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Wigner function for number and phase

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Various quasiprobability distributions have been developed in the past using the Hilbert space of the single-mode light field. The development of a quasiprobability distribution associated with a phase operator has previously been impossible because of the absence of a unique Hermitian phase operator defined on the Hilbert space. Recently, however, Pegg and Barnett [Europhys. Lett. **6**, 483 (1988); Phys. Rev. A **39**, 1665 (1989)] and Barnett and Pegg [J. Mod. Optics **36**, 7 (1989)] introduced a new formalism that does allow the construction of a Hermitian phase operator and associated phase eigenstates. In this paper we develop a quasiprobability distribution associated with the number and phase operators of the single-mode light field in the new formalism. The new distribution, which we call the number-phase Wigner function, has properties analogous to the Wigner function. We also derive the number-phase Wigner representation of number states, phase states, general physical states, coherent states, and the squeezed vacuum. We find this new representation has features that are related to the number and phase properties of states. For example, the number-phase Wigner representation of a number state is nonzero only on a circle, while the representation of a phase state is only nonzero along a radial line.

I. INTRODUCTION

Various quasiprobability distributions (QPD's) have been defined using the Hilbert space of the single-mode light field. These functions display the statistical properties of field states and also give c-number formulations of the quantum dynamics of the field. Well-known examples of QPD's include the Q function[1] and the Wigner function.[3] These particular QPD's allow the statistical nature of the quadrature amplitude operators $\hat{X}$ and $\hat{Y}$ to be represented graphically. A case in point is Yuen's[3] Q representation of the ideal-squeezed states, which is a two-dimensional Gaussian function. This function gives a vivid picture of the "squeezing" of the uncertainty from one quadrature amplitude to the other that characterizes these states. The Q function and the Wigner function have also been used to illustrate the statistical nature of the phase of the field in two ways. Firstly, the so-called "measured-phase" operators[4] are proportional to the quadrature amplitude operators and so these QPD's display the measured-phase properties of states. Secondly, the polar angle of the Wigner function has been interpreted phenomenologically as the phase angle of large-amplitude fields.[5,6] However, both of these illustrations have the disadvantage that they are not based on a Hermitian phase operator corresponding to phase angle. This problem stems from the fact that no unique Hermitian phase operator has yet been found for the Hilbert space itself.[7-9]

Recently, Pegg and Barnett introduced a new quantum-mechanical formalism for describing the phase of a single-mode field.[7-9] This new formalism allows the construction of a Hermitian phase operator $\hat{\phi}_\theta$ with properties normally associated with a phase angle. The new phase formalism is based on a linear space Ψ spanned by the $(s+1)$ number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. A complete description of the single-mode field involves an infinite set of number states and here this corresponds to the limit of infinite s. An essential feature of the new formalism is the method of taking this limit when calculating physical properties, such as expectation values. These properties are first calculated with s finite and only then is the limit of infinite s taken. The space Ψ is also spanned by the $(s+1)$ orthonormal phase states

$$|\theta_m\rangle \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle \ , \tag{1.1}$$

where

$$\theta_m \equiv \theta_0 + m\Delta$$

and

$$\Delta \equiv 2\pi/(s+1) \ ,$$

for $m=0,1,\ldots,s$. Here θ_0 is arbitrary and Δ is the step in phase between successive phase states in this basis. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \ , \tag{1.2}$$

and has eigenvalues in the interval $[\theta_0, \theta_0+2\pi]$, which is closed at the lower end. This operator is conjugate to the number operator $\hat{N}$ which is defined by

$$\hat{N} \equiv \sum_{n=0}^{s} n|n\rangle\langle n| \ .$$

We wish to develop a QPD that can display the field-phase properties associated with the Hermitian phase operator, as opposed to the quadrature-amplitude properties displayed by the usual QPD's. The question arises as to whether a QPD in the new formalism should be defined for infinite or finite s. Our approach is to define the QPD for the finite $(s+1)$-dimensional space Ψ. All calculations of physical properties can then be performed first with s finite before the infinite-s limit is taken. This approach is consistent with the Pegg-Barnett formalism, and it is doubtful whether a number-phase QPD based on an infinite-state space is possible.

Wootters[10] recently defined a QPD called the *discrete* Wigner function for systems whose state space is finite (e.g., spin systems). This function has properties analogous to the Wigner function defined on the infinite Hilbert space. It gives a representation of the statistical nature associated with an arbitrary pair of conjugate observables. In this paper we apply Wootters's discrete Wigner function to the single-mode field and construct a function associated with the number and phase operators in the new formalism. For convenience we call this function the *number-phase Wigner function*. We also examine the number-phase Wigner representation of various states with well-known properties to establish the way it displays their number and phase properties.

II. THE NUMBER-PHASE WIGNER FUNCTION

In this section we use Wootters's discrete Wigner function to define the number-phase Wigner function. The discrete Wigner function is defined on an N-dimensional space.[10] Although N can be arbitrary, the analysis of this function becomes particularly simple for the special cases where N is a prime number. In this paper, we take N to be prime. The discrete Wigner function is defined for N prime as

$$W_{n,m} \equiv N^{-1} \langle \hat{A}(n,m) \rangle \ , \tag{2.1}$$

where

$$\langle k | \hat{U} \hat{A}(n,m) \hat{U}^{\dagger} | l \rangle \equiv \bar{\delta}_{2n,k+l} \exp[2\pi i m(k-l)/N] \tag{2.2}$$

for $N>2$. Here $\hat{U}$ is an arbitrary unitary operator, and the states $|k\rangle$ for $k=0,1,\ldots,(N-1)$ form a complete orthonormal basis. In Eq. (2.2) n and m are integers and $\bar{\delta}_{i,j}$ is a *periodic* Kronecker δ which we define as

$$\bar{\delta}_{i,j} \equiv \bar{\delta}_{i,j+N} \equiv \bar{\delta}_{i+N,j}$$

for all values of i and j, and

$$\bar{\delta}_{i,j} \equiv \delta_{i,j}$$

for i and j in the range $[0,N-1]$. The overbar distinguishes $\bar{\delta}_{i,j}$ from the usual Kronecker $\delta_{i,j}$. Thus, for example, $\bar{\delta}_{N+3,2N-8} = \bar{\delta}_{3,N-8} = \delta_{3,N-8} = 0$ for $N>11$.

For our case $N=s+1$. We choose $\hat{U}$ to be the phase-shift operator $\exp(-i\theta_0\hat{N})$ and define

$$\hat{A}_{N\phi}(n,m) \equiv \sum_{k=0}^{s} \sum_{l=0}^{s} \bar{\delta}_{2n,k+l} \exp[im(k-l)\Delta] \times \exp(i\theta_0\hat{N}) |k\rangle\langle l| \exp(-i\theta_0\hat{N}) \ , \tag{2.3}$$

where $\Delta = 2\pi/(s+1)$. Using the result

$$\bar{\delta}_{2n,k+l} = (s+1)^{-1} \sum_{p=0}^{s} \exp[ip(k+l-2n)\Delta] \ , \tag{2.4}$$

the expression $\theta_m = \theta_0 + m\Delta$, and the definition of the phase states (1.1), we find

$$\begin{aligned}\hat{A}_{N\phi}(n,m) &= (s+1)^{-1} \sum_{k=0}^{s} \sum_{l=0}^{s} \sum_{p=0}^{s} \exp(-ip2n\Delta) \exp(ik\theta_{m+p}) |k\rangle\langle l| \exp(-il\theta_{m-p}) \\ &= \sum_{p=0}^{s} \exp(-ip2n\Delta) |\theta_{m+p}\rangle\langle\theta_{m-p}| \ .\end{aligned} \tag{2.5}$$

This operator is Hermitian: on substituting $p=s+1-q$ and using the periodic property of the phase states

$$|\theta_m\rangle = |\theta_m \pm 2\pi\rangle = |\theta_{m\pm(s+1)}\rangle \ ,$$

we obtain

$$\begin{aligned}\hat{A}_{N\phi}(n,m) &= |\theta_m\rangle\langle\theta_m| + \sum_{q=1}^{s} \exp(i2nq\Delta) |\theta_{m-q}\rangle\langle\theta_{m+q}| \\ &= \hat{A}_{N\phi}(n,m)^{\dagger} \ .\end{aligned}$$

We define the number-phase Wigner function according to Eq. (2.1) by

$$W_{N\phi}(n,\theta_m) \equiv (s+1)^{-1} \langle \hat{A}_{N\phi}(n,m) \rangle \ , \tag{2.6}$$

where the indices n,m range over $0,1,\ldots,s$. For our analysis we have taken N to be a prime number and here this corresponds to $(s+1)$ being an arbitrarily large prime number. To find physical results from $W_{N\phi}$ we let $(s+1)$ tend to infinity through the prime numbers.[11] We also note that $W_{N\phi}$ is real because $\hat{A}_{N\phi}$ is Hermitian. Equations (2.5) and (2.6) can be compared with Wigner's original function for the harmonic oscillator:

$$W(q,p) = \left\langle \pi^{-1} \int_{-\infty}^{\infty} \exp(-2iqy) |p+y\rangle\langle p-y| dy \right\rangle \ ,$$

where $|p\rangle$ are the momentum eigenstates.

The mathematical properties of the number-phase Wigner function are precisely those of the discrete Wigner function. The main property that we exploit in this paper concerns the probability interpretation of $W_{N\phi}$. We now derive this property here; the details of other properties can be found in Wootters's paper.[10] In analogy with *integrating* Wigner's original function $W(p,q)$ over either p or q to obtain a probability density, we *sum* $W_{N\phi}(n,\theta_m)$ over one of the indices n or m to produce a probability distribution. For instance, consider first the operator sum

$$(s+1)^{-1}\sum_{n=0}^{s}\hat{A}_{N\phi}(n,m)$$
$$=(s+1)^{-1}\sum_{n=0}^{s}\sum_{p=0}^{s}\exp(-2inp\Delta)|\theta_{m+p}\rangle\langle\theta_{m-p}| \ . \tag{2.7}$$

The sum over n of $\exp(-2inp\Delta)$ is $(s+1)\bar{\delta}_{2p,0}$. We note[12] that because $(s+1)$ is a prime number, $2p\bmod(s+1)$ is zero only for p zero or a multiple of $(s+1)$, and thus $\bar{\delta}_{2p,0}$ is unity for these values of p and zero otherwise. Hence the right-hand side of Eq. (2.7) reduces to $|\theta_m\rangle\langle\theta_m|$. Taking the expectation value of both sides of Eq. (2.7) yields, from Eq. (2.6),

$$\sum_{n=0}^{s}W_{N\phi}(n,\theta_m)=\langle(|\theta_m\rangle\langle\theta_m|)\rangle$$
$$=P(\theta_m)\ , \tag{2.8}$$

which is the probability that a measurement of the phase operator $\hat{\phi}_\theta$ will yield a value of θ_m. The sum of $(s+1)^{-1}\hat{A}_{N\phi}(n,m)$ over the other index m is found from Eq. (2.3) to be

$$\sum_{m=0}^{s}(s+1)^{-1}\hat{A}_{N\phi}(n,m)$$
$$=\sum_{k=0}^{s}\sum_{l=0}^{s}\bar{\delta}_{2n,k+l}\delta_{k,l}\exp(i\theta_0\hat{N})|k\rangle\langle l|\exp(-i\theta_0\hat{N})$$
$$=\sum_{k=0}^{s}\bar{\delta}_{2n,2k}\,|k\rangle\langle k|\ .$$

Because $(s+1)$ is prime and therefore odd, $\bar{\delta}_{2n,2k}$ is nonzero only for $k=n$, and thus the right-hand side is $|n\rangle\langle n|$. Taking the expectation value of both sides yields, therefore, from Eq. (2.6),

$$\sum_{m=0}^{s}W_{N\phi}(n,\theta_m)=\langle(|n\rangle\langle n|)\rangle=P_n\ , \tag{2.9}$$

which is the probability that a measurement of photon number will yield a value of n photons.

In analogy with the usual treatment of Wigner's original function we represent the number-phase Wigner function as $(s+1)^2$ discrete points in the three-dimensional space. The points in cylindrical coordinates (r,θ,z) are given by $(n,\theta_m,W_{N\phi}(n,\theta_m))$ for $n,m=0,1,\ldots,s$. The phase probability distribution $P(\theta_m)$ is, from Eq. (2.8), the sum of the z components of the $(s+1)$ points above the radial line $\theta=\theta_m$, while the number probability distribution P_n is, from Eq. (2.9), the sum of the z components of the $(s+1)$ points above the circle $r=n$. Thus the radial coordinate $r=n$ and the polar angle $\theta=\theta_m$ are associated with the number and phase operators, and hence the number and phase properties, respectively. In the following sections we derive the number-phase Wigner representation of various states, whose number and phase properties are well known, to establish the actual manner in which these properties are expressed.

III. THE $W_{N\phi}$ REPRESENTATION OF NUMBER AND PHASE STATES

The number-phase Wigner representations of number and phase states provide the clearest examples of how the number and phase properties of states are expressed by this function. The $W_{N\phi}$ representation of the number state $|k\rangle$ is found from Eqs. (2.6) and (2.3) to be

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\bar{\delta}_{2n,2k}\ ,$$

or, because $(s+1)$ is odd,

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\delta_{n,k}\ .$$

Thus $W_{N\phi}$ is only nonzero on the circle of radius $r=k$. This circle is illustrated in Fig. 1. $W_{N\phi}$ is defined at $(s+1)$ equidistant points on this circle; as $s\to\infty$ these points become dense. The value of $W_{N\phi}$ is $(s+1)^{-1}$ at each of these points and is zero elsewhere. Clearly $W_{N\phi}$ exhibits the characteristic properties of number states, that is, well-defined photon number and random phase.

The $W_{N\phi}$ representation of the phase state $|\theta_k\rangle$, where $0\le k\le s$, is found from Eqs. (2.6) and (2.5) to be

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\sum_{p=0}^{s}\exp(-2inp\Delta)\bar{\delta}_{m+p,k}\bar{\delta}_{m-p,k}\ ,$$

where $0\le m\le s$. For $p=0$ the product $\bar{\delta}_{m+p,k}\bar{\delta}_{m-p,k}$ is equal to $\delta_{m,k}$. For all other values of p, k cannot be equal to both $m+p$ and $m-p$; nor can k equal both $m+p+j(s+1)$ and $m-p+l(s+1)$, where j and l are integers, because $(s+1)$ is odd and $p\le s$. Thus this product is zero for $p\neq 0$ and hence

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\delta_{m,k}\ . \tag{3.1}$$

In Fig. 2 we again plot only those points in the x-y plane for which $W_{N\phi}$ is nonzero. We find that $W_{N\phi}$ is nonzero only on the radial line $\theta=\theta_k$ and this vividly illustrates the random photon number and the well-defined phase

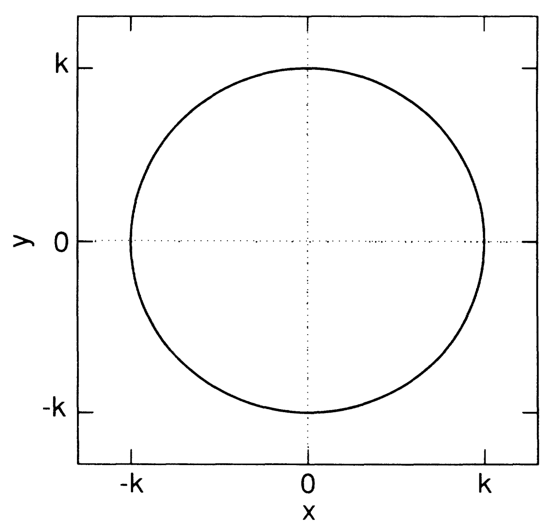

FIG. 1. Circle in the x-y plane on which the $W_{N\phi}$-representation of the number state $|k\rangle$ is nonzero.

properties of $|\theta_k\rangle$, which is an eigenstate of $\hat{\phi}_\theta$. Here when $s\to\infty$, the points in Fig. 2 are not dense but remain separated by the distance corresponding to a difference in photon number of unity, and the line of points extends to infinity.

Not all phase states[7–9] are necessarily eigenstates of $\hat{\phi}_\theta$ with eigenvalues between θ_0 and $\theta_0+2\pi$. By shifting the value of θ_0 by $\lambda\Delta$ where $0<\lambda<1$ we can create a new basis set of $(s+1)$ phase states $|\theta_j+\lambda\Delta\rangle$ with eigenvalues shifted from the eigenvalues of $\hat{\phi}_\theta$. The question now arises as to the nature of $W_{N\phi}$ for a field in a phase state $|\theta_j+\lambda\Delta\rangle$. Using the overlap

$$\langle\theta_j+\lambda\Delta|\theta_m\rangle=(s+1)^{-1}\sum_{k=0}^{s}\exp[ik(\theta_m-\theta_j-\lambda\Delta)]\ ,$$

we find

$$W_{N\phi}(n,\theta_m)=(s+1)^{-3}\sum_{p=0}^{s}\exp(-i2np\Delta)\sum_{k=0}^{s}\exp[ik(\theta_{m+p}-\theta_j-\lambda\Delta)]\sum_{l=0}^{s}\exp[-il(\theta_{m-p}-\theta_j-\lambda\Delta)]\ . \tag{3.2}$$

After some manipulation we obtain

$$W_{N\phi}(n,\theta_m)\Delta^{-1}=\frac{\sin(\lambda\pi)}{\pi(s+1)}[\sin(\lambda\pi+M\theta)-\cos(\lambda\pi+M\theta)\cot\theta]\ , \tag{3.3}$$

where $M\equiv 2n \bmod (s+1)$ and $\theta\equiv\theta_m-\theta_j-\lambda\Delta$. The details of this calculation are given in the Appendix. The singularities in the $\cot\theta$ factor ensure that the right-hand side is nonvanishing as s tends to infinity for $\theta=a\Delta$ and $\theta=a\Delta+\pi$, that is, $\theta_m=\theta_j+\lambda\Delta+a\Delta$ and $\theta_m=\theta_j+\lambda\Delta+a\Delta+\pi$, where a is a real number independent of s. For all other values of θ the right-hand side approaches zero as $s\to\infty$. Thus $W_{N\phi}(n,\theta_m)\Delta^{-1}$ has relatively large values only for $\theta_m\approx\theta_j$ and $\theta_m\approx\theta_j+\pi$. For arbitrarily large s, the relatively large values of $W_{N\phi}(n,\theta_m)\Delta^{-1}$ for $\theta_m\approx\theta_j$ indicate that the phase of the state $|\theta_j+\lambda\Delta\rangle$ is approximately θ_j. The question remains as to what information about the phase (if any) is conveyed by the relatively large values of $W_{N\phi}(n,\theta_m)\Delta^{-1}$ for $\theta_m\approx\theta_j+\pi$. To address this question we examine the phase probability density $P(\theta_m)\Delta^{-1}$,

$$P(\theta_m)\Delta^{-1}=\sum_{n=0}^{s}W_{N\phi}(n,\theta_m)\Delta^{-1}\ , \tag{3.4}$$

because it represents all the information available about the phase. Before evaluating the sum in Eq. (3.4) completely consider first the sum of the pair of terms $[W_{N\phi}(n,\theta_m)+W_{N\phi}(n+\frac{1}{2}s,\theta_m)]\Delta^{-1}$ for $n\le\frac{1}{2}s$. We find from Eq. (3.3) that this sum approaches zero in the region where $\theta_m\approx\theta_j+\pi$ as s tends to infinity. Thus the values of $W_{N\phi}(n,\theta_m)\Delta^{-1}$ and $W_{N\phi}(n+\frac{1}{2}s,\theta_m)\Delta^{-1}$ tend to cancel each other in the sum in Eq. (3.4) in this region. This leads us to conclude that the relatively large values of $W_{N\phi}(n,\theta_m)\Delta^{-1}$ for $\theta_m\approx\theta_j+\pi$ do not convey any information about the phase because they contribute no detail to $P(\theta_m)\Delta^{-1}$.

Evaluating completely the sum in Eq. (3.4) eventually gives

$$P(\theta_m)\Delta^{-1}=\frac{\sin^2(\lambda\pi)}{2\pi(s+1)\sin^2[\frac{1}{2}(\theta_m-\theta_j-\lambda\Delta)]}\ . \tag{3.5}$$

As $s\to\infty$ we find that for $\theta_m=a\Delta+\theta_j$, where a is independent of s,

$$P(\theta_m)\Delta^{-1}\to\frac{(s+1)\sin^2(\lambda\pi)}{2\pi^3(a-\lambda)^2}\ ,$$

while for $\theta_m=b+\theta_j$, where b is independent of s,

$$P(\theta_m)\Delta^{-1}\to 0\ .$$

Evidently, in the infinite-s limit the density $P(\theta_m)\Delta^{-1}$ is zero for all θ_m except for an infinitesimal region near $\theta_m=\theta_j$ where the density is infinitely large. Furthermore, because the distribution $P(\theta_m)$ is normalized for all s then the density $P(\theta_m)\Delta^{-1}$ is also normalized in the limit $s\to\infty$ according to

$$1=\lim_{s\to\infty}\sum_{m=0}^{s}[P(\theta_m)\Delta^{-1}]\Delta\ .$$

This limit can be compared to the integral of a Dirac δ function. Moreover, the right-hand side of Eq. (3.5) is

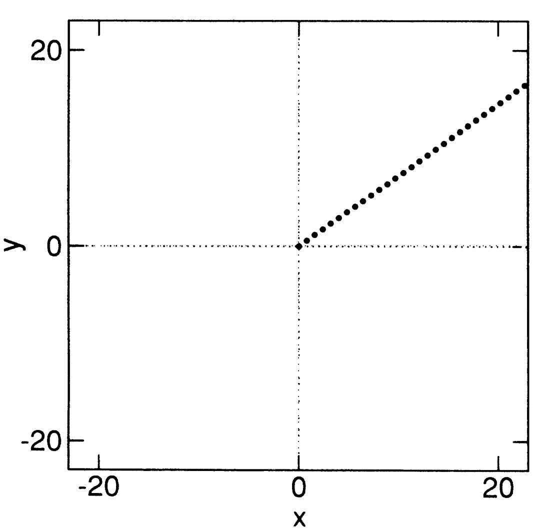

FIG. 2. Points on the x-y plane for which the $W_{N\phi}$-representation of the phase state $|\theta_k\rangle$ is nonzero. These points lie on the radial line that makes an angle of θ_k to the x axis.

periodic in θ_m with a period of 2π. Thus we conclude that the phase probability density $P(\theta_m)\Delta^{-1}$ for the phase state $|\theta_j+\lambda\Delta\rangle$ has periodic Dirac δ-function behavior in the infinite-s limit. This behavior clearly shows that the phase state $|\theta_j+\lambda\Delta\rangle$ gives well-defined values of phase when operated on by $\hat{\phi}_\theta$ even though it is not an eigenstate of $\hat{\phi}_\theta$.

IV. THE $W_{N\phi}$ REPRESENTATION OF PHYSICAL STATES

We illustrated in Sec. III the way in which the number and phase properties of the number and phase states are expressed by the $W_{N\phi}$ function. The number state belongs to a wide class of states called physical states. This class of states is important because it represents states that can be prepared physically. It includes nearly all states used in quantum optics, with the notable exception being the phase states themselves, for which the expectation value of energy is infinite. Formally, physical states $|p\rangle$,

$$|p\rangle \equiv \sum_{n=0}^{s} d_n |n\rangle \ , \tag{4.1}$$

are those states that have finite moments of the number operator $\langle \hat{N}^q\rangle$ for any given q in the limit of infinite s.[6-9] That is, states for which

$$\langle \hat{N}^q\rangle = \lim_{s\to\infty} \sum_{n=0}^{s} |d_n|^2 n^q < B_q \ , \tag{4.2}$$

where B_q is some bound.

The $W_{N\phi}$ representation of physical states is found, by a similar calculation to that which leads to Eq. (A1) in the Appendix, to be

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\sum_{k=0}^{M} d_k^* d_{M-k}\exp[i(2k-M)\theta_m]+(s+1)^{-1}\sum_{k=M+1}^{s} d_k^* d_{M-k+s+1}\exp[i(2k-M-s-1)\theta_m] \ , \tag{4.3}$$

where $M\equiv 2n \bmod (s+1)$. From Eq. (4.2) it follows that

$$|d_n|^2 < B_q m^{-q}$$

and hence

$$|d_k^* d_{l-k}| < B_q[k(l-k)]^{-(1/2)q} \ ,$$

for $l>k$. Using $k(l-k)\ge \frac{1}{2}l$ for $l>k$ and $k\neq 0$ we get

$$|d_k^* d_{l-k}| < B_q(\tfrac{1}{2}l)^{-(1/2)q},$$

from which it follows that the magnitude of the second term on the right-hand side of Eq. (4.3) is less than $B_q[\frac{1}{2}(M+s+1)]^{-(1/2)q}$ for any given q. Thus, by choosing sufficiently large s, this term is negligible and we can approximate Eq. (4.3) to any desired accuracy by

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\sum_{k=0}^{M} d_k^* d_{M-k}\exp[i(2k-M)\theta_m] \ . \tag{4.4}$$

This simple expression gives the number-phase Wigner function for physical states in general. We now examine two particular examples of physical states: the coherent state $|\alpha\rangle$ and the squeezed vacuum[13] $|0,\xi\rangle$. The phase properties of these states depend on the amplitude $|\alpha|$ and squeezing $|\xi|$ parameters. As these parameters are increased from zero the phase of each state transforms from being completely random to a more well-defined value.[8,14]

A. Coherent state

Substituting the number-state coefficients for $|\alpha\rangle$,

$$d_n=\alpha^n(n!)^{-1/2}\exp(-\tfrac{1}{2}|\alpha|^2) \ ,$$

into Eq. (4.4) yields eventually

$$W_{N\phi}(n,\theta_m)=(s+1)^{-1}\Lambda(n,r)\Phi(n,\theta_m,\phi) \ ,$$

where $\Lambda(n,r)\equiv r^M\exp(-r^2)/l!$,

$$\Phi(n,\theta_m,\phi)\equiv \sum_{k=0}^{M}\frac{l!\cos[(2k-M)(\theta_m-\phi)]}{(k!)^{1/2}[(M-k)!]^{1/2}} \ , \tag{4.5}$$

$\alpha=r\exp(i\phi)$, $M\equiv 2n \bmod (s+1)$, and l is the largest integer not exceeding $\frac{1}{2}M$. We notice here that $W_{N\phi}$ is the product of a constant $(s+1)^{-1}$, an amplitude-r-dependent factor $\Lambda(n,r)$ and a phase-ϕ-dependent factor $\Phi(n,\theta_m,\phi)$. This factorization provides an alternative viewpoint as to why more intense coherent states have more well-defined phases. The factor $\Lambda(n,r)$ is related to the photon-number probability distribution $P_n=r^{2n}\exp(-r^2)/n!$ by

$$\Lambda(n,r)=P_n, \quad 0\le n\le \tfrac{1}{2}s \tag{4.6a}$$

$$\Lambda(\tfrac{1}{2}s+n,r)=rP_{n-1}, \quad 0<n\le \tfrac{1}{2}s \ . \tag{4.6b}$$

Here we find that as n increases from zero, $\Lambda(n,r)$ decays to negligible values as n approaches $\frac{1}{2}s$ and then it undergoes a "revival" from $n=\frac{1}{2}s+1$. This should perhaps not be unexpected because in the definition of $W_{N\phi}(n,\theta_m)$, Eqs. (2.3) and (2.6), the dependence on n appears only as $\bar{\delta}_{2n,j}$, which has the property

$$\bar{\delta}_{2n+s,j}=\bar{\delta}_{2n-1,j}$$

for all n and j. Thus $\bar{\delta}_{2n,j}$ is *almost* cyclic in n and it is this property that gives rise to the revival in $\Lambda(n,r)$. We also find from Eq. (4.5) the symmetry property that $\Phi(n,\theta_m,\phi)$ equals $\Phi(n,\theta_m+\pi,\phi)$ for $n\le\frac{1}{2}s$ and $-\Phi(n,\theta_m+\pi,\phi)$ for $n>\frac{1}{2}s$. In Fig. 3 we have plotted

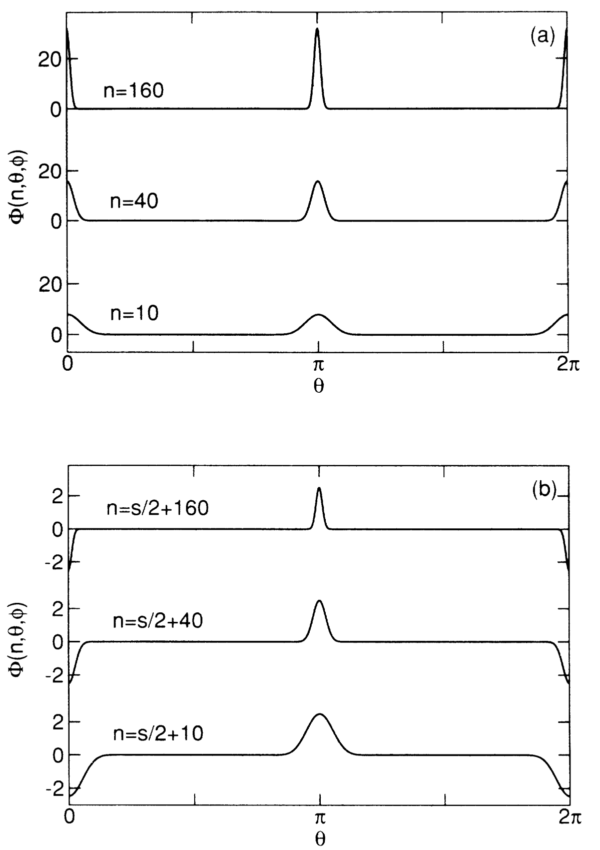

FIG. 3. Phase-dependent factor $\Phi(n,\theta,\phi)$ as a function of the continuous parameter θ for the coherent state $|\alpha\rangle$ with $\arg(\alpha)=\pi$ for (a) $n=10$, 40, and 160, and (b) $n=\frac{1}{2}s+10$, $\frac{1}{2}s+40$, and $\frac{1}{2}s+160$.

$\Phi(n,\theta,\phi)$ as a function of the continuous parameter θ for selected values of n for the coherent state with $\phi=\pi$ and with an arbitrarily large value of s. The $(s+1)$ values of $\Phi(n,\theta_m,\phi)$ for $m=0$ to s form a subset of discrete points on these curves. The periodic behavior of Φ is clearly visible here. The curves have a peak near $\theta=\pi$, which is the expected phase of the coherent state, and peaks (for $n\le\frac{1}{2}s$) or troughs (for $n>\frac{1}{2}s$) near $\theta=0$. We also note that all peaks and troughs become progressively narrower as n increases from 0 to $\frac{1}{2}s$ and, separately, from $\frac{1}{2}s+1$ to s. The phase-probability distribution $P(\theta_m)$ is given by summing $W_{N\phi}(n,\theta_m)$ over n, i.e.,

$$P(\theta_m)=(s+1)^{-1}\sum_{n=0}^{s}\Lambda(n,r)\Phi(n,\theta_m,\phi) . \tag{4.7}$$

In this sum the amplitude-dependent function $\Lambda(n,r)$ acts as a weighting factor. According to Eqs. (4.6a) and (4.6b), $\Lambda(n,r)$ is related to a Poisson distribution with a mean of r^2, and so the most significant contribution to $P(\theta_m)$ will be from terms for which $n\approx r^2$ and $n\approx r^2+\frac{1}{2}s$. In the sum of these terms in Eq. (4.7) the peaks and troughs of $\Phi(n,\theta_m,\phi)$ near $\theta_m=0$ tend to cancel each other while peaks near $\theta_m=\pi$ reinforce each other. Thus $P(\theta_m)$ will exhibit a narrow peak near $\theta_m\approx\pi$ and will be relatively flat elsewhere. As r is increased, the sum in Eq. (4.7) will be dominated by terms involving $\Phi(n,\theta_m,\phi)$ with larger n, that is, terms with narrower peaks in θ_m, giving $P(\theta_m)$ a narrower peak near $\theta_m\approx\pi$. This explains why coherent states have more well-defined phases as the intensity r^2 is increased: the amplitude factor $\Lambda(n,r)$, which is related to the photon-number probability distribution, determines in part the phase-probability distribution through the heavier weighting of $\Phi(n,\theta_m\phi)$ with narrower peaks in Eq. (4.7).

B. Squeezed state

The squeezed vacuum $|0,\xi\rangle$ is also a physical state[6] and so its number-phase Wigner representation can also be found using the approximate expression (4.4). The number-state coefficients of $|0,\xi\rangle$ are[3,5]

$$d_{2n}=\frac{(-\tanh t)^n}{2^n n!}\left[\frac{(2n)!}{\cosh t}\right]^{1/2}\exp(in\eta)$$

and $d_{2n+1}=0$, where $\xi=t\exp(i\eta)$. After some manipulation we find $W_{N\phi}$ for $|0,\xi\rangle$ is approximately

$$W_{N\phi}=(s+1)^{-1}\Lambda'(n,t)\Phi'(n,\theta_m,\eta) ,$$

where

$$\Lambda'(n,t)\equiv\frac{(\tanh t)^n(2q)!}{\cosh t\; q!^2 2^{2q}},\quad n\le\tfrac{1}{2}s$$

$$\Lambda'(n,t)\equiv 0,\quad n>\tfrac{1}{2}s , \tag{4.8}$$

$$\Phi'(n,\theta_m,\eta)\equiv\frac{q!^2(-1)^n}{(2q)!2^{n-2q}}\sum_{k=0}^{n}\frac{[(2k)!(2n-2k)!]^{1/2}}{k!(n-k)!}\times\cos[(n-2k)(\eta-2\theta_m)] , \tag{4.9}$$

and q is the largest integer not exceeding $\frac{1}{2}n$. Thus $W_{N\phi}$ for the squeezed vacuum also factorizes into three factors: a constant $(s+1)^{-1}$, a factor Λ' dependent on t, and a phase-η-dependent factor Φ'. The factor $\Lambda'(n,r)$ steadily decreases with increasing n. It is not difficult to show that Λ' is related to the photon-number probability distribution P_n for the squeezed vacuum by

$$\Lambda'(2n,t)=P_{2n},\quad 2n\le\tfrac{1}{2}s$$

$$\Lambda'(2n+1,t)=(\tanh t)P_{2n},\quad (2n+1)\le\tfrac{1}{2}s .$$

We note that, unlike Λ for the coherent state, Λ' does *not* undergo any revival for $n>\frac{1}{2}s$. This can be traced to the fact that the odd-number state coefficients of the squeezed vacuum are zero. On the other hand, we can see from Eq. (4.9) that Φ' possesses the following symmetries:

$$\Phi'(n,\theta_m,\eta)=\begin{cases}\Phi'(n,\theta_m+\frac{1}{2}\pi,\eta) & \text{for even } n,\\ -\Phi'(n,\theta_m+\frac{1}{2}\pi,\eta) & \text{for odd } n,\\ \Phi'(n,\theta_m+\pi,\eta) & \text{for all } n .\end{cases}$$

These symmetries are clearly visible in the plots of

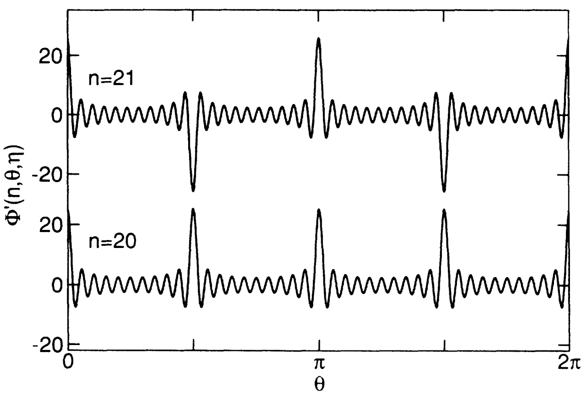

FIG. 4. Two curves illustrating the general properties of the phase-dependent factor $\Phi'(n,\theta,\eta)$ as a function of the continuous parameter θ for the squeezed vacuum $|0,\xi\rangle$ with $\arg(\xi)=\pi$. The $s+1$ values of $\Phi'(n,\theta_m,\eta)$ for $m=0$ to s form a discrete subset of points on these curves.

$\Phi'(n,\theta,\eta)$ for $n=20$ and 21 in Fig. 4 for the squeezed vacuum with $\eta=\pi$. These two curves illustrate the general properties of Φ'. Although not shown here, we find that the relatively large peaks at $\theta=0$ and π become narrower for larger n, with widths of the order of π/n. The peaks for n even and troughs for n odd at $\theta=\pi/2$ and $3\pi/2$ tend to cancel each other in the sum

$$P(\theta_m)=(s+1)^{-1}\sum_{n=0}^{s}\Lambda'(n,t)\Phi'(n,\theta_m,\eta)\ . \qquad (4.10)$$

Thus these peaks and troughs do not convey any significant phase information. Perhaps surprisingly, we find approximately 40 smaller oscillations in each curve in Fig. 4. In general, it turns out that there are approximately $2n$ such oscillations in the curve of $\Phi'(n,\theta,\eta)$. However, in the sum in Eq. (4.10), $\Phi'(n,\theta_m,\eta)$ is weighted by $\Lambda'(n,t)$, which decays relatively slowly with increasing n, and so many terms involving $\Phi'(n,\theta,\eta)$ of different frequencies contribute to $P(\theta_m)$. Thus the smaller oscillations tend to cancel each other and make no significant contribution to the phase information. Hence $P(\theta_m)$ has peaks at $\theta_m\approx 0$ and π only, and is relatively flat in other regions. Finally we note that as the squeezing parameter t increases, the ratios

$$\frac{\Lambda'(2n+2,t)}{\Lambda'(2n,t)}=\frac{\Lambda'(2n+3,t)}{\Lambda'(2n+1,t)}=\frac{2n+1}{2n+2}(\tanh t)^2$$

also increase and this implies $\Lambda'(n,t)$ decays more slowly with n. Thus the contribution to $P(\theta_m)$ from values of $\Phi'(n,\theta_m,\eta)$ with larger n becomes more significant with increased squeezing, and so the two peaks in $P(\theta_m)$ will become narrower. Hence, as the squeezing is increased the phase of the squeezed vacuum becomes more well defined at two values and this can be compared to the similar effect found for the single-peaked phase distribution of the coherent state.

In summary, we note that the coherent states and the squeezed vacuum exhibit a similar mechanism for reducing the phase uncertainty. We found that as the amplitude or the squeezing parameter is increased, values of $W_{N\phi}(n,\theta_m)$ with narrower peaks in θ_m contribute more significantly to the phase-probability distributions $P(\theta_m)$ for these states. Thus these distributions have narrower peaks, which indicates more well-defined phases. We also found that the $W_{N\phi}$ representation of these states has oscillations with θ_m, as illustrated in Figs. 3 and 4, that do not contribute any detail to $P(\theta_m)$. This can be compared to a similar effect we found in Sec. III for the phase state $|\theta_j+\lambda\Delta\rangle$.

V. CONCLUSION

In this paper we have developed the number-phase Wigner function $W_{N\phi}$ by using Wootters's discrete Wigner function to represent a single-mode light field in the Pegg-Barnett formalism. Unlike Wigner's original function, which is defined everywhere on a plane, the number-phase Wigner function $W_{N\phi}(n,\theta_m)$ is defined only on dense subsets of points on circles of integer radius centered on the origin. These points (n,θ_m) correspond to a polar representation of the eigenvalues of the number and phase operators, that is, n are the eigenvalues of $\hat{N}$, and θ_m are the eigenvalues of $\hat{\phi}_\theta$.

We have found that the number-phase Wigner function gives a picture of the number and phase properties of states. For example, $W_{N\phi}$ for the number state $|n\rangle$ is nonzero only along the circle of radius n, while for the phase eigenstate $|\theta_m\rangle$, $W_{N\phi}$ is nonzero only along the radial line, which is at an angle of θ_m to the x axis. A reasonably simple expression was obtained for the number-phase Wigner representation of a general physical state. This expression was used to find the $W_{N\phi}$ representation of the coherent state and the squeezed vacuum. The phase properties of these states were found to be expressed by their respective $W_{N\phi}$ representations as follows. Although $W_{N\phi}$ for these states may contain many oscillations with θ_m, only particular peaks convey the phase information. These particular peaks are narrower for larger n. As the amplitude (or squeezing) is increased, values of $W_{N\phi}(n,\theta_m)$ with larger n contribute more significantly to the phase-probability distribution $P(\theta_m)$, and thus the peaks in $P(\theta_m)$ become narrower, giving a more well-defined phase.

In conclusion, we have introduced the number-phase Wigner function and shown how it gives a new way of viewing the number and phase properties of states in the new formalism.

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APPENDIX

Here we give the details of the derivation of Eq. (3.3) from Eq. (3.2). Performing the sum over p in Eq. (3.2), using $\theta_{m+p}=\theta_m+p\Delta$ and Eq. (2.4), yields

$$W_{N\phi}(n,\theta_m)=(s+1)^{-2}\sum_{n=0}^{s}\sum_{l=0}^{s}\bar{\delta}_{k+l,2n}\exp[i(k-l)(\theta_m-\theta_j-\lambda\Delta)] .$$

We can replace $2n$ by $M\equiv 2n \bmod (s+1)$ in the index of the $\bar{\delta}$ function because $\bar{\delta}$ is periodic with a period of $(s+1)$. Separating the sum over k into two parts then gives

$$W_{N\phi}(n,\theta_m)=(s+1)^{-2}\left[\sum_{k=0}^{M}\sum_{l=0}^{s}\bar{\delta}_{k+l,M}\exp[i(k-l)\theta]+\sum_{k=M+1}^{s}\sum_{l=0}^{s}\bar{\delta}_{k+l,M}\exp[i(k-l)\theta]\right],$$

where for convenience we have written $\theta\equiv\theta_m-\theta_j-\lambda\Delta$. In the first sum $k\le M$ and so

$$\bar{\delta}_{k+l,M}=\bar{\delta}_{l,M-k}=\delta_{l,M-k} ,$$

while in the second sum $k>M$ and thus

$$\bar{\delta}_{k+l,M}=\bar{\delta}_{l,M-k}=\delta_{l,M-k+s+1} .$$

Hence we find

$$W_{N\phi}(n,\theta_m)=(s+1)^{-2}\left[\sum_{k=0}^{M}\exp[i(2k-M)\theta]+\sum_{k=M+1}^{s}\exp[i(2k-M-s-1)\theta]\right]. \tag{A1}$$

Replacing the dummy summation index k in the second sum with $q=k-M-1$ gives

$$W_{N\phi}(n,\theta_m)=(s+1)^{-2}\left[\sum_{k=0}^{M}\exp[i(2k-M)\theta]+\sum_{q=0}^{s-M-1}\exp[i(2q-s+M+1)\theta]\right].$$

The right-hand side now contains two geometric series of the general form

$$\sum_{k=0}^{j}\exp(ik2\theta)\exp(-ij\theta)=\sin[(j+1)\theta]/\sin\theta ,$$

where j is an integer. Replacing the geometric series with their closed expressions then gives

$$\begin{aligned}W_{N\phi}(n,\theta_m)&=(s+1)^{-2}\{\sin[(M+1)\theta]/\sin\theta+\sin[(s-M)\theta]/\sin\theta\}\\&=2(s+1)^{-2}\sin[\tfrac{1}{2}(s+1)\theta]\times\cos\{[\tfrac{1}{2}(s-1)-M]\theta\}/\sin\theta .\end{aligned} \tag{A2}$$

We note that $\theta=\theta_m-\theta_j-\lambda\Delta=(m-j-\lambda)\Delta$ and $\Delta=2\pi/(s+1)$, and so

$$\sin[\tfrac{1}{2}(s+1)\theta]=-(-1)^{m-j}\sin(\lambda\pi) ,$$

$$\begin{aligned}\cos\{[\tfrac{1}{2}(s-1)-M]\theta\}&=(-1)^{m-j}\cos[\lambda\pi+(M+1)\theta]\\&=(-1)^{m-j}[\cos(\lambda\pi+M\theta)\cos\theta-\sin(\lambda\pi+M\theta)\sin\theta] .\end{aligned}$$

Substituting these two results into Eq. (A2) and then dividing by Δ yields Eq. (3.3):

$$W_{N\phi}(n,\theta_m)\Delta^{-1}=\frac{\sin(\lambda\pi)}{\pi(s+1)}[\sin(\lambda\pi+M\theta)-\cos(\lambda\pi+M\theta)\cot\theta] .$$

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