Wave-packet evolution in the damped oscillator

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We demonstrate that, for a simple model of the damped oscillator, both the wave packet in the position representation and oscillator expectation values are insensitive to the rapid decay of off-diagonal coherences. This rapid decay is, however, seen in the von Neumann entropy for the oscillator and the wave packet in the number representation.

I. INTRODUCTION

There have recently been several proposals for the generation of optical quantum superposition states.[1–4] These schemes are based on the propagation of the field state through a nonlinear medium, usually a medium with a Kerr-type nonlinearity. These anharmonic oscillators induce a bifurcation of the field Q function in phase space[5] indicating the establishment of a quantum superposition state. It has been recognized for some time that even the presence of a small amount of dissipation is sufficient to reduce this pure superposition state to a mixture on a time scale much shorter than a typical decay time.[6–8] This rapid collapse of the pure quantum state to a mixture is of considerable importance in discussions of quantum measurement theory[9–11] where the role of the environment is to induce this collapse.

In this paper we study the decay of a field mode initially prepared in a superposition of coherent states. Specifically, we shall assume that the density operator for the field mode obeys a zero temperature master equation in the Born-Markov approximations. This model was first studied by Walls and Milburn[7] and later by Savage and Walls.[8] The decay properties of the superposition have also been studied by Kennedy and Walls when the reservoir is prepared in a multimode squeezed vacuum state.[12] These authors have shown that the off-diagonal terms in the field density operator, expressed in a coherent state basis, are weighted with a factor which rapidly suppresses these coherences. The effect of this decay on observable quantities has not been emphasized, however, and it is one of the purposes of the present work to clarify this issue.

The ideal parameter with which to characterize the decay of a pure state to a mixture is the von Neumann entropy.[13] This quantity is positive for a mixed state and zero for a pure state. The rapid destruction of the coherences to form a mixture should be reflected in the evolution of the field entropy. In this paper we exploit the method used in the calculation of the field entropy in the Jaynes-Cummings problem[14] to evaluate the entropy of the damped oscillator as it evolves to a mixture (we use the terms field mode and oscillator interchangeably in this paper). We then examine the evolution of the diagonal elements of the field density operator in the position representation. We obtain a Fokker-Planck equation for the motion of this wave packet and show that the terms arising from the off-diagonal coherence *do not* exhibit a rapid decay. This behavior is also shown to occur for field expectation values for which the terms arising from the off-diagonal coherence do not decay on a faster timescale than the other terms arising from the diagonal elements. While this insensitivity to the off-diagonal decay appears to be at odds with earlier treatments we shall show that it is, in fact, consistent.

II. SOLUTION OF THE MASTER EQUATION

The master equation in the interaction picture for a damped oscillator at zero temperature and under the Born-Markov approximations is given by[15]

$$\frac{\partial\rho}{\partial t}=\frac{\gamma}{2}(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a) \tag{1}$$

where a, and $a^{\dagger}$ are the annihilation and creation operators for the oscillator and γ is the decay constant. It should also be noted that the coupling to the environment is through a coordinate-coordinate coupling and the above master equation has been derived under the rotating-wave approximation. Following Barnett and Knight[16] we define the superoperators $\hat{J}$ and $\hat{L}$ by their action on the density operator:

$$\hat{J}\rho=\gamma a\rho a^{\dagger}\ , \tag{2}$$

$$\hat{L}\rho=-\frac{\gamma}{2}(a^{\dagger}a\rho+\rho a^{\dagger}a)\ . \tag{3}$$

The formal solution of the master equation (1) can now be written as

$$\rho(t)=\exp[(\hat{J}+\hat{L})t]\rho(0)\ . \tag{4}$$

The exponential evolution operator can be disentangled (see Appendix) to yield the relation[17]

$$\exp[(\hat{J}+\hat{L})t]=\exp(\hat{L}t)\exp\left[\frac{\hat{J}}{\gamma}(1-e^{-\gamma t})\right]\ . \tag{5}$$

This relation together with Eqs. (2) and (3) imply that the action of this evolution operator on an off-diagonal element of a density operator expressed in a coherent state basis can be written as

$$\exp[(\hat{J}+\hat{L})t]|\alpha\rangle\langle\beta| = \langle\beta|\alpha\rangle^{1-\exp(-\gamma t)}|\alpha e^{-\gamma t/2}\rangle\langle\beta e^{-\gamma t/2}| \quad (6)$$

where $|\alpha e^{-\gamma t/2}\rangle$ is a coherent state of amplitude $\alpha e^{-\gamma t/2}$. This is in agreement with the result of Walls and Milburn[7] obtained by a different method. We see from this expression that off-diagonal elements are weighted by a factor $\langle\beta|\alpha\rangle^{1-\exp(-\gamma t)}$ which for real α and β and short times can be written as $\exp(-|\alpha-\beta|^2\gamma t/2)$. It is apparent that the off-diagonal elements are rapidly dephased at a rate governed by the separation of the coherent states.

For the rest of this paper we shall be exclusively concerned with an initial oscillator state described by a superposition of just two coherent states $|\alpha\rangle$ and $|\beta\rangle$. All of the results, except that for the oscillator entropy, can easily be generalized to superpositions of arbitrary numbers of coherent states if so desired. The initial density operator is therefore given by

$$\rho(0)=N(|\alpha\rangle\langle\alpha|+|\beta\rangle\langle\beta|+|\alpha\rangle\langle\beta|+|\beta\rangle\langle\alpha|) \quad (7)$$

where N is the normalization constant. The full time-dependent density operator is given by

$$\rho(t)=N\sum_{\alpha,\beta}\langle\beta|\alpha\rangle^{1-\exp(-\gamma t)}|\alpha e^{-\gamma t/2}\rangle\langle\beta e^{-\gamma t/2}| \quad (8)$$

where the sum is to be taken over the coherent states in the original superposition. This expression was originally obtained by Walls and Milburn.[7] If we write $\alpha e^{-\gamma t/2}=\tilde{\alpha}$ then the solution (8) can be written in the form

$$\rho(t)=N\sum_{\alpha,\beta}\frac{\langle\beta|\alpha\rangle}{\langle\tilde{\beta}|\tilde{\alpha}\rangle}|\tilde{\alpha}\rangle\langle\tilde{\beta}| . \quad (9)$$

In the next section we calculate the entropy of the oscillator when its initial state is given by (7).

III. ENTROPY OF THE OSCILLATOR

The von Neumann entropy of a quantum system is defined in terms of the density operator by[13]

$$S=-\operatorname{Tr}(\rho\ln\rho) \quad (10)$$

where we have set Boltzmann's constant equal to unity. This quantity is zero if ρ describes a pure state and is positive if ρ describes a mixed state. The entropy therefore measures deviations from pure state behavior. For isolated systems S is time independent due to the unitarity of the evolution operator. For open systems such as the damped oscillator, however, the evolution is not unitary and the entropy becomes time dependent. These properties make the von Neumann entropy an ideal parameter with which to characterize the rapid decay of off-diagonal coherences in the damped oscillator.

In general the calculation of the entropy (10) for a given system is nontrivial requiring the diagonalization of the density operator. For the initial state considered above, however, we can employ the method used for the diagonalization of the Jaynes-Cummings field density operator.[14] It is apparent from (9) that an eigenstate of the oscillator density operator must be of the form

$$|\psi\rangle=c_\alpha|\tilde{\alpha}\rangle+c_\beta|\tilde{\beta}\rangle . \quad (11)$$

If λ is the eigenvalue, then the eigenvalue equation can be written as

$$\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix}\begin{bmatrix} c_\alpha \\ c_\beta \end{bmatrix}=\lambda\begin{bmatrix} c_\alpha \\ c_\beta \end{bmatrix} \quad (12)$$

where we have written the elements of this matrix as

$$M_{\alpha\alpha}=1+\langle\beta|\alpha\rangle=M^*_{\beta\beta} , \quad (13)$$

$$M_{\alpha\beta}=\langle\tilde{\alpha}|\tilde{\beta}\rangle+\frac{\langle\beta|\alpha\rangle}{\langle\tilde{\beta}|\tilde{\alpha}\rangle}=M^*_{\beta\alpha} . \quad (14)$$

The diagonal elements of this matrix are time independent but the off-diagonal elements are functions of time. Let us now suppose that the amplitudes c_α and c_β can be written as

$$c_\alpha=\bar{c}_\alpha e^{i\Omega/2} , \quad (15)$$

$$c_\beta=\bar{c}_\beta e^{-i\Omega/2} , \quad (16)$$

and the phase Ω is defined by

$$M_{\alpha\beta}=|M_{\alpha\beta}|e^{i\Omega} . \quad (17)$$

By eliminating λ from the above matrix equation and using the substitutions (15)–(17) we arrive at the following:

$$\bar{c}_\alpha^{\,2}-\bar{c}_\beta^{\,2}=\Delta\bar{c}_\alpha\bar{c}_\beta , \quad (18)$$

$$\Delta=|M_{\alpha\beta}|^{-1}(M_{\alpha\alpha}-M_{\beta\beta}) \quad (19)$$

but $M_{\alpha\alpha}$ and $M_{\beta\beta}$ are complex conjugates and so Δ is purely imaginary and we write it in the form $\Delta=i\bar{\Delta}$. This in turn implies that $\bar{c}_\alpha$ and $\bar{c}_\beta$ are complex conjugates. If we write

$$\bar{c}_\alpha=e^{i\xi/2} , \quad (20)$$

$$\bar{c}_\beta=\pm e^{-i\xi/2} \quad (21)$$

where ξ can be related to Δ through Eq. (18), we obtain the eigenvalues

$$\lambda^{(\pm)}=\tfrac{1}{2}(M_{\alpha\alpha}+M_{\beta\beta})\pm|M_{\alpha\beta}|(1-\tfrac{1}{4}\bar{\Delta}^2)^{1/2} . \quad (22)$$

It should be noted that this expression can be obtained from a straightforward diagonalization of (12) although the method outlined here also gives the form of the eigenstates in a more direct manner. The normalization constant N must be included in (22) and this can be determined from (22) or from (7) and we find that

$$N=(M_{\alpha\alpha}+M_{\beta\beta})^{-1}=[2(1+\operatorname{Re}\langle\alpha|\beta\rangle)]^{-1} . \quad (23)$$

Including this term in (22) yields the following form for the eigenvalues of the density operator for the damped oscillator [given that the initial condition is described by Eq. (7) thereby restricting the space to just two coherent states]:

$$\lambda^{(\pm)}=\tfrac{1}{2}\{1\pm[1-4N^2(|M_{\alpha\alpha}|^2-|M_{\alpha\beta}|^2)]^{1/2}\} . \quad (24)$$

The oscillator entropy is now given by the simple form

$$S=-\lambda^{(+)}\ln\lambda^{(+)}-\lambda^{(-)}\ln\lambda^{(-)} . \quad (25)$$

For short times and for $|\alpha-\beta|$ large we find that the eigenvalues can be approximated by

$$\lambda^{(\pm)}\approx\tfrac{1}{2}(1\pm e^{-(1/2)|\alpha-\beta|^2\gamma t}) \quad (26)$$

so that the "distance" between the two coherent states determines the rate of change of these eigenvalues. The entropy is sensitive to the rapid decay of the off-diagonal coherence and evolves on two time scales. There is a rapid increase to its maximum value at a rate determined by the distance between the two coherent states and a subsequent relaxation to the vacuum at the normal decay rate. This behavior is shown in Fig. 1 where the entropy is plotted for various values of α and β. It should be noted that if the oscillator is prepared in a single coherent state then it remains in a coherent state throughout its subsequent evolution and the entropy remains at zero. This is a peculiarity of the fact that the reservoir is modeled as a collection of harmonic oscillators, coupled in rotating-wave approximation to our system oscillator, with a zero temperature distribution, that is, all of the reservoir oscillators are in coherent states of zero amplitude. Although the oscillator entropy is sensitive to this rapid decay of the off-diagonal coherence it is pertinent to enquire as to whether there are other quantities which are similarly sensitive. In the next section we examine the evolution of the wave packet $\langle x|\rho|x\rangle$ and show that various factors cancel to yield a motion which is *not* sensitive to the rapid decay of the off-diagonal coherence.

IV. EVOLUTION OF THE WAVE PACKET

The evolution of the wave packet $\langle x|\rho|x\rangle$ can be calculated directly from the solution (8) but for the moment we shall consider the equation of motion obeyed by a general wave packet. The annihilation and creation operators can be reexpressed in terms of the position and momentum operators $\hat{x}$ and $\hat{p}$ by

$$a=(2\hbar\omega)^{-1/2}(\omega\hat{x}+i\hat{p}) ,$$
$$a^{\dagger}=(2\hbar\omega)^{-1/2}(\omega\hat{x}-i\hat{p}) . \quad (27)$$

In the position representation the momentum operator becomes the differential operator $\hat{p}=-i\hbar\partial/\partial x$ and the wave packet $\rho(x,t)=\langle x|\rho(t)|x\rangle$ obeys the equation of motion

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\gamma}{2}\left[D\frac{\partial^2}{\partial x^2}\rho(x,t)+\frac{\partial}{\partial x}[x\rho(x,t)]\right] \quad (28)$$

where we have written the diffusion constant D as

$$D=\frac{\hbar}{2\omega} . \quad (29)$$

This parameter sets the scale of the fluctuations in position. This equation is, not surprisingly, a Fokker-Planck equation for the distribution $\rho(x,t)$. If we change variables to $\tau=\gamma t/2$ and write $\xi(x,\tau)=e^{-\tau}\rho(x,\tau)$ the equation of motion becomes

$$\frac{\partial}{\partial\tau}\xi(x,\tau)=D\frac{\partial^2}{\partial x^2}\xi(x,\tau)+x\frac{\partial}{\partial x}\xi(x,\tau) . \quad (30)$$

The first term on the right-hand side of this equation

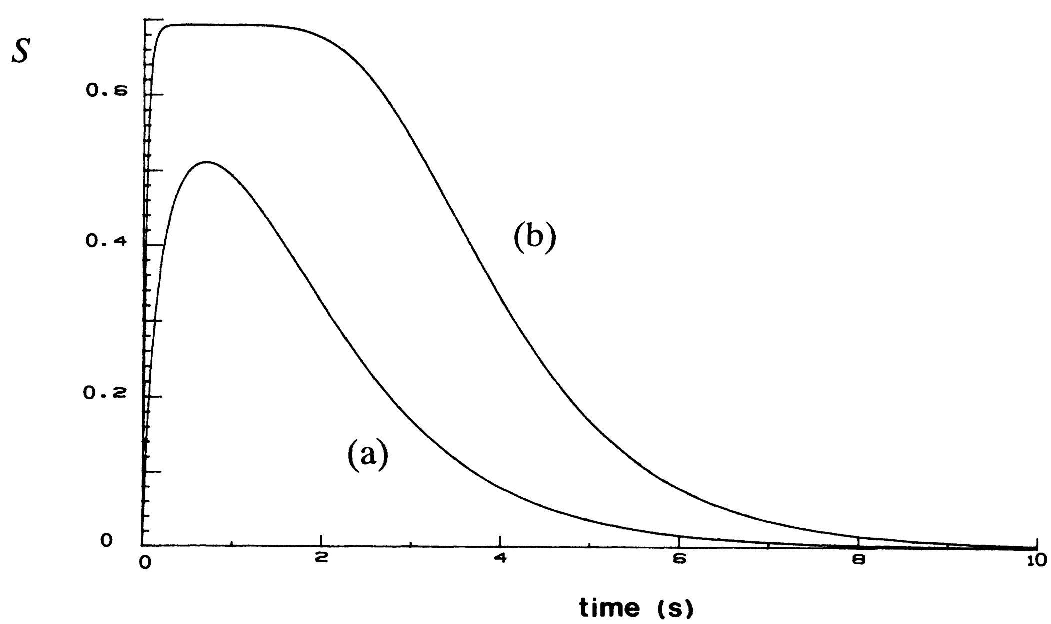

FIG. 1. The von Neumann entropy for the oscillator is plotted as a function of time for the initial amplitudes (a) $\alpha=2$, $\beta=4$; (b) $\alpha=2$, $\beta=7$. We have set the decay constant equal to unity.

leads to a diffusive motion increasing the width, or fluctuation, of an initial distribution but leaving the mean value unchanged. The effect of the second term is not so obvious. In order to see its effect we shall temporarily neglect the diffusive motion so that the differential equation (30) becomes

$$\frac{\partial}{\partial\tau}\xi(x,\tau)=x\frac{\partial}{\partial x}\xi(x,\tau)\ . \tag{31}$$

Changing variables so that $x=e^y$ we find that the general solution of (31) is of the form $\xi(x,\tau)=f(\tau+y)$. Let us consider our initial condition to be a Gaussian function of width σ and mean $\bar{x}$ so that

$$\xi(y,0)=\exp\left[-\frac{(e^y-\bar{x})^2}{2\sigma^2}\right]. \tag{32}$$

With this initial condition the full time-dependent solution of (31) is

$$\xi(x,\tau)=\exp\left[-\frac{(x-\bar{x}e^{-\tau})^2}{2(\sigma e^{-\tau})^2}\right], \tag{33}$$

which is still a Gaussian function but with a time-dependent mean value and width. The mean value decays to zero and the width of the distribution is narrowed. We shall see shortly that if our initial state is a coherent state then the narrowing due to this second term exactly balances the diffusive motion. In this case the mean value still decays to zero but the width of the wave packet is unchanged.

In terms of the diffusion constant the coherent state wave function in the position representation is given by

$$\langle x|\alpha\rangle=(2\pi D)^{-1/4}e^{-(1/2)(|\alpha|^2-\alpha^2)}\times\exp\left[-\left[\frac{x}{2\sqrt{D}}-\alpha\right]^2\right] \tag{34}$$

and the wave packet associated with this state is

$$|\langle x|\alpha\rangle|^2=(2\pi D)^{-1/2}\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\text{Re}\alpha\right]^2\right], \tag{35}$$

which is a Gaussian distribution with a width proportional to D. Let us assume, for the time being, that the amplitudes of our coherent states are real. The solution of (28) with the initial condition (35) is

$$\rho_\alpha(x,t)=(2\pi D)^{-1/2}\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\alpha e^{-\gamma t/2}\right]^2\right], \tag{36}$$

which is a Gaussian of the same width as the initial distribution but the motion of the wave packet is towards the origin. All we have done in fact is to replace the amplitude α with $\alpha e^{-\gamma t/2}$ in the initial distribution. This is consistent with the solution (8). Remembering that we are only considering real amplitudes the initial wave packet arising from the interference, or off-diagonal, terms is

$$\rho_{\text{int}}(x,0)=\left[\frac{2}{\pi D}\right]^{1/2}e^{-(1/2)(\alpha-\beta)^2}\times\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\tfrac{1}{2}(\alpha+\beta)\right]^2\right], \tag{37}$$

which is a Gaussian of the same width as the initial wave packets due to the diagonal elements. The initial wave packet for the state (7) is made up of the sum of three Gaussians of equal widths and we have

$$\rho(x,0)=\rho_\alpha(x,0)+\rho_\beta(x,0)+\rho_{\text{int}}(x,0)\ . \tag{38}$$

The equation of motion (28) is a *linear* partial differential equation and *cannot* distinguish the interference contribution from the diagonal contributions. Apart from constants, the motion of the interference contribution is identical with that of a wave packet due to an initial single coherent state with the same mean value. The interference contribution, therefore, *does not* decay on a faster time scale than the diagonal contributions. The full solution for real initial amplitudes α and β is

$$\rho(x,t)=(2\pi D)^{-1/2}N\sum_{\alpha,\beta}e^{-(1/2)(\alpha-\beta)^2}\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\tfrac{1}{2}(\alpha+\beta)e^{-\gamma t/2}\right]^2\right]. \tag{39}$$

This is a solution of (28) with the correct initial conditions and this can be verified by direct substitution.

Our results appears to contradict Eq. (2.20) of the original paper by Walls and Milburn[7] which, if we neglect free evolution terms, states that for two initial coherent states of amplitudes α and $-\alpha$, where α is real, the wave-packet evolution can be written in the form

$$\langle x|\rho(t)|x\rangle=I_+^2+I_-^2+2I_+I_-e^{-|\alpha|^2[1-\exp(-\gamma t)]}\ . \tag{40}$$

There appears to be a damping factor which rapidly suppresses the interference terms at a rate governed by the distance between the two states. The cross term I_+I_-, however, contains a time dependence which exactly cancels this factor leaving a residual evolution which occurs on the same time scale as the diagonal terms. This is true for arbitrary initial amplitudes α and β as we shall demonstrate below. The above result (39) is therefore consistent with that of Walls and Milburn.[7] From Eq. (9) we find that the wave packet can be written as

$$\rho(x,t)=N\sum_{\alpha,\beta}\frac{\langle\beta|\alpha\rangle}{\langle\tilde{\beta}|\tilde{\alpha}\rangle}\langle x|\tilde{\alpha}\rangle\langle\tilde{\beta}|x\rangle \tag{41}$$

where the overlap terms are given from (34) as

$$\langle x|\tilde{\alpha}\rangle\langle\tilde{\beta}|x\rangle=(2\pi D)^{-1/2}\times\exp[-\tfrac{1}{2}(|\tilde{\alpha}|^2-\tilde{\alpha}^2+|\tilde{\beta}|^2-\tilde{\beta}^{*2})]\times\exp\left[-\left[\frac{x}{2\sqrt{D}}-\tilde{\alpha}\right]^2-\left[\frac{x}{2\sqrt{D}}-\tilde{\beta}^*\right]^2\right] \tag{42}$$

which can be rearranged to give

$$\langle x|\tilde{\alpha}\rangle\langle\tilde{\beta}|x\rangle=(2\pi D)^{-1/2}\langle\tilde{\beta}|\tilde{\alpha}\rangle\times\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\tfrac{1}{2}(\tilde{\alpha}+\tilde{\beta}^*)\right]^2\right]. \tag{43}$$

The overlap $\langle\tilde{\beta}|\tilde{\alpha}\rangle$ *exactly* cancels the corresponding term occurring as a weighting factor in (9) so that the wave packet becomes

$$\rho(x,t)=(2\pi D)^{-1/2}N\times\sum_{\alpha,\beta}\langle\beta|\alpha\rangle\exp\left[-2\left[\frac{x}{2\sqrt{D}}-\tfrac{1}{2}(\tilde{\alpha}+\tilde{\beta}^*)\right]^2\right] \tag{44}$$

and the interference terms evolve on the same time scale as the diagonal terms. We now show that this insensitivity to the off-diagonal decay is also apparent in field expectation values.

V. EXPECTATION VALUES AND RELATIVE ENTROPY

We have seen that the wave packet in the position representation is not sensitive to the rapid destruction of the off-diagonal coherences. It is of some importance, therefore, to ask whether this behavior is also seen in the field observables. We consider normally-ordered expectation values $\langle a^{\dagger n}a^m\rangle$ neglecting the free evolution terms. From (9) we can write the density operator in a number state basis as

$$\rho(t)=N\sum_{\alpha,\beta}\sum_{n,m=0}^{\infty}\frac{\langle\beta|\alpha\rangle}{\langle\tilde{\beta}|\tilde{\alpha}\rangle}\langle n|\tilde{\alpha}\rangle\langle\tilde{\beta}|m\rangle|n\rangle\langle m|\ . \tag{45}$$

Evaluating the scalar products in this expression we find that

$$\rho(t)=N\sum_{\alpha,\beta}\sum_{n,m=0}^{\infty}\langle\beta|\alpha\rangle\exp(-\tilde{\alpha}\tilde{\beta}^*)\frac{\tilde{\alpha}^n(\tilde{\beta}^*)^m}{(n!m!)^{1/2}}|n\rangle\langle m|\ . \tag{46}$$

The expectation value $\langle a^{\dagger n}a^m\rangle$ is given by

$$\langle a^{\dagger n}a^m\rangle=\mathrm{Tr}[\rho(t)a^{\dagger n}a^m]\ . \tag{47}$$

Using the identity

$$a^{\dagger r}a^s|k\rangle=\begin{cases}\left[\dfrac{k!(k-s+r)!}{[(k-s)!]^2}\right]^{1/2}|k-s+r\rangle & (s<k)\\ 0 & (s\geq k)\end{cases} \tag{48}$$

we find that when the trace in (47) is taken over the number states of the field the expectation value can be written as

$$\langle a^{\dagger r}a^s\rangle=N\sum_{\alpha,\beta}\sum_{k,n,m=0}^{\infty}\langle\beta|\alpha\rangle\exp(-\tilde{\alpha}\tilde{\beta}^*)\times\left[\frac{k!(k-s+r)!}{[(k-s)!]^2}\right]^{1/2}\times\frac{\tilde{\alpha}^n(\tilde{\beta}^*)^m}{(n!m!)^{1/2}}\delta_{nk}\delta_{m,k-s+r}\ . \tag{49}$$

This reduces to the expression

$$\langle a^{\dagger r}a^s\rangle=N\sum_{\alpha,\beta}\sum_{k=0}^{\infty}\langle\beta|\alpha\rangle\exp(-\tilde{\alpha}\tilde{\beta}^*)\frac{\tilde{\alpha}^k(\tilde{\beta}^*)^{k-s+r}}{(k-s)!}\ . \tag{50}$$

The summation over k can be performed and we obtain the result

$$\langle a^{\dagger r}a^s\rangle=N\sum_{\alpha,\beta}\langle\beta|\alpha\rangle\tilde{\alpha}^s(\tilde{\beta}^*)^r \tag{51}$$

and once again we see that the terms arising from the off-diagonal coherences in the original state (7) decay on the same time scale as those arising from the diagonal elements. It would have been more convenient for the calculation of this expectation value to have taken the trace over a coherent state basis and this does indeed yield the above result. However, we shall need the density operator in the number state basis for the calculation of the relative Shannon entropy in the number state basis.

The relative Shannon entropy is a parameter which measures the deviation of one probability distribution from another. If $p(n)$ is one distribution and $g(n)$ another then the relative entropy between these distributions is defined to be[18]

$$S(g|p)=\sum_{n=0}^{\infty}p(n)[\ln p(n)-\ln g(n)]\ . \tag{52}$$

This quantity measures the deviation of the distribution $g(n)$ from that of $p(n)$ and measures the information difference between a "true" distribution $p(n)$ and an "estimated" distribution $g(n)$. Equation (52) is not the most general form for the relative entropy in quantum mechanics but is sufficient for our purposes here. Let us consider two density operators in the number state basis. If these are labeled ρ and σ then the relative Shannon entropy in this basis is given by

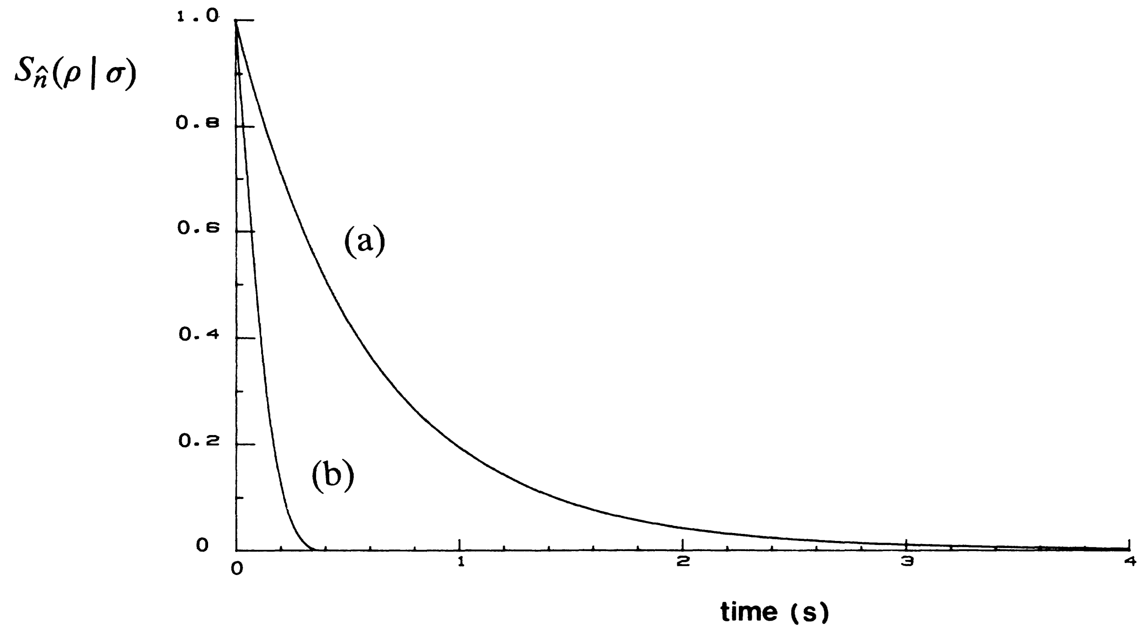

FIG. 2. The relative Shannon entropy in the number state basis is plotted as a function of time for the initial amplitudes (a) $\alpha=2$, $\beta=4$; (b) $\alpha=2$, $\beta=7$. We have set the decay constant equal to unity.

$$S_{\hat{n}}(\rho|\sigma)=\sum_{n=0}^{\infty}\langle n|\sigma|n\rangle(\ln\langle n|\sigma|n\rangle-\ln\langle n|\rho|n\rangle)\ . \tag{53}$$

Let us now suppose that ρ is given by (8) and σ is given by

$$\sigma=M(|\tilde{\alpha}\rangle\langle\tilde{\alpha}|+|\tilde{\beta}\rangle\langle\tilde{\beta}|)\ . \tag{54}$$

If the "wave packet" in the number state basis is sensitive to the decay of the off-diagonal coherences then the relative entropy of these two distributions should show a rapid decay from its maximum to zero on a time scale governed by the separation of the coherent states in the initial superposition.

The diagonal elements of the density operators ρ and σ in a number state basis are, from (46), given by

$$\langle n|\rho|n\rangle=N\sum_{\alpha,\beta}\langle\beta|\alpha\rangle\exp(-\alpha\beta^{*}e^{-\gamma t})\frac{(\alpha\beta^{*})^{n}}{n!}e^{-n\gamma t}\ ,$$
$$\langle n|\sigma|n\rangle=\tfrac{1}{2}\sum_{\lambda=\alpha,\beta}\exp(-|\lambda|^{2}e^{-\gamma t})\frac{|\lambda|^{2n}}{n!}e^{-n\gamma t}\ . \tag{55}$$

The calculation of the relative entropy in the number state basis requires the logarithm of the ratio of these two quantities. Denoting this ratio by μ_n we find that

$$\mu_n=\frac{\langle n|\rho|n\rangle}{\langle n|\sigma|n\rangle}=2N(1+\xi_n) \tag{56}$$

where we have written

$$\xi_n=\frac{\langle\beta|\alpha\rangle\exp(-\alpha\beta^{*}e^{-\gamma t})(\alpha\beta^{*})^{n}}{|\alpha|^{2n}\exp(-|\alpha|^{2}e^{-\gamma t})+|\beta|^{2n}\exp(-|\beta|^{2}e^{-\gamma t})}\ . \tag{57}$$

This quantity is, in fact, sensitive to the decay of the off-diagonal coherence. To see this we choose the initial amplitudes to be α and $-\alpha$ where α is real. With these initial amplitudes ξ_n can be written as

$$\xi_n=\tfrac{1}{2}(-1)^{n}\langle-\alpha|\alpha\rangle\exp(2|\alpha|^{2}e^{-\gamma t})\ . \tag{58}$$

The relative entropy in the number state basis is, therefore, sensitive to the rapid decay of the off-diagonal coherences. This is shown in Fig. 2 where we have used the same initial amplitudes as in Fig. 1. The greater the separation between the initial amplitudes the faster the decay of the relative entropy to zero.

VI. CONCLUSIONS

The study of the decay of quantum superposition states is of considerable current interest. As we have mentioned there have been several ingenious schemes proposed for their production and detection. Furthermore, the study of the decay of such superpositions is important in fundamental theories of quantum measurement. It has been noted that environmental influences are sufficient to cause the reduction of the pure superposition state to a mixture on a time scale much faster than that of a typical system decay time. We have confirmed this conclusion using a simple model of a damped oscillator by showing that the von Neumann entropy for the oscillator is sensitive to the decay of the off-diagonal coherences. We have also shown, however, that the wave packet in the position representation is *not* sensitive to the decay of these coherences and terms arising from the off-diagonal elements decay on the same time scale as those arising from the diagonal elements. The terms which suppress the off-diagonal coherences in the expression for the density operator are canceled when the wave packet is considered. This behavior is also observed for the field expectation values, the influence of the off-diagonal terms persisting over a typical decay time. The relative entropy for the "wave packet" in the number state representation is, however, sensitive to the decay of the off-diagonal coherences. This is in contrast to the behavior of the wave packet in the position representation and underlines the necessity for a careful choice of basis. The persistence of the off-diagonal coherences for the field expectation values gives some hope that effects due to the interference terms may be observed in the laboratory.

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APPENDIX

The formal solution of the master equation is

$$\rho(t) = \exp[(\hat{J}+\hat{L})t]\rho(0) \tag{A1}$$

where $\hat{J}$ and $\hat{L}$ are defined in Eq. (2) and (3). By writing

$$e^{\lambda(\hat{J}+\hat{L})} = e^{\lambda\hat{L}}F(\lambda) \tag{A2}$$

we obtain the differential equation

$$\frac{d}{d\lambda}F(\lambda) = e^{-\lambda\hat{L}}\hat{J}e^{\lambda\hat{L}}F(\lambda) \ . \tag{A3}$$

The commutation relation between the superoperators $\hat{J}$ and $\hat{L}$ is given by[17]

$$[\hat{J},\hat{L}] = -\gamma\hat{J} \ . \tag{A4}$$

Using the relation

$$e^{-\lambda\hat{L}}\hat{J}e^{\lambda\hat{L}} = \hat{J} - \lambda[\hat{L},\hat{J}] + \frac{\lambda^2}{2!}[\hat{L},[\hat{L},\hat{J}]] + \cdots \tag{A5}$$

and Eq. (A4) we obtain the differential equation

$$\frac{d}{d\lambda}F(\lambda) = \hat{J}e^{-\lambda\gamma}F(\lambda) \ . \tag{A6}$$

With the boundary condition $F(0)=1$ we obtain the solution

$$F(\lambda) = \exp\left[\frac{\hat{J}}{\gamma}(1-e^{-\lambda\gamma})\right] \ . \tag{A7}$$

The solution of this damping problem, and others, by the superoperator method was discussed by Barnett.[17]

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