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Magnetic-field effects on the motion of a charged particle in a heat bath

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In a recent paper, Ford, Lewis, and O'Connell [Phys. Rev. A **37**, 4419 (1988)] considered a charged quantum particle moving in an arbitrary potential and linearly coupled to a heat bath, and they showed that the macroscopic equation describing the time development of the particle motion is in the form of a quantum generalized Langevin equation. We generalize these results to include the presence of an external magnetic field. We find that the magnetic field manifests itself in the presence of an additional term in the Langevin equation, which is the quantum generalization of the Lorentz force, but the magnetic field does not affect the memory function nor the random force appearing in the quantum Langevin equation. It follows that the noise-noise autocorrelation function, as well as the nonequal time commutator of the noise, is the same as that in the absence of a magnetic field. The case of a blackbody radiation heat bath is shown to be easily analyzed as a special case of our general formalism.

I. INTRODUCTION

The problem of a quantum particle coupled to a quantum-mechanical heat bath can be formulated in terms of the quantum Langevin equation. The quantum Langevin equation is a macroscopic equation corresponding to a reduced description of the system in which the coupling with the heat bath is described by two terms: an operator-valued random force $F(t)$ with mean zero, and a mean force characterized by a memory function $\mu(t)$.

Ford, Lewis, and O'Connell[1] (FLO) have shown that the most general quantum Langevin equation can be realized by the independent-oscillator (IO) model of a heat bath. It is a simple and convenient model with which to calculate. Yet by suitably choosing the distribution of the frequencies and force constants for the independent oscillators, one can represent the most general positive real function, and through it the general macroscopic description of the heat-bath problem.

In this paper, we extend the work of FLO to include the presence of a static external magnetic field. What we find is that the only influence of the magnetic field on a charged particle occurs through the addition of an extra term in the quantum Langevin equation (which is the quantum version of the classic Lorentz force), and that the memory function and the random force are unchanged by the magnetic field. A similar problem has previously been considered by Marathe,[2] but that work did not include an external potential; the derivation of the equation of motion implied a special gauge for the vector potential $\mathbf{A}$ and a special choice of the memory function was made in calculating such quantities as the noise-noise autocorrelation function.

In Sec. II we give a general, gauge-independent calculation of the contribution of the external magnetic field to the quantum Langevin equation in the IO model. As has been stressed by FLO, although we utilize the IO model, the equations obtained transcend this model. Next, we calculate the noise-noise autocorrelation function, as well as the nonequal time commutator of the noise, for an arbitrary memory function. In Sec. III we present our conclusions and we discuss briefly the blackbody radiation field heat-bath model (BBR) as an example of the generality of the results we have obtained.

II. THE INDEPENDENT-OSCILLATOR MODEL IN A MAGNETIC FIELD

Our working model is the IO model, in which a charged particle moves in an external magnetic field and in an arbitrary potential, and is linearly coupled to a large (eventually infinite) number of heat-bath particles.[1] The Hamiltonian of the system is then

$$H=\frac{1}{2m}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]^2+V(\mathbf{r})+\sum_j\left[\frac{\mathbf{p}_j^2}{2m_j}+\frac{1}{2}m_j\omega_j^2(\mathbf{q}_j-\mathbf{r})^2\right], \tag{1}$$

where e, m, $\mathbf{p}$, and $\mathbf{r}$ are the charge, mass, momentum, and position of the particle, respectively, and $V(\mathbf{r})$ denotes the external potential. The jth heat-bath particle has a mass m_j, frequency ω_j, position $\mathbf{q}_j$, and momentum $\mathbf{p}_j$. The vector potential $\mathbf{A}(\mathbf{r})$ is related to the magnetic field $\mathbf{B}(\mathbf{r})$ by the equation

$$\mathbf{B}(\mathbf{r})=\nabla_\mathbf{r}\times\mathbf{A}(\mathbf{r}). \tag{2}$$

The commutation rules for the various position and momentum operators are, as usual,

$$[r_\alpha,p_\beta]=i\hbar\delta_{\alpha\beta},\quad [q_{j\alpha},p_{k\beta}]=i\hbar\delta_{jk}\delta_{\alpha\beta}, \tag{3}$$

and all other commutators vanish.

Without the $\mathbf{A}$ field, (1) is just the Hamiltonian considered in the FLO paper.[1] In the presence of an external magnetic field, the motion of the charged particle is generally three dimensional. This necessitates the vector notations in the Hamiltonian. In the following vector analysis, the greek indices stand for three spatial directions (i.e., $\alpha,\beta,\ldots=1,2,3$) and the Roman indices i,j,k denote the different heat-bath particles.

The Heisenberg equations of motion for the heat-bath particles from (1) are

$$\begin{aligned}\dot{\mathbf{q}}_j&=[\mathbf{q}_j,H]/i\hbar=\mathbf{p}_j/m_j,\\ \dot{\mathbf{p}}_j&=[\mathbf{p}_j,H]/i\hbar=-m_j\omega_j^2(\mathbf{q}_j-\mathbf{r}).\end{aligned} \tag{4}$$

These combine to give

$$\ddot{\mathbf{q}}_j+\omega_j^2\mathbf{q}_j=\omega_j^2\mathbf{r}, \tag{5}$$

where the dot denotes the derivative with respect to t.

For the charged particle, the equations of motion are

$$\mathbf{v}\equiv\dot{\mathbf{r}}=[\mathbf{r},H]/i\hbar=\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]\Big/m, \tag{6}$$

$$\begin{aligned}\dot{p}_\alpha&=[p_\alpha,H]/i\hbar\\ &=\frac{1}{2mi\hbar}\left[p_\alpha,\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]^2\right]-\partial_\alpha V+\sum_j m_j\omega_j^2(q_{j\alpha}-r_\alpha),\end{aligned} \tag{7}$$

where $\partial_\alpha\equiv\partial/\partial r_\alpha$ is the spatial derivative.

The first term on the right-hand side of (7) may be written as

$$\frac{1}{2mi\hbar}\left[p_\alpha,\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]^2\right]=\frac{e}{2c}[v_\beta\partial_\alpha A_\beta+(\partial_\alpha A_\beta)v_\beta], \tag{8}$$

where the Einstein summation convention applies to repeated indices. Now

$$\begin{aligned}(\partial_\alpha A_\beta)v_\beta&=v_\beta\partial_\alpha A_\beta+\frac{1}{m}[\partial_\alpha A_\beta,p_\beta]\\ &=v_\beta\partial_\alpha A_\beta+\frac{i\hbar}{m}\partial_\alpha\partial_\beta A_\beta,\end{aligned} \tag{9}$$

and

$$(\mathbf{v}\times\mathbf{B})_\alpha=v_\beta\partial_\alpha A_\beta-v_\beta\partial_\beta A_\alpha. \tag{10}$$

Combining (8), (9), and (10), we have

$$\frac{1}{2mi\hbar}\left[p_\alpha,\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]^2\right]=\frac{e}{c}(\mathbf{v}\times\mathbf{B})_\alpha+\frac{e}{c}v_\beta\partial_\beta A_\alpha+\frac{i\hbar e}{2mc}\partial_\alpha\partial_\beta A_\beta. \tag{11}$$

In vector form, (7) thus becomes

$$\dot{\mathbf{p}}=-\nabla V(\mathbf{r})+\sum_j m_j\omega_j^2(\mathbf{q}_j-\mathbf{r})+\frac{e}{c}(\mathbf{v}\times\mathbf{B})+\frac{e}{c}(\mathbf{v}\cdot\nabla)\mathbf{A}+\frac{i\hbar e}{2mc}\nabla(\nabla\cdot\mathbf{A}). \tag{12}$$

Similarly,

$$\dot{\mathbf{A}}(\mathbf{r})=\frac{\partial\mathbf{A}}{\partial t}+[\mathbf{A},H]/i\hbar=(\mathbf{v}\cdot\nabla)\mathbf{A}+\frac{i\hbar}{2m}\nabla^2\mathbf{A}, \tag{13}$$

where we have used the static condition $\partial\mathbf{A}/\partial t=0$.

Eliminating the momentum variables in (6) and (12), and using (13) we get

$$m\ddot{\mathbf{r}}=-\nabla V(\mathbf{r})+\sum_j m_j\omega_j^2(\mathbf{q}_j-\mathbf{r})+\frac{e}{c}(\mathbf{v}\times\mathbf{B})+\frac{i\hbar e}{2mc}[\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}]. \tag{14}$$

But, from electromagnetism, we know that

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\frac{4\pi}{c}\mathbf{j},$$

where $\mathbf{j}$ is the source current of the external magnetic field. In practice, it lies outside the region where the charged particle moves. Thus the last term in (14) vanishes and (14) becomes

$$m\ddot{\mathbf{r}}=-\nabla V(\mathbf{r})+\sum_j m_j\omega_j^2(\mathbf{q}_j-\mathbf{r})+\frac{e}{c}(\mathbf{v}\times\mathbf{B}). \tag{15}$$

Here we see that the only effect of the magnetic field is the $e/c(\mathbf{v}\times\mathbf{B})$ term, which is the quantum generalization of the classic Lorentz force. We note that (15) is gauge independent.

The retarded solution of (5) is

$$\mathbf{q}_j(t)=\mathbf{q}_j^h(t)+\mathbf{r}(t)-\int_{-\infty}^{t}dt'\cos[\omega_j(t-t')]\dot{\mathbf{r}}(t'), \tag{16}$$

where $\mathbf{q}_j^h(t)$ is the general solution of the homogeneous equation of (5) ($\mathbf{r}\equiv 0$).

Substituting (16) in (15) we get the quantum generalized Langevin equation

$$m\ddot{\mathbf{r}}+\int_{-\infty}^{t}dt'\mu(t-t')\dot{\mathbf{r}}(t')+\nabla V(\mathbf{r})-\frac{e}{c}(\dot{\mathbf{r}}\times\mathbf{B})=\mathbf{F}(t)\ , \tag{17}$$

with the memory function and the random force the same as those given in the FLO paper:

$$\mu(t)=\sum_j m_j\omega_j^2\cos(\omega_j t)\Theta(t)\ , \tag{18}$$

$$\mathbf{F}(t)=\sum_j m_j\omega_j^2\mathbf{q}_j^h(t)\ . \tag{19}$$

Thus (17) is the same as the FLO result except for the last term in the left-hand side of (17). One immediate conclusion is that the symmetric autocorrelation as well as the nonequal time commutator of $\mathbf{F}(t)$ are the same as in the absence of the **B** field:[1]

$$\tfrac{1}{2}\langle F_\alpha(t)F_\beta(t')+F_\beta(t')F_\alpha(t)\rangle =\delta_{\alpha\beta}\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Re}[\tilde{\mu}(\omega+i0^+)]\hbar\omega\coth\left[\frac{\hbar\omega}{2kT}\right]\times\cos[\omega(t-t')]\ , \tag{20}$$

$$[F_\alpha(t),F_\beta(t')] =\delta_{\alpha\beta}\frac{2}{i\pi}\int_0^\infty d\omega\,\mathrm{Re}[\tilde{\mu}(\omega+i0^+)]\hbar\omega\sin[\omega(t-t')]\ , \tag{21}$$

where

$$\tilde{\mu}(z)\equiv\int_0^\infty dt\,e^{izt}\mu(t)\ , \tag{22}$$

and

$$\mathrm{Re}[\tilde{\mu}(\omega+i0^+)] =\frac{\pi}{2}\sum_j m_j\omega_j^2[\delta(\omega-\omega_j)+\delta(\omega+\omega_j)]\ . \tag{23}$$

III. CONCLUSIONS

We have seen that the equation of motion of a charged particle in a heat bath, moving in an arbitrary potential and in an external magnetic field, can still be written in the form of a quantum generalized Langevin equation, with the influence of the magnetic field being exhibited solely by a single extra term, which is the quantum version of the Lorentz force.

In contrast to the corresponding results of Marathe,[2] our results are very general in the sense that (a) they are gauge invariant (a special choice of gauge is implied for the vector potential **A** in Ref. 2); (b) they include the case of an arbitrary external potential $V(\mathbf{r})$; and (c) they apply to any choice of memory function [whereas in Ref. 2 a specific choice of $\mu(t)$ was made, as can be seen from Eq. (2.9) of that paper, and noting that the memory function there is denoted by $K(t)$].

The generality of our results has one immediate consequence, viz., they can be applied to get the corresponding results in a case of much physical interest, viz., the blackbody radiation heat bath.[1,3] By means of a series of unitary transformations, FLO have shown the equivalence of the BBR and IO heat-bath models in the absence of a magnetic field.[1] It turns out that exactly the same transformations apply in the present case. The key point is that the unitary transformations leave **r** unchanged, so that the $-e/c(\dot{\mathbf{r}}\times\mathbf{B})$ term in the equation of motion also remains unchanged. In other words, in the case of the BBR heat bath, we can use (17) as it stands, with the explicit forms for $\mu(t)$ and $\mathbf{F}(t)$ being unchanged from the $B=0$ results [see FLO, Eqs. (5.16) and (5.12) for the explicit respective expressions].

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