex-conjugate eigenvalues with $\exp[\phi(\mathbf{q})]$ $=\max_l[|\exp(\phi_{\mathbf{q}}^{(l)})|]$ for $l=0,1$, i.e., $\exp(\phi_{\mathbf{q}}^{(0)})=e^{\phi(\mathbf{q})+i\Omega(\mathbf{q})}$ and $\exp(\phi_{\mathbf{q}}^{(1)})=e^{\phi(\mathbf{q})-i\Omega(\mathbf{q})}$, then $M_{\mathbf{q}}(n)$ is asymptotically written as

$$M_{\mathbf{q}}(n) \simeq 2|J_{\mathbf{q}}^{(0)}|e^{\phi(\mathbf{q})n}\cos\{[\Omega(\mathbf{q})+\theta(\mathbf{q})]n\} ,$$

for a large n, where we put $J_{\mathbf{q}}^{(0)}=|J_{\mathbf{q}}^{(0)}|e^{i\theta(\mathbf{q})}$. This is inconsistent with the positivity of $M_{\mathbf{q}}(n)$ (Ref. 16). Therefore the eigenvalue giving $\max_l[|\exp(\phi_{\mathbf{q}}^{(l)})|]$ is not degenerate.

[30]The double-time-correlation function $C_{\mu\nu}(n)$ [Eq. (A1)] is related with the memory kernel $(n)_1$ via[18]

$$C_{\mu\nu}(n+1)=\frac{1}{2}\lim_{\mathbf{q}\to 0}\frac{\partial^2(n)_1}{\partial q_\mu \partial q_\nu}$$

for $n \geq 0$. Namely, $[\mu]_1$, the Laplace transform of $(n)_1$, is expanded, for a small $|\mathbf{q}|$, as

$$[\mu]_1=\sum_l{}'(e^{i\omega_0^{(l)}+\gamma_0^{(l)}}-\mu)^{-1}\sum_\delta\sum_\nu K_{\delta\nu}^{(l)}q_\delta q_\nu ,$$

where we have employed (A5).

[31]This time series can be a candidate of the stochastic model for the chaos-induced diffusion (Ref. 32) in the extended one-dimensional map $X_{n+1}=X_n+p\sin(2\pi X_n)$, p being the pumping parameter, especially slightly above unity. This map for p slightly above unity corresponds to a small h, r being finite in the model (9.1).

[32]T. Geisel and J. Nierwetberg, Phys. Rev. Lett. **48**, 7 (1982); S. Grossmann and H. Fujisaka, Phys. Rev. A **26**, 1779 (1982); H. Fujisaka and S. Grossmann, Z. Phys. B **48**, 261 (1982); M. Schell, S. Fraser, and R. Kapral, Phys. Rev. A **26**, 504 (1982). Furthermore, for the recent development on the studies of chaos-induced diffusion, see H. Fujisaka, M. Inoue, and H. Uchimura, Prog. Theor. Phys. **72**, 23 (1984).

[33]One should remark that the results in Fig. 6 and Table III are obtained by first calculating the relevant quantities for a small but finite h and then by taking the limit $h \to 0$. If one takes the limit $h \to 0$ first, then the characteristic function is obtained as $M_{\mathbf{q}}(n)=(e^{-q_1 n}+e^{(q_1+q_2)n})/2$, and the phase III is not present.

[34]P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).

[35]K. Honda and M. Matsushita (unpublished).

[36]For "static" scaling laws of λ_q, $\alpha(q)$, and $\sigma(\alpha)$, see H. Fujisaka, M. Inoue, and H. Uchimura, Prog. Theor. Phys. **72**, 23 (1984); M. Inoue and H. Fujisaka, Phys. Rev. B **32**, 277 (1985); H. Ishii, H. Fujisaka, and M. Inoue, Phys. Lett. A **116**, 257 (1986); H. Uchimura, H. Fujisaka, and M. Inoue, Prog. Theor. Phys. **77**, 1344 (1987); H. Fujisaka, R. Hagihara, and M. Inoue, Phys. Rev. A **38**, 3680 (1988); T. Yamane, T. Yamada, and H. Fujisaka, Prog. Theor. Phys. **80**, 588 (1988); M. Inoue, H. Fujisaka, and O. Yamaki, Phys. Lett. A **132**, 403 (1988); M. Inoue, O. Yamaki, and H. Fujisaka, Prog. Theor. Phys. **83**, 22 (1990). See also T. Kobayashi *et al.* in Ref. 28.

[37]For "dynamic" scaling laws of $\Xi_q(\omega)$, see H. Fujisaka, A. Yamaguchi, and M. Inoue, Prog. Theor. Phys. **81**, 1146 (1989); A. Yamaguchi, H. Fujisaka, and M. Inoue, Phys. Lett. A **135**, 320 (1989).

[38]J. P. Crutchfield and K. Young, Phys. Rev. Lett. **63**, 105 (1989).

[39]In terms of functions $\xi_l(\boldsymbol{\alpha}) \equiv [U(\boldsymbol{\alpha})^{-1}(\mathbf{r}^{(L)}-\boldsymbol{\alpha})]_l$ $(l=1,2,\ldots,L-1)$ [Eq. (B21)] and $\xi_L(\boldsymbol{\alpha}) \equiv 1$, one can prove the equality $\pi_l(\boldsymbol{\alpha})=\xi_l(\boldsymbol{\alpha})/Z(\boldsymbol{\alpha})$ $(l=1,2,\ldots,L)$ where $\{\pi_l(\boldsymbol{\alpha})\}$ are defined in (B26) and (B27), and $Z(\boldsymbol{\alpha}) \equiv \sum_{k=1}^{L}\xi_k(\boldsymbol{\alpha})$. Combining this equality with the inequality $\xi_l > 0$, one finds $\pi_l(\boldsymbol{\alpha}) > 0$ for $l=1,2,\ldots,L$.