Perturbation expansion of closed-time-path Green's functions

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The incoming interaction picture is introduced in accordance with the adiabatic hypothesis. A generalized Wick's theorem and, accordingly, a real-time formalism of perturbation theory at finite temperatures are constructed for the closed-time-path Green's functions. This new approach is illustrated in terms of the simple case of an electron-phonon system.

I. INTRODUCTION

Because of the difficulties in evaluating infinite sums over discrete frequencies and in performing analytical continuations in the imaginary time formalism of perturbation theory at finite temperatures, it is desirable to develop a real-time one. Many authors have attempted this.[1,2] We would like to present an alternative based on a new interaction picture and a generalized Wick's theorem. The new interaction picture in which the free fields are proved to be the incoming fields coincides with the Heisenberg one at $t=-\infty$. Therefore, it is exactly the incoming one.

As we are going to show, the interaction picture employed in the book of Abrikosov and his coauthors[3] is, indeed, the one coinciding with the Heisenberg picture at $t=-\infty$. To our knowledge, Abrikosov *et al.* first made use of it, and it was followed widely thereafter in many later works, in particular, in the theory of closed-time-path Green's functions (CTPGF).[3,4] However, it was not recognized as a novel one but, on the contrary, was incorrectly identified as the usual one, coinciding with the Heisenberg picture at $t=0$.[2,5] The incoming fields are also well known in quantum field theory,[6] but they have never been introduced manifestly in connection with picture transformations.

We will first show in Sec. II that the interaction picture defined to coincide with the Heisenberg picture at $t=-\infty$ is no longer the usual one, but the incoming one. Section III formulates a generalized Wick theorem, utilizing special properties of incoming operators. In that section a Dyson equation will be generally established, too. Both the generalized Wick theorem and the Dyson equation are expressed in terms of CTPGF. The CTPGF are then transformed into the retarded Green's functions for ease of practical calculations. Lastly, applications of this real-time approach to the simple problem of electron-phonon interaction are discussed in Sec. IV, for illustrative purposes.

II. THE INCOMING INTERACTION PICTURE

In the book of Abrikosov and his coauthors,[3] the interaction and Heisenberg pictures are related through the following transformations:

$$|a,t\rangle_I = U(t,-\infty)|a\rangle_H \ , \tag{2.1}$$

$$A_I(x) = U(t,-\infty)A_H(x)U^{-1}(t,-\infty) \ , \tag{2.2}$$

where $|a,t\rangle$ stands for the state vector and $A(x)$, the field operator. The subscripts in (2.1) and (2.2) are self-evident. In Eqs. (2.1) and (2.2) the evolution operator U is given by

$$U(t,t_0) = T\exp\left[-i\int_{t_0}^{t} H_{\text{int}I}(t')dt'\right] \ , \tag{2.3}$$

with $H_{\text{int}I}(t)$ the interaction Hamiltonian in interaction picture. The interaction picture prescribed by (2.1)–(2.3) has been widely used in statistical physics, in particular, in the theory of CTPGF.[2,4] However, there remains an open question which has escaped the attention of many authors. What is the explicit form of $H_{\text{int}I}(t)$? Or, does it actually take the usual form

$$H_{\text{int}I}(t) = e^{iH_0 t}H_{\text{int}}e^{-iH_0 t} \tag{2.4}$$

in Ref. 3? This point needs clarification.

For this purpose, let us reexamine the picture transformations. We first write the relations between the Schrödinger and Heisenberg pictures

$$|a\rangle_H = e^{iHt}|a,t\rangle_S \ , \tag{2.5}$$

$$A_H(\mathbf{x},t) = e^{iHt}A_S(\mathbf{x})e^{-iHt} \ . \tag{2.6}$$

We then introduce the interaction picture which is related to the Schrödinger picture through the relations

$$|a,t\rangle_I = R(t)|a,t\rangle_S \ , \tag{2.7}$$

$$A_I(\mathbf{x},t) = R(t)A_S(\mathbf{x})R^{-1}(t) \ , \tag{2.8}$$

where $R(t)$ is a unitary operator to be determined. We require $R(t)$ to obey the customary equation

$$\frac{\partial R(t)}{\partial t}=iR(t)H_{0S} \tag{2.9}$$

or

$$\frac{\partial R^{\dagger}(t)}{\partial t}=-iH_{0S}R^{\dagger}(t)\ , \tag{2.10}$$

since $R(t)$ must be unitary. The equation

$$i\frac{\partial}{\partial t}|a,t\rangle_I=H_{\mathrm{int}I}(t)|a,t\rangle_I \tag{2.11}$$

can then be derived from Eq. (2.7), in which

$$H_{\mathrm{int}I}(t)=R(t)H_{\mathrm{int}S}R^{-1}(t)\ . \tag{2.12}$$

A formal solution of Eq. (2.11) is

$$|a,t\rangle_I=u(t,t_0)|a,t_0\rangle_I\ , \tag{2.13}$$

with

$$u(t,t_0)=T\exp\left[-i\int_{t_0}^{t}H_{\mathrm{int}I}(t')dt'\right]\ . \tag{2.14}$$

Taking the partial derivative of Eq. (2.8) with respect to time leads, with the help of Eqs. (2.9) and (2.10), to

$$i\frac{\partial}{\partial t}A_I(x)=[A_I(x),H_{0I}(t)]\ , \tag{2.15}$$

$$H_{0I}(t)=R(t)H_{0S}R^{-1}(t)\ . \tag{2.16}$$

The solutions of Eq. (2.9) depend on initial conditions. The usual way is to assign

$$R(0)=I\ , \tag{2.17}$$

which yields the known solution

$$R(t)=e^{iH_{0S}t}\ . \tag{2.18}$$

This is just the unitary operator transforming the Schrödinger picture into the usual interaction picture. In this case, the three pictures, the interaction, Schrödinger, and Heisenberg, all coincide at $t=0$. However, the usual interaction picture defined by the operator (2.18) is not the proper one useful in finite-temperature statistical physics, e.g., in the theory of CTPGF. A better choice is to define another interaction picture coinciding with the Heisenberg picture at $t=-\infty$, in accordance with the adiabatic hypothesis. In other words, one can impose the initial condition, requiring

$$|a,-\infty\rangle_I=|a\rangle_H\ . \tag{2.19}$$

It follows from Eq. (2.13) that

$$|a,t\rangle_I=u(t,-\infty)|a\rangle_H\ . \tag{2.20}$$

The transformation for the operator is, accordingly,

$$A_I(x)=u(t,-\infty)A_H(x)u^{-1}(t,-\infty)\ . \tag{2.21}$$

Substitution of Eq. (2.5) into Eq. (2.20) and comparison with Eq. (2.7) yields (where "in" denotes incoming)

$$R^{\mathrm{in}}(t)=u(t,-\infty)e^{iHt}\ . \tag{2.22}$$

Equations (2.12) and (2.22) mean that Eq. (2.14) is a functional equation to be solved self-consistently. Its solution can be easily obtained by inspection. Let us set

$$u(t,-\infty)=e^{iHt}U(0,-\infty)e^{-iHt}\ , \tag{2.23}$$

with the operator $U(0,-\infty)$ still defined by Eqs. (2.3) and (2.4), then

$$R^{\mathrm{in}}(t)=e^{iHt}U(0,-\infty)\ , \tag{2.24}$$

which is a solution of Eq. (2.9), can be verified with the help of the relation[6]

$$HU(0,-\infty)=U(0,-\infty)H_0\ . \tag{2.25}$$

From Eq. (2.23) we have

$$u(t,t_0)=e^{iHt}U(0,-\infty)e^{-iH(t-t_0)}U^{-1}(0,-\infty)e^{-iHt_0}\ , \tag{2.26}$$

which is just the solution of the differential equation

$$\frac{\partial}{\partial t}u(t,t_0)=-iH_{\mathrm{int}I}^{\mathrm{in}}(t)u(t,t_0)\ , \tag{2.27}$$

satisfying the condition $u(t_0,t_0)=I$, as expected, with

$$H_{\mathrm{int}I}^{\mathrm{in}}(t)=e^{iHt}U(0,-\infty)H_{\mathrm{int}S}U^{-1}(0,-\infty)e^{-iHt}\ . \tag{2.28}$$

The operator (2.28) is evidently different from that given by Eq. (2.4), implying that the operator (2.24) is an independent solution of Eq. (2.9) under different initial conditions and must not be confused with the operator (2.18). The important thing is that the free field in this interaction picture is

$$\begin{aligned}A_I(x)&=e^{iHt}U(0,-\infty)e^{-iHt}A_H(x)e^{iHt}U^{-1}(0,-\infty)e^{-iHt}\\&=A_{\mathrm{in}}(x)\end{aligned} \tag{2.29}$$

which is just the in-field known in quantum field theory[6] having the properties

$$A_I(\mathbf{x},t)=e^{iHt}A_I(\mathbf{x},0)e^{-iHt}\ , \tag{2.30}$$

$$\frac{\partial A_I(x)}{\partial t}=i[H,A_I(x)]\ . \tag{2.31}$$

Equation (2.31) is also the free field equation of motion, because

$$H_{0I}(t)=e^{iHt}U(0,-\infty)H_0U^{-1}(0,-\infty)e^{-iHt}=H\ . \tag{2.32}$$

So far we have proved that the correct interaction picture prescribed by Eqs. (2.1)–(2.3) is the one defined by the operator (2.24), which we call the incoming interaction picture, with the U matrix employed in the book of Abrikosov *et al.*[3] In the above discussion we distinguish the incoming interaction picture from the usual one with help of a superscript "in."

It is easier to work out the perturbation expansion in the incoming interaction picture even in the case of zero temperature. For instance, a full Green's function

$$G(x_1,x_2,\ldots,x_n)=\langle 0|TA_H(x_1)A_H(x_2)\cdots A_H(x_n)|0\rangle \tag{2.33}$$

can be expressed in terms of incoming operators

$$G(x_1,x_2,\ldots,x_n) = \langle 0|S^{\dagger}TA_{\rm in}(x_1)A_{\rm in}(x_2)\cdots A_{\rm in}(x_n)S|0\rangle , \quad (2.34)$$

where

$$S=u(\infty,-\infty) . \quad (2.35)$$

In the absence of an external field, $|0\rangle$ and $S|0\rangle$ differ only by a phase factor

$$S|0\rangle = e^{iL}|0\rangle . \quad (2.36)$$

Equation (2.34) is then reduced to

$$G(x_1,x_2,\ldots,x_n) = \langle 0|TA_{\rm in}(x_1)A_{\rm in}(x_2)\cdots A_{\rm in}(x_n)S|0\rangle_c , \quad (2.37)$$

where the subscript c means that the vacuum diagrams are to be omitted in the expansion of (2.37). The scattering problems can be treated similarly. We consider, for instance, the annihilation of positron-electron pair in QED. The scattering amplitude is

$$\lim_{\substack{t'\to\infty \\ t\to-\infty}} \langle 0|a_{\lambda_1}(k_1,t')a_{\lambda_2}(k_2,t')c^{\dagger}_{r_1}(p_1,t)d^{\dagger}_{r_2}(p_2,t)|0\rangle , \quad (2.38)$$

where a_λ stands for the destruction operator of a photon with polarization λ, and $c^{\dagger}_r, d^{\dagger}_r$ are the creation operators of an electron and positron of spin r, respectively. All of them are Heisenberg operators. We then write (2.38) in terms of incoming operators to give

$$\lim_{\substack{t'\to\infty \\ t\to-\infty}} \langle 0|Ta^{\rm in}_{\lambda_1}(k_1,t')a^{\rm in}_{\lambda_2}(k_2,t')c^{\rm in^{\dagger}}_{r_1}(p_1,t)d^{\rm in^{\dagger}}_{r_2}(p_2,t)S|0\rangle_c \quad (2.39)$$

with

$$S=T\exp\left[-i\int d^4x(-ie)\bar{\psi}_{\rm in}(x)A_{\rm in}(x)\psi_{\rm in}(x)\right] . \quad (2.40)$$

The calculation of (2.39) is now a routine procedure, and the result, apart from an unimportant phase factor, is identical to that in ordinary quantum field theory.

We conclude, therefore, that, for the perturbation expansions at zero temperature, the two interaction pictures, the incoming one and the usual one defined by the operator (2.9), are equivalent. However, as we are going to show, they are quite different for the perturbation expansions at finite temperatures.

III. A GENERALIZED WICK THEOREM

Wick's theorem is essential for perturbation expansions. We give here a generalized Wick theorem in real-time formalism at finite temperatures. We consider a thermal equilibrium system with a grand-canonical density matrix

$$\rho=\exp[\beta F-\beta(H-\mu N)] , \quad (3.1)$$

where F is the thermodynamic potential of the system, μ the chemical potential, and

$$N=\int d^3x\,\varphi^{\dagger}(x)\varphi(x) , \quad (3.2)$$

the particle number operator. The Green's function is defined by

$$G(x_1,x_2,\ldots,x_n) = {\rm Tr}[\rho TA_H(x_1)A_H(x_2)\cdots A_H(x_n)] . \quad (3.3)$$

It is impossible to express Eq. (3.3) in terms of incoming operators as in Eq. (2.37) because there does not exist a simple relation like Eq. (3.36) in problems at finite temperatures. The way out is to introduce the CTPGF, and Eq. (3.3) is then generalized to

$$G_p(x_1,x_2,\ldots,x_n) = {\rm Tr}[\rho T_p A_H(x_1)A_H(x_2)\cdots A_H(x_n)] , \quad (3.4)$$

where T_p is the ordering operator along the closed time path $(-\infty\to\infty\to-\infty)$. It is now ready to transform Eq. (3.4) into the incoming interaction picture to give

$$G_p(x_1,x_2,\ldots,x_n) = {\rm Tr}[\rho T_p A_{\rm in}(x_1)A_{\rm in}(x_2)\cdots A_{\rm in}(x_n)S_p] , \quad (3.5)$$

where

$$S_p=T_p\exp\left[-i\int_p H^{\rm in}_{\rm int}(t)dt\right] \quad (3.6)$$

is the S matrix along the closed time path, with the understanding that the interaction on the positive time branch is distinguishable from that on the negative time branch under the effect of T_p. It can be shown,[7] by conservation of particle number, that

$$[N,U(t,t_0)]=0 , \quad (3.7)$$

which leads to

$$N=\int d^3x\,\varphi^{\dagger}_{\rm in}(x)\varphi_{\rm in}(x) , \quad (3.8)$$

as can be seen from Eq. (2.26).

Equations (3.8) and (2.30) give rise to the useful relation

$$\rho^{-1}A_{\rm in}(x)\rho=e^{-\lambda\beta\mu}A_{\rm in}(\mathbf{x},t-i\beta) , \quad (3.9)$$

in which either $\lambda=1$ when $A_{\rm in}(x)=\varphi^{\dagger}_{\rm in}(x)$ or $\lambda=-1$ when $A_{\rm in}(x)=\varphi_{\rm in}(x)$. One obtains from Eq. (3.9)

$$\rho A_{\rm in}(x)=K(t,\lambda)[\rho,A_{\rm in}(x)]_{\mp} , \quad (3.10)$$

with

$$K(t,\lambda)=\left[1\mp\exp\left[-i\beta\frac{\partial}{\partial t}-\lambda\beta\mu\right]\right]^{-1} . \quad (3.11)$$

Starting from the identity

$$\mathrm{Tr}\{[\rho, A(x_1)]_{\mp} A(x_2)\cdots A(x_n)\} = \mathrm{Tr}\{\rho[A(x_1), A(x_2)]_{\mp} A(x_3)\cdots A(x_n)\}$$
$$\pm \mathrm{Tr}\{\rho A(x_2)[A(x_1), A(x_3)]_{\mp} A(x_4)\cdots A(x_n)\}$$
$$+ \cdots (\pm 1)^n \mathrm{Tr}\{\rho A(x_2)\cdots A(x_{n-1})[A(x_1), A(x_n)]_{\mp}\} , \tag{3.12}$$

we arrive, by using Eq. (3.10), at

$$\mathrm{Tr}[\rho A_{\mathrm{in}}(x_1)A_{\mathrm{in}}(x_2)\cdots A_{\mathrm{in}}(x_n)] = K(t_1,\lambda)(\mathrm{Tr}\{\rho[A_{\mathrm{in}}(x_1), A_{\mathrm{in}}(x_2)]_{\mp} A_{\mathrm{in}}(x_3)\cdots A_{\mathrm{in}}(x_n)$$
$$+ \cdots (\pm 1)^n \mathrm{Tr}\{\rho A_{\mathrm{in}}(x_2)\cdots A_{\mathrm{in}}(x_{n-1})[A_{\mathrm{in}}(x_1), A_{\mathrm{in}}(x_n)]_{\mp}\}) . \tag{3.13}$$

In particular,

$$\mathrm{Tr}[\rho A_{\mathrm{in}}(x_1)A_{\mathrm{in}}(x_2)] = K(t_1,\lambda)[A_{\mathrm{in}}(x_1), A_{\mathrm{in}}(x_2)]_{\mp}$$
$$= \langle A_{\mathrm{in}}(x_1)A_{\mathrm{in}}(x_2)\rangle . \tag{3.14}$$

It is easy to see from Eq. (3.10) that

$$\mathrm{Tr}[\rho A_{\mathrm{in}}(x)] = 0 . \tag{3.15}$$

We, therefore, need only to consider the case of even n in Eq. (3.13), which is reduced, by the repeated application of Eq. (3.10), to

$$\mathrm{Tr}[\rho A_{\mathrm{in}}(x_1)\cdots A_{\mathrm{in}}(x_n)] = \sum_{p(j_1\cdots j_n)} (\pm 1)^P \langle A_{\mathrm{in}}(x_{j_1})A_{\mathrm{in}}(x_{j_2})\rangle \cdots \langle A_{\mathrm{in}}(x_{j_{n-1}})A_{\mathrm{in}}(x_{j_n})\rangle , \tag{3.16}$$

where $p(j_1\cdots j_n)$ is the permutation to arrange $(1,\ldots,n)$ into the order $(j_1,\ldots,j_n)$. We obtain finally from Eq. (3.16) the generalized Wick theorem

$$\mathrm{Tr}[\rho T_p A_{\mathrm{in}}(x_1)\cdots A_{\mathrm{in}}(x_n)] = \sum_{p(j_1\cdots j_n)} (\pm 1)^P \langle T_p A_{\mathrm{in}}(x_{j_1})A_{\mathrm{in}}(x_{j_2})\rangle \cdots \langle T_p A_{\mathrm{in}}(x_{j_{n-1}})A_{\mathrm{in}}(x_{j_n})\rangle , \tag{3.17}$$

in which the minus sign is for bosons and the plus sign for fermions. The generalized Wick theorem (3.17) and the perturbative expression (3.5) form the basis of our new perturbation theory at finite temperatures in terms of real-time CTPFG.

Before going on further to applications of this new approach, it is useful to discuss a generalized Dyson equation. The generating functional of Green's functions serves this purpose properly, and it can be constructed generally, in the interaction picture, as

$$Z(J,J^{\dagger}) = \mathrm{Tr}\left[\rho T_p \exp\left[iI_{\mathrm{int}}(\varphi_i,\varphi_i^{\dagger}) + i\int_p d^4x[J^{\dagger}(x)\varphi_i(x) + \varphi_i^{\dagger}(x)J(x)]\right]\right] , \tag{3.18}$$

where we have set $\varphi_i(x) \equiv \varphi_{\mathrm{in}}(\chi)$ and I_{int} is defined by

$$I_{\mathrm{int}}(\varphi_i,\varphi_i^{\dagger}) = -\int_p d^4x \mathcal{H}_{\mathrm{int}}[\varphi_i(x),\varphi_i^{\dagger}(x)] , \tag{3.19}$$

with $\mathcal{H}_{\mathrm{int}}$ the density of the interaction Hamiltonian of the system. As usual, Eq. (3.18) can be expressed as

$$Z(J,J^{\dagger}) = \exp\left[iI_{\mathrm{int}}\left[\frac{-i\delta}{\delta J^{\dagger}}, \frac{\mp i\delta}{\delta J}\right]\right]\mathrm{Tr}\left[\rho T_p \exp\left[i\int_p d^4x[J^{\dagger}(x)\varphi_i(x) + \varphi_i^{\dagger}(ix)J(x)]\right]\right] , \tag{3.20}$$

which can be reduced, with the help of the Wick theorem (3.17) to

$$Z(J,J^{\dagger}) = \exp\left[iI_{\mathrm{int}}\left[\frac{-i\delta}{\delta J^{\dagger}}, \frac{\mp i\delta}{\delta J}\right]\right]\exp\left[-i\int_p d^4x d^4y J^{\dagger}(x)G_p^0(x,y)J(y)\right] , \tag{3.21}$$

in which

$$G_p^0(x,y)=-i\langle T_p\varphi_i(x)\varphi_i^\dagger(y)\rangle \tag{3.22}$$

is the free propagator along the closed time path. The full propagator is given by

$$G_p(x,y)=\left.\frac{\mp\delta^2\ln Z(J,J^\dagger)}{i\delta J^\dagger(x)\delta J(y)}\right|_{J=J^\dagger=0} . \tag{3.23}$$

Performing the functional derivative of Eq. (3.21), the following equation can be derived straightforwardly:

$$G_p(x,y)=G_p^0(x,y)+\int_p d^4z_1d^4z_2G_p^0(x,z_1)\times R(z_1,z_2)G_p^0(z_2,y) , \tag{3.24}$$

or in matrix notation,

$$G_p=G_p^0+G_p^0RG_p^0 , \tag{3.25}$$

with $R(x,y)$ denoting

$$R(x,y)=-i\langle T_pj_i(x)j_i^\dagger(y)S_p\rangle -\left\langle T_p\frac{\delta^2I_{\rm int}(\varphi_i,\varphi_i^\dagger)}{\delta\varphi_i(y)\delta\varphi_i^\dagger(x)}S_p\right\rangle , \tag{3.26}$$

where

$$j_i(x)=-\frac{\delta I_{\rm int}(\varphi_i,\varphi_i^\dagger)}{\delta\phi_i^\dagger(x)} , \tag{3.27}$$

$$j_i^\dagger(x)=\mp\frac{\delta I_{\rm int}(\varphi_i,\varphi_k^\dagger)}{\delta\varphi_i(x)} , \tag{3.28}$$

and

$$S_p=T_p\exp[iI_{\rm int}(\varphi_i,\varphi_i^\dagger)] . \tag{3.29}$$

Equations (3.26) and (3.27) apply to both bosons and fermions, whereas in Eq. (3.28) the minus (plus) sign should be taken in the case of bosons (fermions). Equation (3.25) can be cast into the usual form of Dyson's equation

$$G_p=G_p^0+G_p^0\Sigma_pG_p , \tag{3.30}$$

where we obtain by comparison of Eqs. (3.30) and (3.25)

$$\Sigma_p(x,y)=\{R(x,y)\}_{\rm 1PI} \tag{3.31}$$

with the subscript 1PI implying the $\Sigma_p(x,y)$ is the one-particle-irreducible part of $R(x,y)$.

The perturbation expansion is in terms of CTPGF. But for practical calculations, it is more convenient to employ the retarded Green's functions. According to the relations between the CTPGF, the retarded, advanced, and causal functions,[8] Eq. (3.30), can be transformed into

$$G_r=G_r^0+G_r^0\Sigma_rG_r , \tag{3.32}$$

where the retarded functions are defined by[8]

$$G_r=G_F-G_{+-} , \tag{3.33}$$

$$\Sigma_r=\Sigma_F-\Sigma_{+-} . \tag{3.34}$$

It is obvious that despite the fact that our theory is formulated in coordinate space, it is valid in momentum space as well.

IV. FEYNMAN RULES FOR ELECTRON-PHONON INTERACTING SYSTEM

For illustrative purposes, we consider the simple problem of electron-phonon system. The interaction Hamiltonian reads

$$H_{\rm int}=\sum_{\mathbf{k},s,\mathbf{q}}g(\mathbf{q})a_s^\dagger(\mathbf{k}+\mathbf{q})a_s(\mathbf{k})\varphi(\mathbf{q}) , \tag{4.1}$$

where a_s and $a_s^\dagger$ are destruction and creation operators of electrons with spin s, and

$$\varphi(\mathbf{q})=b(\mathbf{q})+b^\dagger(-\mathbf{q}) \tag{4.2}$$

is the phonon operator. The coupling constant has the property

$$g^*(\mathbf{q})=g(-\mathbf{q}) . \tag{4.3}$$

According to (3.30), the Dyson equations are (for the sake of simplicity, the electron spin indices will be omitted hereafter)

$$G_p(\mathbf{k},t_1-t_2)=G_p^0(\mathbf{k},t_1-t_2)+\int_p d\tau_1d\tau_2G_p^0(\mathbf{k},t_1-\tau_1)\times\Sigma_p(\mathbf{k},\tau_1-\tau_2)\times G_p(\mathbf{k},\tau_2-t_2) . \tag{4.4}$$

$$D_p(\mathbf{q},t_1-t_2)=D_p^0(\mathbf{q},t_1-t_2)+\int_p d\tau_1d\tau_2D_p^0(\mathbf{q},t_1-\tau_1)\times\Pi_p(\mathbf{q},\tau_1-\tau_2)\times D_p(\mathbf{q},\tau_2-t_2) , \tag{4.5}$$

where

$$G_p^0(\mathbf{k},t_1-t_2)=-i\langle T_pa_i(\mathbf{k},t_1)a_i^\dagger(\mathbf{k},t_2)\rangle , \tag{4.6}$$

$$G_p(\mathbf{k},t_1-t_2)=-i\langle T_pa(\mathbf{k},t_1)a^\dagger(\mathbf{k},t_2)\rangle \tag{4.7}$$

are the free and full electron propagators, respectively, and

$$D_p^0(\mathbf{q},t_1-t_2)=-i\langle T_p\varphi_i(\mathbf{q},t_1)\varphi_i^\dagger(\mathbf{q},t_2)\rangle , \tag{4.8}$$

$$D_p(\mathbf{q},t_1,t_2)=-i\langle T_p\varphi(\mathbf{q},t_1)\varphi^\dagger(\mathbf{q},t_2)\rangle \tag{4.9}$$

are the free and full phonon propagators. The self-energy parts are given by

$$\Sigma_p(\mathbf{k},t_1-t_2)=\{-i\langle T_pj_i(\mathbf{k},t_1)j_i^\dagger(\mathbf{k},t_2)S_p\rangle\}_{\rm 1PI} , \tag{4.10}$$

$$\Pi_p(\mathbf{q},t_1-t_2)=\{-i\langle T_pQ_i(\mathbf{q},t_1)Q_i^\dagger(\mathbf{q},t_2)S_p\rangle\}_{\rm 1PI} \tag{4.11}$$

with

$$j(\mathbf{k},t)=\sum_{\mathbf{q}}g^*(\mathbf{q})a(\mathbf{k}+\mathbf{q},t)\varphi^\dagger(\mathbf{q},t) , \tag{4.12}$$

$$j^\dagger(\mathbf{k},t)=\sum_{\mathbf{q}}g(\mathbf{q})a^\dagger(\mathbf{k}+\mathbf{q},t)\varphi(\mathbf{q},t) , \tag{4.13}$$

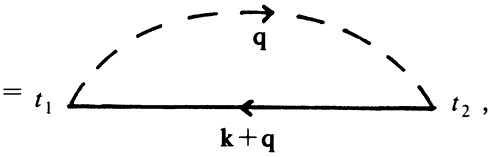
$$Q(\mathbf{q},t)=\sum_{\mathbf{k}}g^*(\mathbf{q})a^\dagger(\mathbf{k}-\mathbf{q},t)a(\mathbf{k},t) , \tag{4.14}$$

$$Q^\dagger(\mathbf{q},t)=\sum_{\mathbf{k}}g(\mathbf{q})a^\dagger(\mathbf{k}+\mathbf{q},t)a(\mathbf{k},t) . \tag{4.15}$$

To the lowest order, the electron self-energy part is

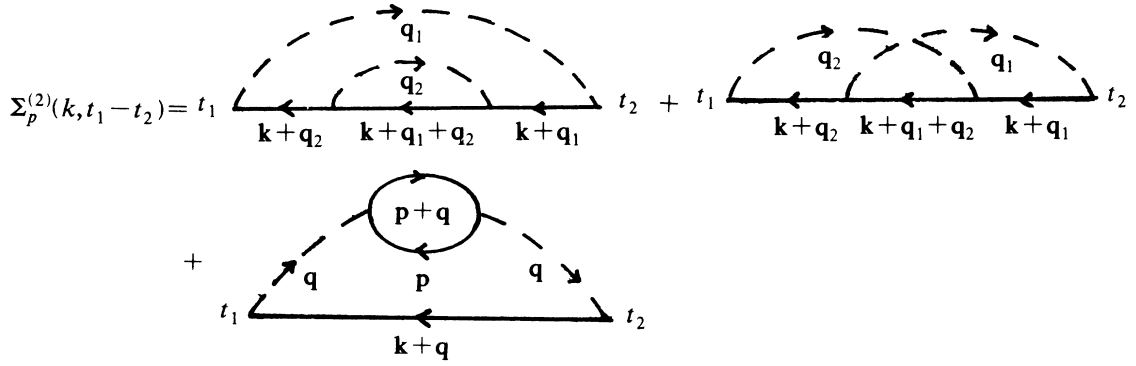
$$\Sigma_p^{(1)}(\mathbf{k},t_1-t_2) = i\sum_{\mathbf{q}} |g(\mathbf{q})|^2 G_p^0(\mathbf{k}+\mathbf{q},t_1-t_2)D_p^0(\mathbf{q},t_2-t_1) , \quad (4.16)$$

which can be easily obtained from Eq. (4.10) by using the generalized Wick theorem (3.17). The Feynman diagram corresponding to (4.16) is then



(4.17)

where the solid and dashed lines represent free electron and phonon propagators, respectively. The next-order contribution to the electron self-energy is given by



(4.18)

It is clear that the Feynman rules are similar to those in the perturbation theory at zero temperature. It is natural because the generalized Wick theorem is, as we have seen in Sec. III, exactly parallel to that at zero temperature.

The single-time Green's functions are now needed for practical calculations. Following the definition (4.6), it is easy to show, with the help of the basic relation (3.14), that

$$G_F^0(k)=[1-f(\mathbf{k})]G_r^0(k)+f(\mathbf{k})G_a^0(k) , \quad (4.19)$$

$$G_{\bar{F}}^0(k)=-[1-f(k)]G_a^0(k)-f(k)G_r^0(k) , \quad (4.20)$$

$$G_{+-}^0(k)=i2\pi f(k)\delta(k_0-E(k)) , \quad (4.21)$$

$$G_{-+}^0(k)=-i2\pi[1-f(k)]\delta(k_0-E(k)) , \quad (4.22)$$

where

$$G_r^0(k)=[k_0-E(\mathbf{k})+i0^+]^{-1}, \quad G_a^0(k)=[k_0-E(\mathbf{k})-i0^+]^{-1} \quad (4.23)$$

are the retarded and advanced free Green's functions, respectively, and $f(k)$ is the Fermi distribution function

$$f(\mathbf{k})=\{1+\exp[\beta E(\mathbf{k})]\}^{-1} , \quad (4.24)$$

with

$$E(\mathbf{k})=\frac{1}{2m}k^2-\mu . \quad (4.25)$$

The phonon propagator can be expressed as

$$D_p^0(\mathbf{q},t_1-t_2)=d_p^0(\mathbf{q},t_1-t_2)+d_p^0(-\mathbf{q},t_2-t_1) , \quad (4.26)$$

where

$$d_p^0(\mathbf{q},t_1-t_2)=-i\langle T_p b_i(\mathbf{q},t_1)b_i^\dagger(\mathbf{q},t_2)\rangle , \quad (4.27)$$

which leads to

$$D_\alpha^0(q)=d_\alpha^0(q)+d_\alpha^0(-q), \quad \alpha=F,\bar{F},r,a \quad (4.28)$$

$$D_{+-}^0(q)=d_{+-}^0(q)+d_{-+}^0(-q) . \quad (4.29)$$

The single-time Green's functions for phonons can be derived in a similar manner with the result

$$d_F^0(q)=[1+n(\mathbf{q})]d_f^0(q)-n(\mathbf{q})d_a^0(q) , \quad (4.30)$$

$$d_{\bar{F}}^0=-[1+n(\mathbf{q})]d_a^0(q)+n(\mathbf{q})d_r^0(q) , \quad (4.31)$$

$$d_{+-}^0(q)=-i2\pi n(q)\delta(q_0-\omega(\mathbf{q})) , \quad (4.32)$$

$$d_{-+}^0(q)=-i2\pi[1+n(\mathbf{q})]\delta(q_0-\omega(\mathbf{q})) , \quad (4.33)$$

where

$$d_r^0(q)=[q_0-\omega(\mathbf{q})+i0^+]^{-1} , \quad d_a^0(q)=[q_0-\omega(q)-i0^+]^{-1} \quad (4.34)$$

and

$$n(\mathbf{q})=\{\exp[\beta\omega(\mathbf{q})]-1\}^{-1} , \quad (4.35)$$

with

$$\omega(\mathbf{q})=|\mathbf{q}| .$$

We are ready now to calculate the electron self-energy. We have from Eqs. (3.34) and (4.16)

$$\Sigma_r^{(1)}(k)=i\int_{-\infty}^{\infty}(2\pi)^{-1}dq_0\sum_{\mathbf{q}}|g(q)|^2[G_r^0(k+q)D_F^0(q)+G_{+-}^0(k+q)D_a^0(q)]\ . \tag{4.36}$$

Substitution of Eqs. (4.21), (4.23), and (4.28) into the above expression yields after simple manipulations

$$\Sigma_r^{(1)}(k)=\sum_{\mathbf{q}}|g(\mathbf{q})|^2\{[f(\mathbf{k}+\mathbf{q})+n(\mathbf{q})][k_0+\omega(\mathbf{q})-E(\mathbf{k}+\mathbf{q})+i0^+]^{-1} +[1-f(\mathbf{k}+\mathbf{q})+n(\mathbf{q})][k_0-\omega(\mathbf{q})-E(\mathbf{k}+\mathbf{q})+i0^+]^{-1}\}\ , \tag{4.37}$$

which coincides with the known result from Matsubara approach.[9] Further, we make a partial summation over electron self-energy diagrams, using the Migdal approximation, to give

$$\Sigma_r(k)=i\int_{-\infty}^{\infty}(2\pi)^{-1}dq_0\times\sum_{\mathbf{q}}|g(\mathbf{q})|^2[G_r(k+q)D_F(q)+G_{+-}(k+q)D_a(q)]\ . \tag{4.38}$$

With the help of the spectral representations

$$G_r(k)=\int_{-\infty}^{\infty}(2\pi)^{-1}dx\,\rho(\mathbf{k},x)(k_0-x+i0^+)^{-1}\ , \tag{4.39}$$

$$D_r(q)=\int_{-\infty}^{\infty}(2\pi)^{-1}dx\,\sigma(\mathbf{q},x)(q_0-x+i0^+)^{-1}\ , \tag{4.40}$$

where the spectral functions are defined by

$$\rho(k)=i[G_{-+}(k)-G_{+-}(k)]\ ,\qquad \sigma(q)=i[D_{-+}(q)-D_{+-}(q)]\ , \tag{4.41}$$

and the following relation

$$G_{+-}(k)=if(k_0)\rho(k),\quad D_F(q)=D_r(q)-i\sigma(q)n(q_0)\ , \tag{4.42}$$

Eq. (4.38) is reduced, after completing the q_0 integration, to

$$\Sigma_r(k)=\sum_{\mathbf{q}}|g(\mathbf{q})|^2(2\pi)^{-2}\int_{-\infty}^{\infty}dx\int_0^{\infty}dy\,\rho(\mathbf{k}+\mathbf{q},x)\sigma(\mathbf{q},y)\{[f(x)+n(y)](k_0+y-x+i0^+)^{-1} +[1-f(x)+n(y)](k_0-y-x+i0^+)^{-1}\}\ , \tag{4.43}$$

which coincides again with the corresponding expression derived in the theory of imaginary-time formalism.[10] It is seen that the cumbersome evaluation of infinite sums and analytical continuations are avoided in the deduction of Eqs. (4.37) and (4.43). Nontrivial applications of this new approach are planned to be published separately.

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