Phase-sensitive amplification in a three-level atomic system

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A linear theory of two-photon amplification by three-level atoms in the cascade configuration is developed, where a coherence is induced between the top and bottom levels, by an external classical driving field. It is shown that this system becomes an ideal parametric amplifier for sufficiently strong driving field, whereas for a weak driving field it is a phase-insensitive amplifier. In between these two extremes, one finds phase-sensitive amplification as well as squeezing for a certain range. The system does not, however, reduce to a model studied previously where the atomic coherence was treated as an initial condition. The system is also studied in a cavity configuration: It is predicted that the oscillator may behave as a two-photon correlated emission laser, i.e., its phase diffusion coefficient vanishes.

I. INTRODUCTION

Ordinarily, amplification of a signal is accompanied by the introduction of noise. For so-called phase-insensitive amplifiers, and at the smallest possible level, such degradation of the signal-to-noise ratio upon amplification is in fact required by the laws of quantum mechanics, to preserve the uncertainty principle.[1] It is possible, however, to envision optical phase-sensitive amplifiers[2] which add unequal amounts of noise to the two quadratures of the signal and/or amplify them by different amounts. (Here "quadratures" means those components of the signal in phase and 90° out of phase, respectively, with some external reference oscillator). Such devices may exhibit a wide range of possibilities regarding the amplification of a signal and the introduction of noise. For instance, if both quadratures are amplified by the same amount, one may still reduce the added noise in one quadrature to virtually zero, thus preserving the signal-to-noise ration for that quadrature intact upon amplification, at the expense of increasing the noise added to the conjugate quadrature. One may also have the total added noise to both quadratures go to zero, provided that the two are amplified by unequal amounts—more precisely, if one is deamplified by the same amount as the other one is amplified. This last instance of a phase-sensitive amplifier corresponds, in fact, to an ideal parametric amplifier, a device which, at optical frequencies, is usually embodied in a nonlinear crystal with a $\chi^{(2)}$ coefficient. This device has already demonstrated its ability to manipulate noise, even down to the quantum level, in a phase-sensitive way, by generating light in a so-called "squeezed state".[3,4]

Mostly in connection with squeezing, the theory of linear amplifiers has received a great deal of attention over the past few years[5] (results for some nonlinear devices have also been presented; see Ref. 6). Still, the number of truly microscopic models for phase-sensitive amplifiers is reduced. One should mention the models for four-wave mixing in atomic systems,[7] which has been also demonstrated experimentally to generate squeezed light,[8] and also the various models of correlated-emission laser (CEL).[9–11] Other systems are presented in Refs. 12 and 13.

Scully and Zubairy proposed recently[14] a model consisting of three-level atoms, in the cascade configuration, prepared in a coherent superposition of the upper and lower states. Such a system would be an example of the first kind of phase-sensitive amplifier mentioned above, namely one which amplifies both quadratures by the same amount but adds vanishing noise to one quadrature. The three-level cascade atom with initial atomic coherence is also the basis for the two-photon CEL,[11] a device which would exhibit a number of interesting properties including, potentially, an arbitrarily high degree of intracavity squeezing, even in the presence of cavity losses.

In this paper we present a study of the three-level cascade system when the atomic coherence is established by driving the atoms (initially pumped incoherently to the upper state) continually with a strong external field. We have found that this system exhibits remarkable differences with the previously studied case[14] where the atomic coherence might be prepared before the interaction by, e.g., a short, strong pulse, so that there would be no external field present during the amplification process itself. This indicates that the two methods for generating the atomic coherence are not at all equivalent: in fact, we find that the system with the driving field studied here does not, under any condition, reduce to the one studied in Ref. 14. Instead, it exhibits, as a function of the driving field strength, a range of possible behavior which is quite interesting in its own right, from a phase-insensitive amplifier for low driving field to an ideal parametric amplifier at the other extreme. This makes it remarkable as an instance of a microscopic model for parametric amplification in an atomic system.

In addition to the amplifier, we have also studied some of the aspects of the possible performance of the system in an oscillator configuration. This, of course, is of interest because of the close connection to the two-photon CEL.[11] Here we find that the system does indeed reproduce many of the features of Ref. 11 (although the details vary) and may in fact be considered as an alternative way to produce a two-photon CEL.

This paper is arranged as follows. In Sec. II the basic model is introduced and the master equation for the evolution of the density matrix of a single-mode field interacting with the atomic system is derived. Section III deals with the amplifier configuration and thus with such quantities as the gain and added noise for the two quadratures, for different values of the driving field strength. The connection with Caves's theory of phase-sensitive amplifiers,[2] of which this particular system may be considered to be a very apt illustration, is also made in Sec. III. Section IV discusses the oscillator configuration, and considers such quantities as phase diffusion, locking, and squeezing.

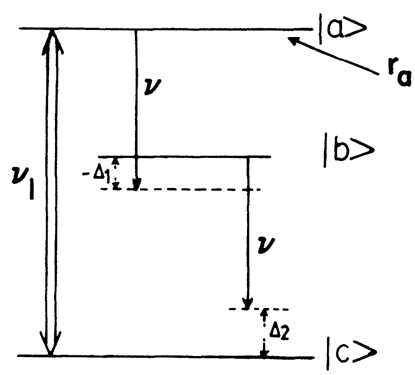

FIG. 1. Energy diagram of a three-level atomic system in cascade configuration.

II. MODEL AND EQUATION OF MOTION FOR THE DENSITY MATRIX

We consider a system of three-level atoms in the cascade configuration (Fig. 1). The transitions from level $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ are dipole allowed but $|a\rangle \rightarrow |c\rangle$ is dipole forbidden. We assume that this transition may be induced by applying a sufficiently strong external field. We treat the $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ transitions quantum mechanically up to second order in the coupling constant and the $|a\rangle \rightarrow |c\rangle$ transition semiclassically to all orders.

The Hamiltonian for the atom-field system is

$$H = H_0 + V \ , \tag{1}$$

where the unperturbed part is

$$H_0 = \sum_{i=a,b,c} \hbar\omega_i |i\rangle\langle i| + \hbar\nu a^\dagger a \ , \tag{2}$$

and the interaction term is

$$V = \hbar g[(|a\rangle\langle b| + |b\rangle\langle c|)a + a^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) - \frac{\hbar\Omega}{2}(e^{-i\phi - i\nu_1 t}|a\rangle\langle c| + e^{i\phi + i\nu_1 t}|c\rangle\langle a|)] \ . \tag{3}$$

Here a and $a^\dagger$ are the destruction and creation operators for the field mode of frequency ν, g is the atom-field coupling constant (assumed equal for both transitions), Ω is the Rabi frequency of the driving classical field, and ϕ and ν_1 are its phase and frequency, respectively. In what follows, we assume exact resonance of the external field with the $|a\rangle \rightarrow |c\rangle$ transition, i.e., $\nu_1 = \omega_a - \omega_c$.

We can obtain the reduced density matrix for the field by taking the trace over the atomic states,

$$\langle n|\rho_F|n'\rangle = \langle a,n|\rho|a,n'\rangle + \langle b,n|\rho|b,n'\rangle + \langle c,n|\rho|c,n'\rangle \ . \tag{4}$$

It is convenient to define the following atom-field states:

$$|1\rangle = |a,n-2\rangle \ , \quad |2\rangle = |b,n-1\rangle \ , \quad |3\rangle = |c,n\rangle \ . \tag{5}$$

We can calculate the density-matrix equation of motion for the field mode along similar lines as shown in Ref. 10. Here we consider that the atoms are injected in level $|a\rangle$ at the rate r_a. We also assume for simplicity equal decay rates γ for all three levels. We calculate the density-matrix elements in the basis (5) and trace as in Eq. (4) to obtain the reduced density-matrix equation of motion

$$\dot{\rho} = -[\beta_{11}^* a a^\dagger \rho_F + \beta_{11}\rho_F a a^\dagger - (\beta_{11} + \beta_{11}^*) a^\dagger \rho_F a] - [\beta_{22}^* a^\dagger a \rho_F + \beta_{22}\rho_F a^\dagger a - (\beta_{22} + \beta_{22}^*) a \rho_F a^\dagger] - [\beta_{12}^* a a \rho_F + \beta_{21}\rho_F a a - (\beta_{12}^* + \beta_{21}) a \rho_F a] e^{-i\Phi} - [\beta_{21}^* a^\dagger a^\dagger \rho_F + \beta_{12}\rho_F a^\dagger a^\dagger - (\beta_{12} + \beta_{21}^*) a^\dagger \rho_F a^\dagger] e^{i\Phi} \ , \tag{6}$$

where

$$\beta_{11} = \frac{g^2}{4} r_a \left[\left[\frac{1}{\gamma} + \frac{1}{\gamma - i\Omega} \right] \frac{1}{\gamma + i(\Delta_1 - \Omega/2)} + \left[\frac{1}{\gamma} + \frac{1}{\gamma + i\Omega} \right] \frac{1}{\gamma + i(\Delta_1 + \Omega/2)} \right] , \tag{7a}$$

$$\beta_{12} = \frac{g^2}{4} r_a \left[\left[\frac{1}{\gamma} - \frac{1}{\gamma - i\Omega} \right] \frac{1}{\gamma - i(\Delta_2 + \Omega/2)} - \left[\frac{1}{\gamma} - \frac{1}{\gamma + i\Omega} \right] \frac{1}{\gamma - i(\Delta_2 - \Omega/2)} \right] , \tag{7b}$$

$$\beta_{21} = \frac{g^2}{4} r_a \left[\left[\frac{1}{\gamma} + \frac{1}{\gamma - i\Omega} \right] \frac{1}{\gamma + i(\Delta_1 - \Omega/2)} - \left[\frac{1}{\gamma} + \frac{1}{\gamma + i\Omega} \right] \frac{1}{\gamma + i(\Delta_1 + \Omega/2)} \right] , \tag{7c}$$

$$\beta_{22} = \frac{g^2}{4} r_a \left[\left[\frac{1}{\gamma} - \frac{1}{\gamma - i\Omega} \right] \frac{1}{\gamma - i(\Delta_2 + \Omega/2)} + \left[\frac{1}{\gamma} - \frac{1}{\gamma + i\Omega} \right] \frac{1}{\gamma - i(\Delta_2 - \Omega/2)} \right] . \tag{7d}$$

with $\Delta_1=\omega_a-\omega_b-\nu$, $\Delta_2=\omega_b-\omega_c-\nu$. The first term in Eq. (6) corresponds to the gain and the second term to the absorption in the system. The third and fourth terms are due to the coherent excitation of the atomic states because of the classical field and are responsible for the phase sensitivity in the system. The phase angle Φ appearing in Eq. (6) equals $\Phi=\phi+(\nu_1-2\nu)t$ and has the character of a phase-matching condition. In the following, we shall assume strict two-photon resonance throughout, i.e., $2\nu=\nu_1$. This implies $\Delta_1+\Delta_2=0$. Then $\Phi=\phi$, the external field reference phase.

III. TWO-PHOTON LINEAR AMPLIFIER

In this section we consider the case of the two-photon linear amplifier, and calculate the time-dependent solution of the equations of motion for various quantities under certain limits. Then we calculate the noise and gain contributions in the two quadratures. From Eq. (6) we have

$$\frac{d\langle a\rangle}{dt}=\rho_1\langle a\rangle+\rho_2\langle a^\dagger\rangle , \tag{8a}$$

$$\frac{d\langle a^\dagger a\rangle}{dt}=(\rho_1+\rho_1^*)\langle a^\dagger a\rangle+\rho_2^*\langle a^\dagger a^\dagger\rangle+\rho_2\langle aa\rangle+\rho_{11} , \tag{8b}$$

$$\frac{d\langle aa\rangle}{dt}=2\rho_1\langle aa\rangle+2\rho_2\langle a^\dagger a\rangle-2\beta_{21}^*e^{i\phi} , \tag{8c}$$

where

$$\rho_1=(\beta_{11}-\beta_{22}^*) , \tag{9a}$$

$$\rho_2=(\beta_{12}-\beta_{21}^*)e^{i\phi} , \tag{9b}$$

$$\rho_{11}=\beta_{11}+\beta_{11}^* . \tag{9c}$$

These equations can be solved exactly. In what follows, we shall assume that we have $\rho_1=\rho_1^*$ and $\rho_2=|\rho_2|$. The first assumption is necessary to separate out the gain and noise terms, and we shall discuss below under which conditions it holds. The second assumption amounts to a choice of the reference phase ϕ, i.e., to a certain choice of quadratures, and it is made for convenience. The solutions to Eqs. (8) then are

$$\langle a\rangle_t=[\cosh(\rho_2t)\langle a\rangle_0+\sinh(\rho_2t)\langle a^\dagger\rangle_0]e^{\rho_1t} , \tag{10a}$$

$$\begin{aligned}\langle a^\dagger a\rangle_t=&[\langle a^\dagger a\rangle_0\cosh(2\rho_2t)+(\tfrac{1}{2}\langle aa\rangle_0+\tfrac{1}{2}\langle a^\dagger a^\dagger\rangle_0)\sinh(2\rho_2t)]e^{2\rho_1t}\\&+\frac{\rho_{11}}{2(\rho_1^2-\rho_2^2)}\{[\rho_1\cosh(2\rho_2t)-\rho_2\sinh(2\rho_2t)]e^{2\rho_1t}-\rho_1\}\\&+\frac{\beta_{21}^*e^{i\phi}+\beta_{21}e^{-i\phi}}{2(\rho_1^2-\rho_2^2)}\{[\rho_2\cosh(2\rho_2t)-\rho_1\sinh(2\rho_2t)]e^{2\rho_1t}-\rho_2\} ,\end{aligned} \tag{10b}$$

$$\begin{aligned}\langle aa\rangle_t=&\left[\frac{\langle aa\rangle_0}{2}[\cosh(2\rho_2t)+1]+\langle a^\dagger a\rangle_0\sinh(2\rho_2t)+\tfrac{1}{2}\langle a^\dagger a^\dagger\rangle_0[\cosh(2\rho_2t)-1]\right]e^{2\rho_1t}\\&+\frac{\rho_{11}}{2(\rho_1^2-\rho_2^2)}\{[\rho_1\sinh(2\rho_2t)-\rho_2\cosh(2\rho_2t)]e^{2\rho_1t}+\rho_2\}\\&+\frac{\beta_{21}e^{-i\phi}}{2(\rho_1^2-\rho_2^2)}\left[\left[\rho_2\sinh(2\rho_2t)-\rho_1\cosh(2\rho_2t)+\frac{1}{2\rho_1}\right]e^{2\rho_1t}+\frac{\rho_2^2}{\rho_1}\right]\\&+\frac{\beta_{21}^*e^{i\phi}}{2(\rho_1^2-\rho_2^2)}\left[\left[\rho_2\sinh(2\rho_2t)-\rho_1\cosh(2\rho_2t)-\frac{1}{2\rho_1}\right]e^{2\rho_1t}+\frac{2\rho_1^2-\rho_2^2}{\rho_1}\right] .\end{aligned} \tag{10c}$$

If we define our quadratures as

$$a_1=\frac{1}{2}(a+a^\dagger) , \tag{11a}$$

$$a_2=\frac{1}{2i}(a-a^\dagger) , \tag{11b}$$

then we get from Eq. (10a) and its complex conjugate

$$\langle a_1\rangle_t=\sqrt{G_1}\langle a_1\rangle_0 , \tag{12a}$$

$$\langle a_2\rangle_t=\sqrt{G_2}\langle a_2\rangle_0 , \tag{12b}$$

where

$$G_1=e^{2(\rho_1+\rho_2)t} , \tag{13a}$$

$$G_2=e^{2(\rho_1-\rho_2)t} , \tag{13b}$$

are the gain factors for the different quadratures. The phase sensitivity is apparent already in that in general $G_1\neq G_2$ unless $\rho_2=0$ (an additional condition which will be discussed below). To calculate the noise in both quadratures we use Eqs. (10b) and (10c) and obtain

$$(\Delta a_1^2)=G_1(\Delta a_1^2)_0+N_1(G_1-1) , \tag{14a}$$

$$(\Delta a_2^2)=G_2(\Delta a_2^2)_0+N_2(G_2-1) , \tag{14b}$$

where

$$N_1=\frac{\beta_{11}+\beta_{22}-(\beta_{12}e^{i\phi}+\beta_{21}e^{-i\phi})+\text{ c.c.}}{8(\rho_1+\rho_2)} \tag{15a}$$

$$N_2=\frac{\beta_{11}+\beta_{22}+(\beta_{12}e^{i\phi}+\beta_{21}e^{-i\phi})+\text{ c.c.}}{8(\rho_1-\rho_2)} \tag{15b}$$

are the added noise terms. In this notation, Caves's theorem[2] for phase-sensitive linear amplifiers becomes

$$|N_1N_2|^{1/2}\geq\frac{1}{4}\frac{|(G_1G_2)^{1/2}-1|}{(G-1)^{1/2}(G_2-1)^{1/2}}, \tag{16}$$

which may be called the "amplifier uncertainty principle". In the following we discuss certain special cases.

A. Zero detuning

In this section we calculate the noise and gain terms in the simplest case of zero detuning, i.e., $\Delta_1=\Delta_2=0$. This condition requires level $|b\rangle$ to lie exactly halfway between $|a\rangle$ and $|c\rangle$. Under this assumption, the condition $\rho_1=\rho_1^*$ is automatically satisfied. To make $\rho_2\exp(i\phi)$ real and positive requires only to set $\phi=-\pi/2$, which amounts to a specific choice of quadratures relative to the classical field. Under these conditions, we find for the different coefficients which appeared in the noise and again terms, the expressions are

$$\beta_{11}=\frac{g^2r_a}{(\gamma^2+\Omega^2)}, \tag{17a}$$

$$\beta_{22}=\frac{3g^2r_a\Omega^2}{4(\gamma^2+\Omega^2)(\gamma^2+\Omega^2/4)}, \tag{17b}$$

$$\beta_{12}e^{i\phi}=\frac{-g^2r_a\Omega(\Omega^2/2-\gamma^2)}{2\gamma(\gamma^2+\Omega^2)(\gamma^2+\Omega^2/4)}, \tag{17c}$$

$$\beta_{21}e^{-i\phi}=\frac{-g^2r_a\Omega}{\gamma(\gamma^2+\Omega^2)}. \tag{17d}$$

The noise and gain terms become

$$N_1=\frac{2\gamma^3+2\gamma\Omega^2+3\gamma^2\Omega}{4(2\gamma^3+\Omega^3+\gamma^2\Omega-\gamma\Omega^2)}, \tag{18a}$$

$$N_2=\frac{2\gamma^2+2\gamma\Omega^2-3\gamma^2\Omega}{4(2\gamma^3-\Omega^3-\gamma^2\Omega-\gamma\Omega^2)}, \tag{18b}$$

and

$$G_1=\exp\left[\frac{\alpha\gamma t(2\gamma^3+\Omega^3+\gamma^2\Omega-\gamma\Omega^2)}{2(\gamma^2+\Omega^2)(\gamma^2+\Omega^2/4)}\right], \tag{19a}$$

$$G_2=\exp\left[\frac{\alpha\gamma t(2\gamma^3-\Omega^3-\gamma^2\Omega-\gamma\Omega^2)}{2(\gamma^2+\Omega^2)(\gamma^2+\Omega^2/4)}\right] \tag{19b}$$

where $\alpha=2g^2r_a/\gamma^2$ is the linear gain coefficient in the absence of the driving field. Equations (18) and (19) simplify considerably in the following two limits, corresponding to two very different kinds of behavior.

(i) When $\Omega\gg\gamma$, i.e., when the Rabi frequency of the classical driving field is much larger than the atomic level width γ, we obtain

$$N_1\to\gamma/2\Omega, \tag{20a}$$

$$N_2\to-\gamma/2\Omega, \tag{20b}$$

$$G_1\to e^{2\alpha\gamma t/\Omega}, \tag{20c}$$

$$G_2\to e^{-2\alpha\gamma t/\Omega}. \tag{20d}$$

The N_2 term becomes negative for the values of $G_2<1$, in order to keep the second term of Eq. (14b) positive. We see that in the limit $\Omega/\gamma\to\infty$, $\alpha t\to\infty$, $\alpha\gamma t/\Omega$ finite, both noises approach zero and $G_1=1/G_2$. Thus for large driving Rabi frequency the system becomes identical to the degenerate parametric amplifier. This may also be checked directly by using the appropriate limit forms of β_{11}, β_{22}, β_{12}, and β_{21} [Eqs. (17)] in the master equation (6). The phase-insensitive terms proportional to β_{11} and β_{22} approach zero faster than the phase-sensitive ones, and in the latter $\beta_{12}\to\beta_{21}$, $\beta_{12}+\beta_{21}^*\to0$, yielding the master equation for a parametric amplifier in the absence of pump depletion.[15] The amplifier uncertainty principle (16) is satisfied in this case with the equal sign (both sides being equal to zero).

(ii) When $\gamma\gg\Omega$, i.e., the atomic width is much larger than the Rabi frequency of the driving classical field, we have

$$N_1\to\tfrac{1}{4}, \tag{21a}$$

$$N_2\to\tfrac{1}{4}, \tag{21b}$$

$$G_1\to e^{\alpha t}, \tag{22a}$$

$$G_2\to e^{\alpha t}, \tag{22b}$$

i.e., equal noise is added to both quadratures and they are amplified with equal gain. Thus we have a phase-insensitive amplifier in this limit, adding the minimum amount of noise required by the amplifier uncertainty principle (16): $(N_1N_2)^{1/2}\geq\frac{1}{4}$.

Between these two extremes we have various kinds of phase-sensitive amplification. In Figs. (2) and (3) we have plotted the noise and gain for both quadratures versus Ω/γ for $\alpha t=1$. For $\Omega<0.81\gamma$, both quadratures are

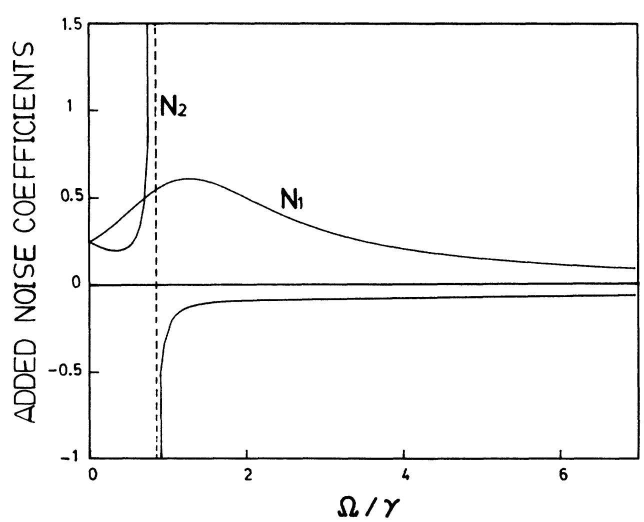

FIG. 2. Added noise coefficients vs Ω/γ.

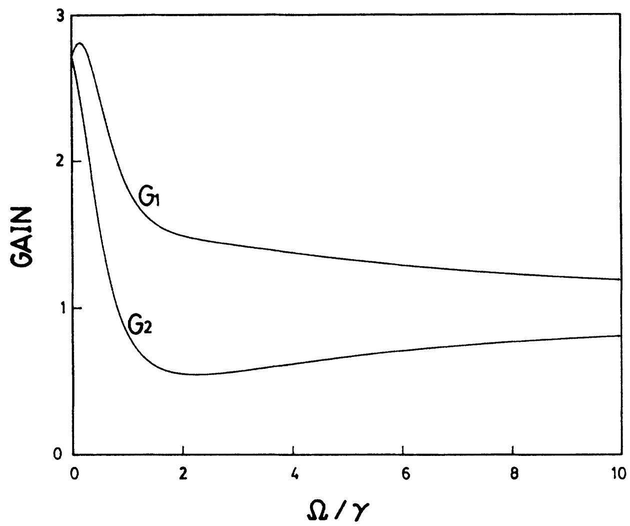

FIG. 3. Gain coefficients vs Ω/γ for $\alpha t = 1$.

amplified. At $\Omega=0.81\gamma$, one has $\rho_1=\rho_2$, so that $G_2=1$. The right-hand side of (16) then becomes infinite. Figure 2 shows how the amplifier uncertainty principle is satisfied in the case as $|N_2|\to\infty$. The product $N_2(G_2-1)$ approaches a finite value proportional to t; thus, for $\Omega=0.81\gamma$, the quadrature a_2 is not amplified, but diffusion-type noise is added to it, so $\langle \Delta a_2\rangle \propto \sqrt{t}$.

Beyond $\Omega=0.81\gamma$, the second quadrature is deamplified [note how N_2 becomes negative so that $N_2(G_2-1)$ remains positive], and for $\Omega>\gamma$ one has $|N_2|<\frac{1}{4}$, which means that a_2 becomes, in fact, squeezed for sufficiently long interaction times, regardless of the initial state. This squeezing is shown in Fig. 4 where we plot $\langle \Delta a_2^2\rangle_t$ versus αt for different values of Ω/γ assuming $\langle \Delta a_2^2\rangle_0=\frac{1}{4}$ (i.e., the initial state is taken to be the vacuum, or a coherent state). As for the conjugate quadrature, Figs. 2 and 3 show that it is always amplified, and with negligible added noise as $\Omega/\gamma\to 0$ (parametric amplifier limit).

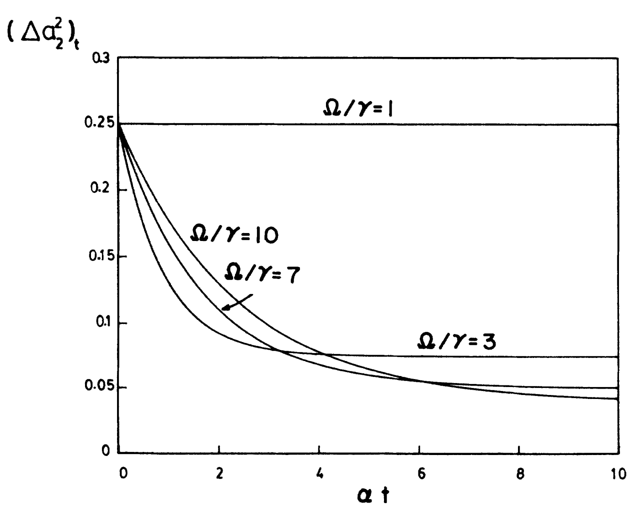

FIG. 4. Squeezing in quadrature a_z vs αt for $\Omega/\gamma=1$, 3, 7, and 10.

B. Nonzero detuning

In Sec. III A we considered the situation when the middle level is exactly halfway between $|a\rangle$ and $|c\rangle$, and the one-photon resonance condition is satisfied. A more general case is $\Delta_1=-\Delta_2=\Delta$, i.e., the middle level is not exactly one-photon resonant but two-photon resonance is still obtained, that is, $\omega_a-\omega_c=2\nu$. In this case, however, $\rho_1=\rho_1^*$ is not automatically guaranteed, with the result that the amplification will in general mix the two quadratures (regardless of how these are defined). To have $\rho_1=\rho_1^*$, we must impose the additional condition

$$\Omega^2=4(\gamma^2+\Delta^2)\ , \tag{23}$$

whereas to make ρ_2 real and positive we must choose the angle ϕ so that

$$e^{i\phi}=-\frac{\gamma\Delta+i(\gamma^2+\Omega^2)}{\Omega(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}\ , \tag{24a}$$

for which

$$\rho_2=r_\alpha g^2\frac{(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}{\gamma(\gamma^2+\Omega^2)}\ . \tag{24b}$$

the value of ρ_1 when Eq. (23) is satisfied is

$$\rho_1=\frac{-r_\alpha g^2}{2}\frac{1}{\gamma^2+\Omega^2}\ , \tag{24c}$$

$$\beta_{11}+\text{ c.c.}=\frac{g^2r_a}{2}\frac{\Omega^2}{\gamma^2(\gamma^2+\Omega^2)}\ , \tag{25a}$$

$$\beta_{22}+\text{ c.c.}=g^2r_a\frac{\gamma^2+\Omega^2/2}{\gamma^2(\gamma^2+\Omega^2)}\ , \tag{25b}$$

$$\beta_{12}+\text{ c.c.}=g^2r_a\frac{\frac{3}{2}\gamma^2+\frac{1}{4}\Omega^2}{\gamma(\gamma^2+\Omega^2)(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}\ , \tag{25c}$$

$$\beta_{21}+\text{ c.c.}=-g^2r_a\frac{3\gamma^2+\frac{7}{4}\Omega^2}{\gamma(\gamma^2+\Omega^2)(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}\ . \tag{25d}$$

The noise terms in both the quadratures are

$$N_1=\frac{[2(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}+3\gamma](\gamma^2+\Omega^2)}{[(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}-\gamma]\gamma(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}\ , \tag{26a}$$

$$N_2=\frac{[2(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}-3\gamma](\gamma^2+\Omega^2)}{[-(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}-\gamma]\gamma(\frac{9}{4}\gamma^2+\Omega^2)^{1/2}}\ , \tag{26b}$$

and the gain terms are

$$G_1=\exp\left[\frac{\alpha\gamma t[-\gamma+(9\gamma^2+4\Omega^2)^{1/2}]}{2(\gamma^2+\Omega^2)}\right]\ , \tag{27a}$$

$$G_2=\exp\left[\frac{\alpha\gamma t[-\gamma-(9\gamma^2+4\Omega^2)^{1/2}]}{2(\gamma^2+\Omega^2)}\right]\ . \tag{27b}$$

Now one always has phase-sensitive amplification, with the quadrature a_2 being deamplified ($G_2<1$) and the quadrature a_1 amplified ($G_1>1$) in all cases. We also find that under the limit $\Omega\gg\gamma$, $G_1/G_2\cong\exp(\alpha\gamma t/\Omega)$, and $N_1=-N_2\cong\Omega/(4\gamma)$ when Eq. (23) is satisfied. This last result indicates that in this limit the system does not behave like a parametric amplifier, but instead introduces large amounts of excess noise.

We have also investigated the limit of very large driving field Ω with Δ_1,Δ_2 arbitrary. When $\Omega\gg\gamma,\Delta_1,\Delta_2$, the general coefficients β_{11}, β_{12}, β_{21}, and β_{22} acquire simple forms, which are, to leading order,

$$\beta_{11}=\frac{g^2r_a(\gamma+2i\Delta_i)}{\gamma\Omega^2}\ , \tag{28a}$$

$$\beta_{12}=i\frac{g^2r_a}{\gamma\Omega}\ , \tag{28b}$$

$$\beta_{21}=i\frac{g^2r_a}{\gamma\Omega}\ , \tag{28c}$$

$$\beta_{22}=\frac{g^2r_a(3\gamma-2i\Delta_2)}{\gamma\Omega^2}\ , \tag{28d}$$

and so

$$\rho_1=\frac{g^2r_a}{\gamma\Omega^2}[-2\gamma+2i(\Delta_1-\Delta_2)]\ , \tag{29a}$$

$$\rho_2=2i\frac{g^2r_a}{\gamma\Omega}e^{i\phi}\ , \tag{29b}$$

which shows that, in this limit, ρ_1 is usually not real under two-photon resonance conditions (i.e., $\Delta_1=-\Delta_2=\Delta$) unless $\Delta\ll\gamma$.

C. Comparison with the system with injected atomic coherence

The previous results show that this system, with external driving field, behaves quite differently from the one studied in Ref. 14, which involved the same level structure but assumed an initial atomic coherence between the levels $|a\rangle$ and $|c\rangle$. That system also showed phase-sensitive amplification, but equal gain factors for both quadratures; thus, the only way to reduce the added noise in one of them was to increase it in the other. There was no parametric limit where both N_1 and N_2 could be made vanishingly small.

In the present system, in order to have equal gains we must have $\rho_2=0$. This turns out to be impossible under two-photon resonance conditions. It can also be shown that even for arbitrary detunings Δ_1 and Δ_2 it is impossible to have simultaneously $\rho_2=0$ and $\rho_1=\rho_1^*$. Thus the system never reduces to the one studied in Ref. 14.

Our results show, therefore, that the way in which an atomic coherence is established may have profound consequences: different ways, which one might have expected *a priori* to be roughly comparable, may result in systems with very different properties.

IV. TWO-PHOTON LASER

In this section we consider some of the properties of a (two-photon) laser whose gain medium is the amplifier studied in Sec. III; i.e., we assume that the amplifier is enclosed in a cavity resonant at frequency ν. We modify our density-matrix equation of motion [Eq. (6)] by adding to it the cavity-loss terms

$$(\dot\rho_F)_{\text{loss}}=-\Gamma(a^\dagger a\rho_F+\rho_F a^\dagger a-2a\rho_F a^\dagger)\ , \tag{30}$$

where Γ is the cavity amplitude decay rate.

Next we convert the density-matrix equation of motion for the field mode in a c-number Fokker-Planck equation in the P representation via the substitution[16]

$$a|\alpha\rangle\langle\alpha|=\alpha|\alpha\rangle\langle\alpha|\ , \tag{31a}$$

$$a^\dagger|\alpha\rangle\langle\alpha|=\left[\frac{\partial}{\partial\alpha}+\alpha^*\right]|\alpha\rangle\langle\alpha|\ . \tag{31b}$$

The resultant Fokker-Planck equation is of the form

$$\frac{\partial P}{\partial t}=\left[\beta_{11}\frac{\partial^2}{\partial\alpha\partial\alpha^*}+(\beta_{11}-\beta_{22}'^*)\frac{\partial}{\partial\alpha}\alpha-(\beta_{12}-\beta_{21}^*)e^{i\phi}\frac{\partial}{\partial\alpha}\alpha^* -\beta_{21}^*e^{i\phi}\frac{\partial^2}{\partial\alpha^2}+\text{ c.c.}\right]P\ . \tag{32a}$$

where

$$\beta_{22}'=\beta_{22}+\Gamma\ . \tag{32b}$$

Introducing the amplitude and phase variables r and θ as

$$\alpha=re^{i\theta}\ , \tag{33}$$

we have

$$\frac{\partial}{\partial\alpha}=\tfrac{1}{2}e^{-i\theta}\left[\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\theta}\right]\ , \tag{34a}$$

$$\frac{\partial}{\partial\alpha^*}=\tfrac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right]\ , \tag{34b}$$

and the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t}=-\frac{1}{2r}\frac{\partial(d_IP)}{\partial r}-\frac{\partial(d_\theta P)}{\partial\theta} +\frac{1}{4r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(d_{II}P)}{\partial r}+\frac{\partial^2(D_{\theta\theta}P)}{\partial\theta^2}+\frac{1}{2r}\frac{\partial^2(D_{I\theta}P)}{\partial r\partial\theta}\ , \tag{35}$$

where the drift and diffusion coefficients are

$$d_I=r^2(\rho_1+\rho_1^*+\rho_2e^{-2i\theta}+\rho_2^*e^{2i\theta})+\rho_{11}\ , \tag{36a}$$

$$d_\theta=\frac{i}{2}(\rho_1-\rho_1^*+\rho_2e^{-2i\theta}-\rho_2^*e^{2i\theta}) +\frac{1}{r^2}(\beta_{21}^*e^{i(\phi-2\theta)}-\beta_{21}e^{-i(\phi-2\theta)})\ , \tag{36b}$$

$$D_{I\theta}=i(\beta_{21}^*e^{i(\phi-2\theta)}-\beta_{21}e^{-i(\phi-2\phi)})\ , \tag{36c}$$

$$D_{II}=r^2(\rho_{11}-\beta_{21}e^{-i(\phi-2\theta)}-\beta_{21}^*e^{i(\phi-2\theta)})\ , \tag{36d}$$

$$D_{\theta\theta}=\frac{1}{4r^2}(\rho_{11}+\beta_{21}e^{-i(\phi-2\theta)}+\beta_{21}^{*}e^{i(\phi-2\theta)}) , \qquad (36e)$$

where ρ_1, ρ_2, and ρ_{11} are given by Eqs. (9a)–(9c) with $\beta_{22}=\beta_{22}'$ given by Eq. (32b). Since we have ignored the saturation of the amplifier medium, this equation is, strictly speaking, valid only in the linear regime, i.e., below threshold. It is known, however, that one may obtain from such an equation some information regarding the above-threshold properties of the laser; in particular, the phase diffusion $D_{\theta\theta}$ and locking d_θ terms above threshold are as given by Eqs. (36b) and (36e), to a good approximation, with r^2 regarded as constant and equal to $\bar{n}$, the average number of photons in the cavity. We concentrate on these terms in what follows. They are of interest especially because of the close relationship between this system and the "two-photon CEL" (Ref. 11) which has the same level structure but injected atomic coherence instead of an external driving field.

From Eq. (36b), the phase-drift term yields (neglecting the small diffusion-induced drift proportional to $1/\bar{n}$)

$$\frac{d\langle\theta\rangle}{dt}=\langle d_\theta\rangle\approx|\rho_2|\,\langle\sin2\theta\rangle , \qquad (37)$$

where we assume again $\rho_1=\rho_1^*$, $\rho_2=|\rho_2|$. From this we can find out the particular locked phase angle choice for which $\langle d_\theta\rangle=0$ and the stability condition $d\langle d_\theta\rangle/d\theta<0$ is satisfied, namely, for $\theta=\pi/2$. The diffusion coefficient is then

$$\begin{aligned}D_{\theta\theta}&=\frac{1}{4\bar{n}}(\rho_{11}-\beta_{21}e^{-i\phi}+\beta_{21}^{*}e^{i\phi}) ,\\ &=\frac{1}{4\bar{n}}(4N_2+1)(\rho_1-|\rho_2|) ,\\ &=\frac{g^2r_a}{2\bar{n}}\frac{1}{\gamma^2+\Omega^2}\left[1-\frac{\Omega}{\gamma}\right] ,\end{aligned} \qquad (38)$$

where the last line applies specifically in the zero-detuning case discussed in Sec. III A. We immediately see that for $\gamma=\Omega$ the CEL condition is obtained, i.e., the spontaneously emitted photons are highly correlated so that the phase diffusion vanishes. When $\Omega>\gamma$ the diffusion coefficient becomes negative; this corresponds to a nonclassical state of the intracavity field mode for which P becomes nonpositive definite (and in general highly singular). Recall that in the limit $\Omega\gg\gamma$ we have essentially an ideal parametric amplifier in the cavity, i.e., an ideal parametric oscillator; this system has been quite thoroughly studied in the literature.[15] For $\gamma\gg\Omega$, on the other hand, we have an ordinary two-photon laser.

In Ref. 11, use we was made of the following equation for the magnitude of the phase fluctuations, above threshold, in the locked regime:

$$\langle\delta\theta^2\rangle=\frac{1}{4\bar{n}}+\frac{\langle D_{\theta\theta}(\theta)\rangle}{|\partial d_\theta(\theta_0)/\partial\theta|} , \qquad (39)$$

where the first term represents the standard phase uncertainty for a coherent state with a large average number of photons $\bar{n}$. A negative $\langle D_{\theta\theta}(\theta)\rangle$ corresponds to squeezing of the phase fluctuations in the intracavity field. In our case, we obtain for (39)

$$\begin{aligned}\langle\delta\theta^2\rangle&=\frac{1}{4\bar{n}}\left[1+\frac{(4N_2+1)(\rho_1-\rho_2)}{2\rho_2}\right] ,\\ &=\frac{1}{8\bar{n}}\left[2+\frac{4\gamma^2+\Omega^2}{\gamma^2+\Omega^2}\left[\frac{\gamma}{\Omega}-1\right]\right] ,\end{aligned} \qquad (40)$$

(again, the last line holds for the zero detuning case). We find (not surprisingly) that the phase is squeezed when $\Omega>\gamma$, which is when $0>N_2>-\frac{1}{4}$ and $\rho_1-\rho_2<0$. Note how the maximum squeezing achievable is a factor of $\frac{1}{2}$ below the coherent state level, in the limit $\Omega/\gamma\to\infty$; this is consistent with the fact that one has a parametric oscillator in that limit, and the familiar result that the squeezing in the intracavity field for a parametric oscillator cannot exceed 50%. More importantly perhaps, this agrees also with the appropriate limit for the two-photon CEL discussed in Ref. 11, namely, the case when $\rho_{bb}=\rho_{ba}=\rho_{ab}=0$ initially [see the discussion about Eq. (14) in Ref. 11]. Thus, in this case the two ways to prepare the atomic coherence appear to lead to equivalent results, although the details vary: for instance, large detunings (equal and opposite) were required in Ref. 11 for maximum phase noise reduction, which here appears to be possible with $\Delta=0$ (one may still think of the present system as being highly detuned in this case, however, if the Stark shift of the levels a and c due to the strong external field is considered). Also, the density-matrix equation for the system of Ref. 11 does not reduce to the parametric amplifier form in this limit.

Note that under the phase locked condition the linear gain is

$$\rho_1+\rho_2>0 , \qquad (41)$$

so we are dealing with an active device at all times. The other very interesting case discussed in Ref. 11 for a two-photon CEL, involving initial atomic coherences ρ_{ba} and ρ_{cb}, falls outside the scope of the present model, since we have no way to set up an initial coherence involving the middle level b.

We conclude that this system offers a possible alternate realization of a two-photon CEL, where the atomic coherence is not initially prepared but dynamically generated by an applied, strong, external field at frequency 2ν. The resulting system offers quite a wide spectrum of behavior as a function of the external field strength: from an ordinary (phase-insensitive) two-photon laser to a CEL and beyond, to an ideal parametric oscillator, and a possible generator of squeezed light over much of this range.

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[1]See, for instance, the discussion by R. J. Glauber in *Frontiers in Quantum Optics,* edited by E. R. Pike and S. Sarkar (Hilger, London, 1986); an early reference is H. A. Haus and J. A. Mullen, Phys. Rev. **128**, 2407 (1962).

[2]C. M. Caves, Phys. Rev. D **26**, 1817 (1982).

[3]Early theoretical predictions include H. Takahashi, Adv. Commun. Syst. **1**, 227 (1965); D. Stoler, Phys. Rev. D **1**, 3217 (1970); **4**, 1925 (1971); H. P. Yuen, Phys. Rev. A **13**, 2226 (1976); see also Ref. 2. More recent references are found in Ref. 4 below.

[4]A survey of experimental results is given by L-A. Wu, M. Xiao, and H. J. Kimble, J. Opt. Soc. Am. B **4**, 1465 (1987).

[5]See R. Loudon, in *Frontiers in Quantum Optics,* edited by E. R. Pike and S. Sarkar (Hilger, London, 1986), p. 42; R. Loudon and T. J. Shepherd, Opt. Acta **31**, 1243 (1984) and references therein.

[6]M. J. Collet and D. F. Walls, Phys. Rev. Lett. **61**, 2442 (1988).

[7]D. A. Holm, M. Sargent III, and B. A. Capron, Opt. Lett. **11**, 443 (1986) and references therein.

[8]See R. E. Slusher, B. Yurke, P. Grangier, A. Laporta, D. F. Walls, and M. Reid, J. Opt. Soc. Am. B **4**, 1453 (1987) for a survey of experimental results and further references to theoretical papers.

[9]M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).

[10]M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987).

[11]M. O. Scully, K. Wodkiewicz, M.S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Vehn, Phys. Rev. Lett. **60**, 1832 (1988).

[12]D. T. Pegg and J. A. Vaccaro, Opt. Commun. **61**, 317 (1987).

[13]K. Zaheer and M. S. Zubairy, Opt. Commun. **69**, 37 (1988).

[14]M. O. Scully and M. S. Zubairy, Opt. Commun. **66**, 303 (1988).

[15]G. J. Milburn and D. F. Walls, Opt. Commun. **39**, 401 (1981); B. Yurke, Phys. Rev. A **29**, 408 (1984); M. J. Collet and C. W. Gardiner, *ibid.* **30**, 1386 (1984).

[16]M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Chap. 16.