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# Relativistic electrodynamics of continuous media

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The relativistic transform of electromagnetic fields and inductions in a nonlinear medium is studied theoretically, within the framework of classical electrodynamics. A covariant formulation of the electrodynamics of nonlinear media is presented. This theoretical formulation is of particular relevance to periodic systems since the spatial symmetries of a given medium, which determine most of its linear and nonlinear electromagnetic properties, as described by group theory are not conserved by the Lorentz transform (space-time symmetry). The Čerenkov radiation process is treated as an example in the rest frame of the interacting electron; it is found that in this case the linear refractive index of the dielectric medium becomes anisotropic and exhibits a singularity at the usual Čerenkov radiation angle.

## I. INTRODUCTION

The interaction of electromagnetic waves with matter can be described according to two different theoretical formulations.[1-6] On the one hand, the electromagnetic properties of the medium may be defined by introducing relations between the fields and the inductions; this approach is usually referred to as the Minkowski formulation.[1,6] Generally, these so-called constitutive relations are nonlinear. The other formulation describes the reaction of the medium to the electromagnetic waves in terms of an induced four-vector current density. As long as the theoretical analysis of the interaction of electromagnetic radiation with matter is performed in the rest frame of the medium under consideration, these two formulations are equivalent. However, whereas the four-vector current-density approach can lead to a covariant description of the electrodynamics of nonlinear media, the relations between fields and inductions become very complicated in any reference frame where the medium is not at rest. This is particularly true in the case of a nonlinear medium. Still, it should be noted that in the rest frame of the medium the constitutive relations describing its electromagnetic properties, which are generally derived from quantum mechanics and group theory, directly reflect the underlying spatial symmetries of the medium and therefore are usually the preferred formulation in classical nonlinear optics.[7-9] In the relativistic case, the difficulty arises from the fact that the Lorentz group conserves space-time symmetries rather than spatial symmetries. For example, it is possible to transform a tetragonal lattice into a cubic one through the Lorentz transform; a spin-polarized relativistic electron beam with the right energy will be sensitive to the magnetic phase transition corresponding to this relativistic symmetry effect, and spin-resonance phenomena should result from such experiments. Similarly, the Čerenkov radiation[10] process in a linear, isotropic dielectric, which is studied in this paper, can be viewed in the rest frame of the interacting electron as resulting from a singularity of the anisotropic refractive index of the medium for electromagnetic waves propagating at the Čerenkov angle.

The purpose of this paper is to compare these two approaches and to describe the electromagnetic properties of nonlinear media in any Galilean reference frame.[6,11,12] This theoretical formulation is particularly relevant to relativistic nonlinear media such as free-electron lasers,[13] astrophysical plasmas,[14] and Čerenkov devices,[10] and to such experiments as the probing of magnetic lattices with spin-polarized beams. In addition, by studying the same phenomenon from different frames of reference, we can gain some insight about the physics underlying electromagnetic phenomena. Finally, it is worth noting that within a relativistic description, nonlinear effects can couple the electric field to itself, but also to the magnetic field, and that one can consider magnetic nonlinearities.

This paper is organized as follows. In Sec. II we briefly review the general terms of the two formulations discussed above. In Sec. III we review the relativistic transform of the relation between fields and inductions in a linear, isotropic medium and we make use of these relations to describe the Čerenkov radiation process in the rest frame of the interacting electron. The alternative induced-source formalism is studied in Sec. IV in the case of a linear, isotropic medium. Section V focuses on the relativistic transform of nonlinear susceptibilities, described by the four-vector current density. Finally, in Sec. VI, conclusions are drawn.

## II. THEORETICAL BACKGROUND

In this section we first briefly review the definition of the fields and inductions, within the context of Maxwell's equations. The electromagnetic interaction is characterized, in the classical theory, by the electric field **E** and the magnetic field **H**. The corresponding electric and magnetic inductions are **D** and **B**, respectively. Maxwell's equations are conventionally separated into two groups. The first group, also called source-free group, corresponds to

$$\nabla\times\mathbf{E}+\partial_t\mathbf{B}=\mathbf{0}\ , \tag{1}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{2}$$

and the second group is described by

$$\nabla\times\mathbf{H}-\partial_t\mathbf{D}=\mathbf{j}\ , \tag{3}$$

$$\nabla\cdot\mathbf{D}=\rho\ . \tag{4}$$

Here, $j_\nu\equiv(c\rho,\mathbf{j})$ is the four-vector current density. Maxwell's equations combine the fields and inductions; the additional relations between the fields and inductions in vacuum are

$$\mathbf{D}=\epsilon_0\mathbf{E}\ , \tag{5}$$

$$\mathbf{B}=\mu_0\mathbf{H}\ , \tag{6}$$

where the permittivity $\epsilon_0$ and the permeability $\mu_0$ of free space are related to the speed of light in vacuum through the well-known equation

$$\epsilon_0\mu_0c^2=1\ . \tag{7}$$

It should be noted here that in classical electrodynamics the vacuum is a linear, isotropic medium. In QED, vacuum nonlinearities appear at the energy threshold for $e\bar{e}$ pair creation. In a medium, the most general relations are nonlinear and anisotropic:

$$\mathbf{D}=\mathbf{D}(\mathbf{E},\mathbf{H})\ , \tag{8}$$

$$\mathbf{B}=\mathbf{B}(\mathbf{E},\mathbf{H})\ . \tag{9}$$

Here we allow the possibility of coupled nonlinear electric and magnetic effects, since relativity requires an equal treatment of electric and magnetic phenomena. In the low-field limit, one can expand the above expressions in a Taylor series and take into account lower-order nonlinearities only; the corresponding polynomial coefficients are the nonlinear susceptibilities. In the rest frame of the medium, these constitutive relations are determined by the structure of the medium at the atomic scale and by its spatial symmetries. The nonlinear susceptibilities can thus be derived from quantum mechanics and group theory; they are generally tensors and describes the macroscopic electromagnetic properties of the nonlinear medium. The nonlinear susceptibilities are semiclassical in the sense that they are averaged over a large number of atomic systems, as optical wavelengths are generally long compared to typical lattice scales.

It is important, however, to note that the constitutive relations are clearly frequency dependent and that, in addition, the most general relations are nonlocal in character, as specified by the Kramers-Kronig relations,[5,8,9] and can be described only through space-time integrals. Here, we make the implicit assumption of steady state, and we assume that the relations between the fields and the inductions can be satisfactorily described by quasilocal expressions.

We now consider the interaction of electromagnetic waves with a nonlinear medium, in the absence of external fields. Two equivalent descriptions are available. In the first approach, we consider Maxwell's equations with no source term ($\rho=0$, $\mathbf{j}=\mathbf{0}$) and describe the electromagnetic properties of the medium through its constitutive relations. We have the following set of equations:

$$\nabla\cdot\mathbf{D}(\mathbf{E},\mathbf{H})=0\ , \tag{10}$$

$$\nabla\cdot\mathbf{B}(\mathbf{E},\mathbf{H})=0\ , \tag{11}$$

$$\nabla\times\mathbf{E}+\partial_t\mathbf{B}(\mathbf{E},\mathbf{H})=\mathbf{0}\ , \tag{12}$$

$$\nabla\times\mathbf{H}-\partial_t\mathbf{D}(\mathbf{E},\mathbf{H})=\mathbf{0}\ . \tag{13}$$

Here, **E** and **H** represent the incoming electromagnetic wave and **D** and **B** represent the reactions of the nonlinear medium; the sources are integrated into the inductions. Equations (10)–(13), together with the constitutive relations (8) and (9), describe electromagnetic phenomena within the framework of the so-called Minkowski formulation.

In the second formulation, we consider Maxwell's equations in vacuum, and we describe the nonlinear reactions of the medium through source terms. The constitutive relations are those of a vacuum, and we now have

$$\nabla\cdot\epsilon_0\mathbf{E}=\rho(\mathbf{E},\mathbf{H})\ , \tag{14}$$

$$\nabla\cdot\mu_0\mathbf{H}=0\ , \tag{15}$$

$$\nabla\times\mathbf{E}+\mu_0\partial_t\mathbf{H}=\mathbf{0}\ , \tag{16}$$

$$\nabla\times\mathbf{H}-\epsilon_0\partial_t\mathbf{E}=\mathbf{j}(\mathbf{E},\mathbf{H})\ . \tag{17}$$

These two sets of equations constitutive two alternative formulations of the electrodynamics of nonlinear media. However, their mathematical properties under transformations of the Lorentz group are quite different. As will be discussed in Sec. IV, the second formulation is covariant because the vacuum constitutive relations are inversible and because the source terms are described by a four-vector. In Sec. III we study the relativistic transform of the first set of equations in the case of a linear, isotropic medium.

## III. LINEAR ISOTROPIC MEDIUM: MINKOWSKI FORMULATION

Here we study the basic interaction of electromagnetic waves with a linear, isotropic medium, within the Minkowski formulation. We thus make use of the relations between fields and inductions in the medium; in other words, we consider Maxwell's equations with no source terms and describe the electromagnetic properties of the scattering medium through its constitutive relations. In this section and in the remainder of the analysis, the primed variables refer to the rest frame of the medium. For the case of a linear, isotropic medium considered in this section, the constitutive relations are given, in the rest frame of the medium, by

$$\mathbf{D}'=\epsilon\mathbf{E}'\ , \tag{18}$$

$$\mathbf{B}'=\mu\mathbf{H}'\ , \tag{19}$$

and Maxwell's equations reduce to

$$\nabla'\cdot\epsilon\mathbf{E}'=0\ , \tag{20}$$

$$\nabla'\cdot\mu\mathbf{H}'=0\ , \tag{21}$$

$$\nabla'\times\mathbf{E}'+\mu\partial_{t'}\mathbf{H}'=\mathbf{0}\ , \tag{22}$$

$$\nabla'\times\mathbf{H}'-\epsilon\partial_{t'}\mathbf{E}'=\mathbf{0}\ . \tag{23}$$

At this point, we briefly review the dispersion of electromagnetic waves, as described in the rest frame of the medium. We can represent the electromagnetic wave by a space-time Fourier transform[15]

$$\mathbf{E}'(x'_\nu)=\frac{1}{(2\pi)^4}\int_{\mathbf{R}^4}\mathcal{E}'(k'_\nu)\exp(ik'_\nu x'^\nu)(dk'_\nu)^4\ , \tag{24}$$

where $x'_\nu\equiv(ct',\mathbf{r}')$ is the four-vector position and $k'_\nu\equiv(\omega'/c,\mathbf{k}')$ is its conjugate, the four-wave vector. We then have the following operational equivalences in conjugate space:

$$\nabla'\equiv-i\mathbf{k}',\quad \partial_{t'}\equiv i\omega'\ . \tag{25}$$

Taking the curl of Eq. (22) and making use of (20) and (23), we obtain

$$(\epsilon\mu\partial_{t'}^2-\nabla'^2)\mathbf{E}'=\mathbf{0}\ . \tag{26}$$

Combining (26) with the operational equivalences defined above, we recover the usual dispersion relation

$$(\epsilon\mu\omega'^2-k'^2)\mathcal{E}'=\mathbf{0}\ . \tag{27}$$

Finally, the refractive index $n'$ is defined as

$$n'=|ck'/\omega'|=c\sqrt{\epsilon\mu}\ . \tag{28}$$

We now study the same basic phenomenon viewed from another reference frame. The invariance of Maxwell's equations under the Lorentz transform, which results directly from the principle of relativity, yields the transformation formulas for the fields and inductions.[1,12] We have

$$\mathbf{E}'=\gamma\left[\mathbf{E}-(1-\alpha)(\mathbf{E}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}+\mathbf{v}\times\mathbf{B}\right]\ , \tag{29}$$

$$\mathbf{H}'=\gamma\left[\mathbf{H}-(1-\alpha)(\mathbf{H}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}-\mathbf{v}\times\mathbf{D}\right]\ , \tag{30}$$

$$\mathbf{D}'=\gamma\left[\mathbf{D}-(1-\alpha)(\mathbf{D}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}+\frac{\mathbf{v}}{c^2}\times\mathbf{H}\right]\ , \tag{31}$$

$$\mathbf{B}'=\gamma\left[\mathbf{B}-(1-\alpha)(\mathbf{B}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}-\frac{\mathbf{v}}{c^2}\times\mathbf{E}\right]\ , \tag{32}$$

where $\mathbf{v}$ is the velocity of the medium relative to the reference frame we consider and $\gamma=1/(1-v^2/c^2)^{1/2}=1/\alpha$ is the relativistic factor. Note that, as the transformations formulas directly result from the relativistic invariance of Maxwell's equations, they combine the fields and inductions. We can now rewrite the constitutive relations (18) and (19) as follows:

$$\mathbf{D}-(1-\alpha)(\mathbf{D}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}+\frac{\mathbf{v}}{c^2}\times\mathbf{H}=\epsilon\left[\mathbf{E}-(1-\alpha)(\mathbf{E}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}+\mathbf{v}\times\mathbf{B}\right]\ , \tag{33}$$

$$\mathbf{B}-(1-\alpha)(\mathbf{B}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}-\frac{\mathbf{v}}{c^2}\times\mathbf{E}=\mu\left[\mathbf{H}-(1-\alpha)(\mathbf{H}\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}-\mathbf{v}\times\mathbf{D}\right]\ . \tag{34}$$

Taking the scalar product of (33) and (34) with $\mathbf{v}$ yields

$$\mathbf{D}\cdot\mathbf{v}=\epsilon\mathbf{E}\cdot\mathbf{v}\ , \tag{35}$$

$$\mathbf{B}\cdot\mathbf{v}=\mu\mathbf{H}\cdot\mathbf{v}\ . \tag{36}$$

Using these identities into (33) and (34), we have

$$\mathbf{D}+\frac{\mathbf{v}}{c^2}\times\mathbf{H}=\epsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B})\ , \tag{37}$$

$$\mathbf{B}-\frac{\mathbf{v}}{c^2}\times\mathbf{E}=\mu(\mathbf{H}-\mathbf{v}\times\mathbf{D})\ . \tag{38}$$

Finally, after some straightforward calculations we can eliminate $\mathbf{B}$ from Eq. (37) and obtain the sought-after constitutive relations

$$\mathbf{D}(\mathbf{E},\mathbf{H})=\chi\frac{\epsilon}{\gamma^2}\mathbf{E}+\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\mathbf{v}\times\mathbf{H}-\epsilon\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\ , \tag{39}$$

$$\mathbf{B}(\mathbf{E},\mathbf{H})=\chi\frac{\mu}{\gamma^2}\mathbf{H}-\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\mathbf{v}\times\mathbf{E}-\mu\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\mathbf{v}(\mathbf{v}\cdot\mathbf{H})\ . \tag{40}$$

Here, we have defined the following dimensionless parameter:

$$\chi=(1-\epsilon\mu v^2)^{-1}\ . \tag{41}$$

These relations can be recast in the following form:

$$\mathbf{D}=\xi\mathbf{E}+\eta\mathbf{v}\times\mathbf{H}-\epsilon\eta\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\ , \tag{42}$$

$$\mathbf{B}=\frac{\mu}{\epsilon}\xi\mathbf{H}-\eta\mathbf{v}\times\mathbf{E}-\mu\eta\mathbf{v}(\mathbf{v}\cdot\mathbf{H})\ , \tag{43}$$

by introducing the coefficients

$$\xi=\epsilon\frac{\chi}{\gamma^2},\quad \eta=\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\ .$$

The new constitutive relations are still linear but, as expected, they combine both electric and magnetic contributions and they are obviously anisotropic. In addition, the noncovariant character of the pseudoscalars $\epsilon$ and $\mu$ appears very clearly. At this point, it is possible to study the propagation of electromagnetic waves in a frame where the scattering medium is not at rest. The geometry of the problem is illustrated in Fig. 1. We start from

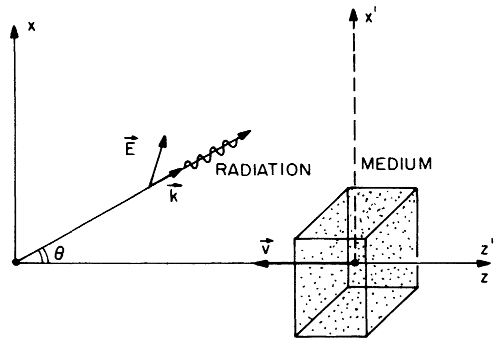


FIG. 1. Geometry of the relativistic transform studied in Secs. III and IV.

Maxwell's equations with no sources, and we make use of the constitutive relations derived above,

$$\nabla\times\mathbf{E}+\partial_t\left[\frac{\mu}{\epsilon}\xi\mathbf{H}-\eta\mathbf{v}\times\mathbf{E}-\mu\eta\mathbf{v}(\mathbf{v}\cdot\mathbf{H})\right]=\mathbf{0}\ , \tag{44}$$

$$\nabla\times\mathbf{H}-\partial_t[\xi\mathbf{E}+\eta\mathbf{v}\times\mathbf{H}-\epsilon\eta\mathbf{v}(\mathbf{v}\cdot\mathbf{E})]=\mathbf{0}\ . \tag{45}$$

Again, we represent the electromagnetic field by a four-dimensional (4D) Fourier transform,

$$\mathbf{E}(x_\nu)=\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\mathscr{E}(k_\nu)\exp(ik_\nu x^\nu)(dk_\nu)^4\ , \tag{46}$$

which yields the usual operational equivalences

$$\nabla\equiv-i\mathbf{k},\quad \partial_t\equiv i\omega\ .$$

Combining Eqs. (44) and (45) to eliminate the magnetic field **H**, we obtain

$$(\mathbf{k}+\omega\eta\mathbf{v})\times(\mathbf{k}+\omega\eta\mathbf{v})\times\mathscr{E}+\omega^2\frac{\mu}{\epsilon}\xi^2\mathscr{E}-\omega^2\mu\xi\eta\mathbf{v}(\mathbf{v}\cdot\mathscr{E})=\mathbf{0}\ . \tag{47}$$

We now define the transverse and parallel components of the following vectors:

$$\mathscr{E}=\hat{\mathbf{z}}\mathscr{E}_{\parallel}+\hat{\mathbf{x}}\mathscr{E}_{\perp}\ , \tag{48}$$

$$\mathbf{k}=\hat{\mathbf{z}}k_{\parallel}+\hat{\mathbf{x}}k_{\perp}\ , \tag{49}$$

where we have defined the $\hat{\mathbf{z}}$ axis so that $\mathbf{v}=\hat{\mathbf{z}}v$ (see Fig. 1). Upon elimination of the amplitudes $\mathscr{E}_{\parallel}$ and $\mathscr{E}_{\perp}$ from Eq. (47), we end up with the following dispersion relation:

$$\omega^2\xi^2\frac{\mu}{\epsilon}-(k_{\parallel}+\omega\eta v)^2=k_{\perp}^2\frac{\xi}{\xi-\epsilon\eta v^2}\ . \tag{50}$$

Defining the propagation angle $\theta$ (see Fig. 1), the dispersion relation can be rewritten as

$$\omega^2\chi^2\frac{\epsilon\mu}{\gamma^2}-\left[k\cos\theta+\omega v\chi\left[\epsilon\mu-\frac{1}{c^2}\right]\right]^2=k^2\sin^2\theta\frac{\chi}{\gamma^2}\ , \tag{51}$$

which clearly shows the anisotropy of the medium.

We first check the relativistic invariance of the dispersion relation. The Lorentz transformation of the four-wave vector gives

$$\omega=\gamma(\omega'+k_{\parallel}'v)\ , \tag{52}$$

$$k_{\parallel}=\gamma\left[k_{\parallel}'+\beta\frac{\omega'}{c}\right]\ , \tag{53}$$

$$k_{\perp}=k_{\perp}'\ . \tag{54}$$

Here, $\beta=v/c$ is the normalized velocity. We can use these expressions in the dispersion relation (50) to obtain, after some straightforward algebra,

$$(\omega'\sqrt{\epsilon\mu}+k_{\parallel}')(-\omega'\sqrt{\epsilon\mu}+k_{\parallel}')=-k_{\perp}'^2\ , \tag{55}$$

which reduces to

$$k_{\parallel}'^2+k_{\perp}'^2=\omega'^2\epsilon\mu\ . \tag{56}$$

Equation (56) is identical to the dispersion relation of electromagnetic waves in a linear, isotropic medium described by Eq. (27).

We now derive the refractive index $n=|ck/\omega|$ from Eq. (51) to obtain

$$n(\theta)=\frac{c\left[\frac{\mu}{\epsilon}\xi^2-\eta^2v^2\right]}{\eta v\cos\theta+\left[(\cos^2\theta)\xi^2\frac{\mu}{\epsilon}+\frac{(\sin^2\theta)\xi\left[\frac{\mu}{\epsilon}\xi^2-\eta^2v^2\right]}{\xi-\epsilon\eta v^2}\right]^{1/2}}\ . \tag{57}$$

There are two simple limiting cases to the above equation. On the one hand, one may consider vacuum ($\epsilon=\epsilon_0$, $\mu=\mu_0$), in which case one finds $n=1$. The other limiting case is obtained by taking $v=0$ ($\gamma=1$); we then verify that $n=n'$. The most interesting feature of Eq. (57), however, is the fact that the index of refraction exhibits a singularity for

$$\tan^2\theta=-\frac{\xi-\epsilon\eta v^2}{\xi}=-\frac{\gamma^2}{\chi}\ . \tag{58}$$

This means that we can expect a strong coupling of the radiation field to a static charge ($\omega=0$) for this particular radiation angle. We can translate this condition into the rest frame of the medium by noting that

$$\tan\theta' \equiv \frac{k'_\perp}{k'_\parallel} = \frac{k_\perp}{\gamma\left[k_\parallel - \beta\dfrac{\omega}{c}\right]} = \frac{\tan\theta}{\gamma\left[1 - \dfrac{\beta}{n\cos\theta}\right]} \tag{59}$$

At the singularity ($n \to \infty$), we find

$$\tan^2\theta' = \frac{\tan^2\theta}{\gamma^2} = \epsilon\mu v^2 - 1 \ , \tag{60}$$

which finally yields

$$\cos\theta' = \frac{1}{n'\beta'} \tag{61}$$

the well-known Čerenkov radiation threshold condition.

We have thus shown that the Čerenkov radiation condition corresponds to a singularity of the refractive index of the interacting medium, similar to that of an atomic transition, in the rest frame of the radiating particle. It is particulary interesting to note that in the rest frame of the test particle we only need to study the refractive index of the medium to infer the possibility of a radiation process, whereas in the rest frame of the medium nothing in the dispersion relation indicates the possibility of Čerenkov radiation, and one has to solve entirely the field equations to derive the Čerenkov threshold condition.[10]

In the general case of a nonlinear medium, the constitutive relations now read

$$\mathbf{D}'(\mathbf{D},\mathbf{H}) = \mathbf{D}'[\mathbf{E}'(\mathbf{E},\mathbf{B}),\mathbf{H}'(\mathbf{H},\mathbf{D})] \ , \tag{62}$$

$$\mathbf{B}'(\mathbf{B},\mathbf{E}) = \mathbf{B}'[\mathbf{E}'(\mathbf{E},\mathbf{B})\mathbf{H}'(\mathbf{H},\mathbf{D})] \ . \tag{63}$$

It is clear that for any complex nonlinear dependence of the inductions $\mathbf{D}'$ and $\mathbf{B}'$ on the fields $\mathbf{E}'$ and $\mathbf{H}'$, the inversion of the above equation will become analytically intractable. For the relativistic description of nonlinear media, the constitutive relation formalism proves to be inadequate, and there are no simple transformation formulae of the nonlinear susceptibilities. This is due to the incompatibility of the 3D (spatial) tensorial character of the nonlinear susceptibilities with the 4D (space-time) aspect of the Lorentz transformation.

## IV. LINEAR ISOTROPIC MEDIUM: INDUCED-SOURCE FORMALISM

In this section we focus our attention on the induced-source formulation; in other words, we now consider Maxwell's equations in vacuum and describe the electromagnetic properties of the interacting medium through a source term as prescribed in Eqs. (14)–(17). To illustrate this derivation, we first start from the basic example of the linear, isotropic medium. In this case, we have the following relation between the electromagnetic field and the induced current density, expressed in the rest frame of the medium

$$\mathbf{j}' = \epsilon_0\sigma\omega'\mathbf{E}' + \lambda\mathbf{k}'\times\mathbf{H}' \ . \tag{64}$$

The vectorial product for the magnetic-field contribution directly results from the polar character of the vector **H**, as opposed to the axial character of **E** and **j**; this, in turn, is correlated to the fact that the origin of magnetic properties in a material is determined by spin effects. We simplify matters further by considering a dielectric medium where $\rho' = 0$. Making use of the 4D Fourier transform, we obtain the dispersion relation in the following form:

$$k'^2 = \frac{\omega'^2}{c^2}\left[\frac{1 - i\sigma}{1 - i\lambda}\right] , \tag{65}$$

where the electric conductivity $\sigma$ and its magnetic analog $\lambda$ are defined as

$$1 - i\sigma = \epsilon/\epsilon_0 \ , \tag{66}$$

$$1 - i\lambda = \mu_0/\mu \ . \tag{67}$$

We can then identify Eq. (65) to the usual dispersion relation for a linear, isotropic medium given by (27).

We now consider the relativistic transformation of the source terms. Combining the dielectric condition and the Lorentz transform of the charge density, we have

$$\rho' = 0 = \gamma\left[\rho - \frac{\mathbf{v}}{c^2}\cdot\mathbf{j}\right] . \tag{68}$$

Similarly, the relativistic transformation of the current density is given by

$$\mathbf{j}' = \mathbf{j} + \gamma\frac{\mathbf{v}}{v^2}[(1-\alpha)(\mathbf{j}\cdot\mathbf{v}) - \rho v^2] \ . \tag{69}$$

Using the dielectric condition (68) into Eq. (69), we obtain a simplified expression of the current density

$$\mathbf{j}' = \mathbf{j} - \frac{\mathbf{v}}{v^2}(1-\alpha)(\mathbf{j}\cdot\mathbf{v}) \ . \tag{70}$$

Taking the scalar product of Eq. (70) with **v** and making use of the relativistic transform of the four-wave vector and electromagnetic fields, together with the expression of the induced linear current density (64) into (68), we obtain the following expression for the charge density:

$$\rho = \frac{\gamma^2}{c^2}[\epsilon_0\sigma(\omega - \mathbf{v}\cdot\mathbf{k})(\mathbf{E}\cdot\mathbf{v}) + \lambda\mathbf{v}\cdot\mathbf{k}\times(\mathbf{H} - \mathbf{v}\times\mathbf{D})] \ . \tag{71}$$

We now use the vacuum constitutive relations to rewrite the charge-density transform as a function of the electromagnetic fields only,

$$\rho = \frac{\gamma^2}{c^2}[\epsilon_0\sigma(\omega - \mathbf{v}\cdot\mathbf{k})(\mathbf{E}\cdot\mathbf{v}) + \lambda\mathbf{v}\cdot\mathbf{k}\times(\mathbf{H} - \epsilon_0\mathbf{v}\times\mathbf{E})] \ . \tag{72}$$

Proceeding in the same way for the current density, we end up with

$$\mathbf{j} = \gamma^2\epsilon_0\sigma(\omega - \mathbf{v}\cdot\mathbf{k})(\mathbf{E} + \mu_0\mathbf{v}\times\mathbf{H}) + \gamma^2\lambda(1-\alpha)\frac{\mathbf{v}}{v^2}[\mathbf{v}\cdot\mathbf{k}\times(\mathbf{H} - \epsilon_0\mathbf{v}\times\mathbf{E})]$$
$$+ \gamma\lambda\left[\left[\mathbf{k} + \gamma\frac{\mathbf{v}}{v^2}[(1-\alpha)(\mathbf{k}\cdot\mathbf{v}) - \beta^2\omega]\right]\times(\mathbf{H} - \epsilon_0\mathbf{v}\times\mathbf{E}) - (1-\alpha)(\mathbf{H}\cdot\mathbf{v})\frac{\mathbf{k}\times\mathbf{v}}{v^2}\right] . \tag{73}$$

Here, again, $\mathbf{v}$ is the velocity of the medium relative to our frame of reference. We can now study the dispersion of electromagnetic waves in the dielectric, linear, isotropic medium considered here. We use the source terms derived above to drive Maxwell's equations in vacuum:

$$\nabla\times\mathbf{E}+\mu_0\partial_t\mathbf{H}=\mathbf{0}\ , \tag{74}$$

$$\nabla\times\mathbf{H}-\epsilon_0\partial_t\mathbf{E}=\mathbf{j}(\mathbf{E},\mathbf{H})\ , \tag{75}$$

$$\nabla\cdot\epsilon_0\mathbf{E}=\rho(\mathbf{E},\mathbf{H})\ . \tag{76}$$

Through 4D Fourier analysis, we obtain in the moving frame

$$\mathcal{E}\left[\frac{\omega^2}{c^2}-k^2\right]=i\left(\mu_0\omega\mathbf{j}(\mathcal{E},\mathcal{H})-\frac{1}{\epsilon_0}\rho(\mathcal{E},\mathcal{H})\mathbf{k}\right) \tag{77}$$

and

$$\epsilon_0\mathbf{k}\cdot\mathcal{E}=\rho(\mathcal{E},\mathcal{H})\ . \tag{78}$$

In the special case of a nonmagnetic material ($\mu=\mu_0$, $\lambda=0$) the expressions for the source terms are greatly simplified:

$$\mathbf{j}(\mathcal{E},\mathcal{H})=\gamma^2\epsilon_0\sigma(\omega-\mathbf{v}\cdot\mathbf{k})(\mathcal{E}+\mu_0\mathbf{v}\times\mathcal{H})\ , \tag{79}$$

$$\rho(\mathcal{E},\mathcal{H})=\frac{\gamma^2}{c^2}\epsilon_0\sigma(\omega-\mathbf{v}\cdot\mathbf{k})(\mathcal{E}\cdot\mathbf{v})\ . \tag{80}$$

Making use of these expressions in Eq. (77), we obtain the dispersion relation in the following form:

$$\frac{\omega^2}{c^2}-k^2=i\sigma\frac{\gamma^2}{c^2}(\omega-\mathbf{v}\cdot\mathbf{k})^2\ , \tag{81}$$

where we recognize the vacuum dispersion on the left-hand side and the usual Doppler-shifted mode on the right-hand side. We note that the left-hand side of Eq. (81) is a scalar representing the magnitude of the four-wave vector, and thus a relativistic invariant. We can then rewrite the dispersion relation as follows:

$$\left[\frac{\omega'^2}{c^2}-k'^2\right]=i\sigma\omega'^2\ , \tag{82}$$

which is clearly identical to (65) for $\lambda=0$, thus demonstrating the relativistic invariance of the dispersive characteristic of the medium.

We now derive the refractive index $n=|ck/\omega|$ from Eq. (81). We start by defining the propagation angle $\theta$ as shown in Fig. 1, and we use the definition of $\sigma$ to rewrite (81) as follows:

$$\omega^2-k^2c^2=\gamma^2\left[1-\frac{\epsilon}{\epsilon_0}\right](\omega-\beta ck\cos\theta)^2\ . \tag{83}$$

From this equation we can easily solve for $k(\omega)$ and obtain the following expression of the refractive index:

$$n(\theta)=\frac{\left[1-\frac{\epsilon}{\epsilon_0}\right]\gamma^2\beta\cos\theta+\left[\left[1-\frac{\epsilon}{\epsilon_0}\right]\gamma^2(\beta^2\cos^2\theta-1)+1\right]^{1/2}}{\left[1-\frac{\epsilon}{\epsilon_0}\right]\gamma^2\beta^2\cos^2\theta+1}\ . \tag{84}$$

In the limiting case of vacuum ($\epsilon=\epsilon_0$), we easily find $n=1$; in addition, for $\beta=0$ ($\gamma=1$), we recover $n=n'=\sqrt{\epsilon/\epsilon_0}$. Again, the refractive index is clearly anisotropic, and it exhibits a singularity for waves propagating at an angle defined by the following equation:

$$\left[\frac{\epsilon}{\epsilon_0}-1\right]\gamma^2\beta^2\cos^2\theta=1\ . \tag{85}$$

To transform this condition on the propagation angle back to the rest frame of the medium, we use the relation between angles derived in Sec. II [Eq. (59)]:

$$\tan^2\theta'=\frac{\tan^2\theta}{\gamma^2}\ , \tag{86}$$

which is valid at the singularity ($n\to\infty$). Using the trigonometric relation between $\tan^2\theta$ and $\cos^2\theta$, we then easily find

$$\frac{1}{\cos^2\theta'}=\beta^2+\frac{1}{\gamma^2\cos^2\theta}\ . \tag{87}$$

Finally, the singularity Eq. (85) yields the following condition in the rest frame of the medium:

$$\frac{1}{\cos^2\theta'}=\beta^2\frac{\epsilon}{\epsilon_0}\ , \tag{88}$$

which is the Čerenkov radiation condition for a dielectric, nonmagnetic medium in the linear, isotropic case. We have thus shown the complete equivalence of the Minkowski formulation and the induced four-vector current-density formalism in the linear regime.

## V. NONLINEAR FORMALISM

We now treat the full nonlinear problem. The general formalism is the following. In the rest frame of the nonlinear medium, the induced-source terms

$$\mathbf{j}'=\mathbf{j}'(\mathbf{E}',\mathbf{H}')\ , \tag{89}$$

$$\rho'=\rho'(\mathbf{E}',\mathbf{H}') \tag{90}$$

describe its nonlinear electromagnetic response. In addition, the relativistic transform of the electromagnetic field yields [see Eqs. (29) and (30)]

$$\mathbf{E}'=\mathbf{E}'(\mathbf{E},\mathbf{B})\ , \tag{91}$$

$$\mathbf{H}'=\mathbf{H}'(\mathbf{H},\mathbf{D})\ . \tag{92}$$

The crucial point of this formulation is that the vacuum constitutive relations are relativistically invariant. Therefore, we have, within this formulation, and in any Galilean reference frame,

$$\mathbf{D}=\epsilon_0\mathbf{E}\ , \tag{93}$$

$$\mathbf{B}=\mu_0\mathbf{H}\ . \tag{94}$$

We can thus transform the four-vector current density and the electromagnetic field and make use of the vacuum constitutive relations to finally obtain the sought-after relativistic description of the nonlinear response of the medium:

$$\begin{aligned}\mathbf{j}(\mathbf{E},\mathbf{H})=&\ \mathbf{j}'[\mathbf{E}'(\mathbf{E},\mu_0\mathbf{H}),\mathbf{H}'(\mathbf{H},\epsilon_0\mathbf{E})]\\ &-\gamma\frac{\mathbf{v}}{v^2}((1-\alpha)\{\mathbf{v}\cdot\mathbf{j}'[\mathbf{E}'(\mathbf{E},\mu_0\mathbf{H}),\mathbf{H}'(\mathbf{H},\epsilon_0\mathbf{E})]\}\\ &-v^2\rho'[\mathbf{E}'(\mathbf{E},\mu_0\mathbf{H}),\mathbf{H}'(\mathbf{H},\epsilon_0\mathbf{E})])\ ,\end{aligned} \tag{95}$$

$$\rho(\mathbf{E},\mathbf{H})=\gamma\left[\rho'[\mathbf{E}'(\mathbf{E},\mu_0\mathbf{H}),\mathbf{H}'(\mathbf{H},\epsilon_0\mathbf{E})]+\frac{\mathbf{v}}{c^2}\cdot\mathbf{j}'[\mathbf{E}'(\mathbf{E},\mu_0\mathbf{H}),\mathbf{H}'(\mathbf{H},\epsilon_0\mathbf{E})]\right]\ . \tag{96}$$

We now address the same problem in a somewhat more detailed way. We consider a dielectric medium with electric nonlinearities. In its rest frame, we have the following expression of the induced nonlinear current density:

$$j_i'=\sum_{\ell=1}^{\infty}\sigma_{i;k,l,\ldots,p}^{(\ell)}E_k'E_l'\cdots E_p'\ . \tag{97}$$

Here the italic indices refer to the three spatial coordinates, and repeated indices are summed over according to Einstein's convention. The integer $\ell$ refers to the order of the nonlinearity. Making use of Eq. (95), together with the dielectric condition ($\rho'=0$), the relativistic transform of the current density yields

$$j_i=j_i'-\gamma\frac{v_i}{v^2}(1-\alpha)v_q j_q'\ . \tag{98}$$

Introducing the expression of the induced nonlinear current density in the rest frame of the medium, we obtain

$$j_i=\sum_{\ell=1}^{\infty}\left[\sigma_{i;k,l,\ldots,p}^{(\ell)}E_k'E_l'\cdots E_p'+\gamma\frac{v_i}{v^2}(1-\alpha)v_q\sigma_{q;k,l,\ldots,p}^{(\ell)}E_k'E_l'\cdots E_p'\right]\ . \tag{99}$$

Finally, making use of the relativistic transform of the electric field and the vacuum constitutive relations, we find

$$\begin{aligned}j_i=\sum_{\ell=1}^{\infty}\Bigg[&\left[\sigma_{i;k,l,\ldots,p}^{(\ell)}+\gamma\frac{v_i}{v^2}(1-\alpha)v_q\sigma_{q;k,l,\ldots,p}^{(\ell)}\right]\gamma^{\ell}\left[E_k-(1-\alpha)\frac{v_k}{v^2}E_kv_k+(\mathbf{v}\times\mu_0\mathbf{H})_k\right]\\ &\times\left[E_l-(1-\alpha)\frac{v_l}{v^2}E_lv_l+(\mathbf{v}\times\mu_0\mathbf{H})_l\right]\cdots\left[E_p-(1-\alpha)\frac{v_p}{v^2}E_pv_p+(\mathbf{v}\times\mu_0\mathbf{H})_p\right]\Bigg]\ .\end{aligned} \tag{100}$$

This expression exhibits an interesting property: terms in $\gamma^{\ell}$ appear for nonlinear effects of order $\ell$ and it appears that in certain reference frames higher-order nonlinearities can contribute strongly to the interaction. This is a direct consequence of the relativistic transformation of the electric field and the nonlinearities of the scattering medium.

## VI. DISCUSSION

We have studied theoretically the relativistic transformation of the electromagnetic fields and inductions in a nonlinear medium. Two different formal descriptions of the electromagnetic properties of nonlinear media are considered in this paper. On the one hand, these electromagnetic properties can be defined by introducing relations between the fields and inductions; this approach is referred to as the Minkowski formulation. Generally, these so-called constitutive relations are nonlinear. In addition, in the most general case, the constitutive relations are frequency dependent and nonlocal. We have assumed here that a quasilocal approximation describes satisfactorily the relations between the fields and inductions in the steady state. The other formulation considered here describes the nonlinear polarization effects in the medium in terms of an induced four-vector current density coupled with the vacuum constitutive relations. In the linear, isotropic case, we have shown that these two formalisms are completely equivalent. In particular, we find with both formulations that the refractive index of a linear, isotropic medium, as described in its rest frame, becomes anisotropic in any other Galilean frame of reference. In addition, this refractive index exhibits a singularity for waves propagating at an angle that is found to correspond to the usual Čerenkov threshold angle. This indicates that for a static charge distribution in the reference frame considered, we can expect a strong coupling to the radiation field giving rise to Čerenkov radiation at the threshold angle.

However, we find that for nonlinear problems, the Minkowski formulation becomes inadequate because the nonlinear constitutive relations are not inversible. This fact reflects the incompatibility of the 3D (spatial) tensorial character of the nonlinear susceptibilities with the 4D (space-time) aspect of covariant transformations. A direct consequence of this consideration is the fact that one can vary the symmetry of a given lattice through the Lorentz transform. Because the electromagnetic properties of a medium depend very strongly on its symmetries, as can be shown by group theory, we expect to observe phenomena such as relativistic electromagnetic phase transitions by probing magnetic lattices with spin-polarized relativistic electron beams. These effects would appear as spin resonances in the scattering of such beams.

The electromagnetic properties of nonlinear media under relativistic transforms can be studied within the framework of the induced four-vector current-density formulation. This results from the fact that in this formulation the constitutive relations are those of vacuum which are both inversible and relativistically invariant, and from the covariant character of the four-vector current density. One can then transform from the rest frame of the nonlinear medium considered to any other convenient Galilean frame of reference. In the case of free-electron lasers (FEL's), for example, one can derive the nonlinear refractive index of a bunched electron beam in its rest frame and then study the interaction of electromagnetic waves with such a medium in the laboratory frame to recover the familiar Doppler up- and down-shifted FEL interaction. This can also be done for relativistic astrophysical plasma. Finally, it is important to realize that by studying the same phenomenon from different frames of reference, we can actually gain some insight about the physics underlying electromagnetic phenomena.

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