Passage-time calculation for the detection of weak signals via the transient dynamics of a laser

Salvador Balle, F. De Pasquale,* and M. San Miguel
Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain
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The passage-time (PT) distribution associated with the switch on of a laser with an injected signal can be used to detect weak signals. We calculate the distribution function of such a distribution. Results for the mean PT interpolate between the limit of no injected signal and a simple logarithmic law for strong signals. It has a scaling form with a single important independent parameter. The variance of the distribution is more sensitive to the presence of the injected signal than the mean. Explicit formulas for the limit of sensitivity of the detection method and for the bandwidth resolution are given.

It was recently proposed by Vemuri and Roy[1] that weak signals could be detected via the transient dynamics of a laser much in the same way as superregenerative detection in radar receivers. The basic physical idea behind the proposal is that weak signals are greatly amplified when used to trigger the decay of an unstable state. This is basically the same mechanism underlying the proposal of using the transient dynamics of a laser as a statistical microscope to probe the initial radiation distribution.[2] The switch-on process of a laser can be characterized by the time at which the intensity reaches a prescribed reference value. This is a stochastic quantity known as passage time (PT). The PT turns out to be highly sensitive to the presence of an injected weak signal. The injected signal reduces the initiation time of the laser. The usefulness of this possibility for detecting weak signals has been proved[3] by indirect measurements of the intensity of an attenuated He-Ne laser acting as an injected signal during the switch-on of an argon-laser pumped single-mode ring dye laser. The experimental results indicate a logarithmic dependence of the initiation time with the intensity of the injected signal. The strength of spontaneous-emission noise gives a bound on the intensities that can be detected. Another bound of detection, given by the bandwidth resolution, indicates that only signals with a small detuning with respect to the laser frequency can be detected. Our purpose in this paper is to provide a theoretical framework of PT calculations where these results can be analyzed.

The theoretical model for the evaluation of the complex, scaled, dimensionless laser field of a single-mode laser in the good cavity limit is given by[1,3]

$$d_t E = -\kappa E + \frac{FE}{1+A(I/F)} + \kappa_e E_e + \xi(t) , \tag{1}$$

where κ is the cavity decay rate, F is the gain parameter, A the saturation parameter, $I=|E|^2$ the intensity of the laser field, E_e the injected field, and κ_e the coupling parameter. Spontaneous-emission noise is modeled by a complex random force $\xi(t)$, taken to be Gaussian of zero mean and correlation

$$\langle \xi(t)\xi^*(t')\rangle = 2\epsilon\, \delta(t-t') . \tag{2}$$

The laser is switched on by changing the loss parameter κ from below threshold to above threshold. We are interested in the time at which the intensity reaches a value I_r taken[3] as 2% of the steady-state value $I_S=(F-\kappa)/A$. In this regime saturation is not important and the process can be described by the linear version of (1)

$$d_t E = +(\gamma_1 + i\gamma_2)E + \kappa_e E_e + \xi(t) , \tag{3}$$

where $\gamma_1 = F-\kappa$, and where we have allowed for a detuning γ_2 between the laser field and the injected signal E_e. The field E_e is taken in (3) as a real number.

The experimental results of Ref. 3 have been compared with numerical simulations of the above model showing good agreement. From the point of view of an analytical calculation, comparison has only been established[1,2] with the results of a Fokker-Planck calculation for $E_e=\gamma_2=0$. The difficulty of a PT calculation for (3) is related to the fact that a closed equation for the intensity I does not exist when E_e or γ_2 are not zero. Equation (3) defines a linear problem, so that the statistics of the field E are easily obtained.[4] However, a straightforward calculation of PT statistics by Fokker-Planck methods[5] becomes rather involved due to boundary conditions when there is more than one relevant variable. This is the case of Eq. (3) in which the intensity and phase of the laser field are dynamically coupled. We follow here an alternative method which focuses on the individual realizations of the stochastic process $E(t)$. The PT statistics are easily calculated by solving first for t as a function of I_r in each realization and then averaging over repeated experiments. This method has already been shown to be useful in the description of the transient statistics of a dye laser,[6] in the calculation of the dependence of PT statistics on a finite sweeping rate of the net gain parameter in the laser switch-on[7] and also in the description of the switching-on of a laser with saturable absorber.[8] Our analytical results give a scaling form for the generating function of the PT moments. This indicates that the PT distribution only depends on two separate parameters. In particular, the dependence on the intensity of the injected signal and on the detuning are not independent. Therefore different measurements for different pairs of these two parameters correspond to the same PT distribution given by our scaling form. We point out that the variance of the PT distribution is a better quantity to detect the presence of an injected signal than the mean PT (MPT). Our results for the MPT interpolate between the limits of no external field and the case of decay dominated by the injected signal in which a simple logarithmic law holds. These results permit us to identify a critical value of the combination of parameters which gives the limit of applicability of the detection method. An expression for the bandwidth resolution is also obtained.

In order to calculate the PT statistics we write the solution of (3) as

$$E(t)=h(t)\exp(\gamma t) , \tag{4}$$

where

$$h(t)=E(0)+\int_0^t dt' e^{-\gamma t'}[\kappa_e E_e+\xi(t')] . \tag{5}$$

$E(0)$ is the initial value of the field at the time at which the cavity losses are switched to above threshold.[9] Thus the value of $E(0)$ is distributed with the stationary distribution associated with (3) but with γ_1 replaced by $-\gamma_0$, where $\gamma_0=\kappa_i-F$ and k_i is the initial loss parameter. This means that

$$E(0)=\int_{-\infty}^{0} dt' e^{(\gamma_0-i\gamma_2)t'}[\kappa_e E_e+\xi(t')] . \tag{6}$$

Since passage times are typically such that $\gamma_1 t \gg 1$ we can replace $h(t)$ in (5) by $h(\infty)=h=h_1+ih_2$. In this case the stochastic process $h(t)$ becomes a time-independent random variable h which plays the role of an effective random initial condition in (4). The random initial condition triggers the decay process and it is exponentially amplified. This amplification allows at later times the detection of the seed of the process. Solving (4) we have

$$t=\frac{1}{2\gamma_1}\ln\frac{I_r}{h_1^2+h_2^2} , \tag{7}$$

so that the statistical properties of the random time t are determined by those of h through the transformation (7). The statistical properties of the bivariate Gaussian process (h_1,h_2) can be obtained from the explicit expression of $h=h_1+h_2$,

$$h=\int_{-\infty}^{0} dt' e^{(\gamma_0-i\gamma_2)t'}[\kappa_e E_e+\xi(t')] + \int_0^{+\infty} dt' e^{-(\gamma_1+i\gamma_2)t'}[\kappa_e E_e+\xi(t')] . \tag{8}$$

One finds that h_1 and h_2 are uncorrelated with a probability distribution

$$P(h_1,h_2)=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(h_1-\langle h_1\rangle)^2+(h_2-\langle h_2\rangle)^2}{2\sigma^2}\right] , \tag{9}$$

where

$$\langle h_1\rangle=\kappa_e E_e\left[\frac{\gamma_1}{\gamma_1^2+\gamma_2^2}-\frac{\gamma_0}{\gamma_0^2+\gamma_2^2}\right] , \tag{10}$$

$$\langle h_2\rangle=-\kappa_e E_e\left[\frac{\gamma_2}{\gamma_1^2+\gamma_2^2}-\frac{\gamma_2}{\gamma_0^2+\gamma_2^2}\right] , \tag{11}$$

$$\sigma^2=\frac{\epsilon}{2\gamma_1}+\frac{\epsilon}{2\gamma_0} . \tag{12}$$

The statistical moments of the PT distribution can be obtained from the generating function $G(2\gamma_1\lambda)$

$$G(2\gamma_1\lambda)\equiv\langle e^{-2\gamma_1\lambda t}\rangle = \int_{-\infty}^{+\infty} dh_1 \int_{-\infty}^{+\infty} dh_2 P(h_1,h_2)\left[\frac{I_r}{h_1^2+h_2^2}\right]^{-\lambda} . \tag{13}$$

Introducing the modulus and phase of h, $h=Re,^{i\phi}$ the integral over the phase can be done explicitly, yielding

$$G(2\gamma_1\lambda)=\int_0^{+\infty} dR\, R\left[\frac{\alpha}{R^2}\right]^{-\lambda} I_0(\beta R)e^{-(R^2+\beta^2)/2} , \tag{14}$$

where $I_0(z)$ stands for the modified Bessel function[10] of zeroth order and $\alpha\equiv I_r/\sigma^2$ and $\beta^2\equiv|\langle h\rangle|^2\sigma^2$, with σ and h given in Eqs. (10)–(12). Using a power-series expansion for I_0, the generating function can be written as

$$\begin{aligned} G(2\gamma_1\lambda)&=\left[\frac{\alpha}{2}\right]^{-\lambda} e^{-\beta^2/2} \times\sum_{m=0}^{\infty}\left[\frac{\beta^2}{2}\right]^m (m!)^{-2}\Gamma(m+\lambda+1) \\ &=\left[\frac{\alpha}{2}\right]^{-\lambda} e^{-\beta^2/2}M(\lambda+1,1,\beta^2/2)\Gamma(\lambda+1) , \end{aligned} \tag{15}$$

where $M(a,b,z)$ is the Kummer confluent hypergeometric function.[10] The scaling of G in (14) or (15) indicates that the PT distribution depends only on the combination of parameters given by α and β. The dependence on α is rather simple and contains the influence of the reference value I_r. The parameter β contains the combined effect of the noise, gain, and detuning parameters and the intensity of the applied field. In practical cases $\gamma_1 < < \gamma_0$ so that α and β take the simple form

$$\alpha=\frac{2\gamma_1 I_r}{\epsilon} , \tag{16}$$

$$\beta=\left[\frac{2(\kappa_e E_e)^2}{\epsilon\gamma_1}\right]^{1/2}\frac{1}{\left[1+\left[\frac{\gamma_2}{\gamma_1}\right]^2\right]^{1/2}} . \tag{17}$$

The generating function admits particularly simple expressions in the limits $\beta=0$ and ∞. For $\beta=0$ it reproduces the result in the absence of injected signal[7]

$$G(2\gamma_1\lambda)=\left[\frac{\alpha}{2}\right]^{-\lambda}\Gamma(\lambda+1)\ , \tag{18}$$

while for $\beta=\infty$, it gives the result for relaxation triggered by large injected signals

$$G(2\gamma_1\lambda)=\left[\frac{\alpha}{\beta^2}\right]^{-\lambda}\ . \tag{19}$$

In this last limit noise plays no important role and G is independent of ϵ. An explicit result for the MPT is easily obtained from (15),

$$\langle 2\gamma_1 t\rangle=\left[-\frac{d}{d\lambda}\ln G(2\gamma_1\lambda)\right]_{\lambda=0}=\ln\left[\frac{\alpha}{2}\right]-e^{-\beta^2/2}\sum_{m=0}^{\infty}\left[\frac{\beta^2}{2}\right]^m(m!)^{-1}\psi(m+1)\ , \tag{20}$$

where $\psi(z)$ is the digamma function.[10] A first important consequence of (20) is that the value of α only gives a constant shift of the MPT for all β, so that

$$\langle 2\gamma_1 t\rangle_{\alpha=\alpha_1}-\langle 2\gamma_1 t\rangle_{\alpha=\alpha_2}=\ln\left[\frac{\alpha_1}{\alpha_2}\right]\ . \tag{21}$$

A plot of $\langle 2\gamma_1 t\rangle$ calculated from (20) for two different values of α evidentiating this fact is shown in Fig. 1. The value $\alpha=1.7945\times10^7$ corresponds to the dye laser parameters of Ref. 1 with $F=1.4\times10^7$ sec^{-1}, $\gamma_{0,1}=|F-k_{0,1}|$, $k_1=1.2\times10^6$ sec^{-1}, $k_0=7\times10^7$ sec^{-1}, $\epsilon=0.004$ sec^{-1}, and $I_r=0.02\gamma_1/A$ with $A=2.6\times10^6$ sec^{-1}. The second value of α corresponds to a spontaneous-emission noise four orders of magnitude smaller. The initial slow decay of $\langle t\rangle$ versus β seen in Fig. 1 corresponds to the regime in which the switch-on is dominated by noise. In this regime (20) is well approximated by

$$\langle 2\gamma_1 t\rangle=\langle 2\gamma_1 t\rangle_{E_e=0}-\frac{\beta^2}{2}+\frac{\beta^4}{16}\ . \tag{22}$$

In the opposite limit of strong signals a good approximation to $\langle 2\gamma_1 t\rangle$ is obtained by the asymptotic expansion of (15):

$$\langle 2\gamma_1 t\rangle=2\gamma_1 T_{\epsilon=0}-\frac{e^{-\beta^2/2}}{\beta^2/2}\ , \tag{23}$$

where

$$T_{\epsilon=0}=(2\gamma_1)^{-1}\ln(\alpha/\beta^2) \tag{24}$$

is the deterministic relaxation time which can be obtained from (19). Equation (20) interpolates between the two limits (22) and (24) above. As an interesting technical point, we mention that in the presence of an injected signal the state of zero macroscopic field above threshold is not an unstable state, since it decays even in the absence of noise. A naive approach to the problem would be a calculation of passage times by a perturbative expansion around a deterministic trajectory for $E_e\neq0$. This method does not work because, as seen in (23), the correction to the deterministic limit is singular in ϵ. However, it is clear that the logarithmic dependence of the initiation time on the intensity of the injected signal observed in the experiments is already obtained in the limit $\epsilon=0$ as given by (24). This deterministic contribution is the dominant one for $\beta\gg1$.

Other moments of the PT distribution can be obtained from (15). For example, the variance of the distribution gives a measure of the reliability of using $\langle t\rangle$ as a quantity to detect injected signals. More important is that it turns out that $(\Delta t)^2=\langle t^2\rangle-\langle t\rangle^2$ is a quantity independent of α which is much more sensitive to the presence of the injected signal than $\langle t\rangle$. From (15) we obtain

$$(2\gamma_1\Delta t)^2=e^{-\beta^2/2}\sum_{m=0}^{\infty}\left[\frac{\beta^2}{2}\right]^m(m!)^{-1}\times[\psi'(m+1)+\psi^2(m+1)]-\left[e^{-\beta^2/2}\sum_{m=0}^{\infty}\left[\frac{\beta^2}{2}\right]^m(m!)^{-1}\psi(m+1)\right]^2\ . \tag{25}$$

A plot of $(\Delta t)^2$ versus β as obtained from (25) is shown in Fig. 2, exhibiting a clear initial plateau and a rapid decrease to $(\Delta t)^2=0$ for $\beta\rightarrow\infty$. The initial plateau corresponds to weak injected signals. In that regime, (25) is well approximated by

$$(2\gamma_1\Delta t)^2=\psi(1)-\frac{\beta^4}{8}\ . \tag{26}$$

The absence of a correction of order β^2 to the limit $\beta=0$ describes the very slow dependence with β. In the opposite limit $\beta\gg1$ the deterministic dynamics dominates and $(\Delta t)^2$ vanishes as

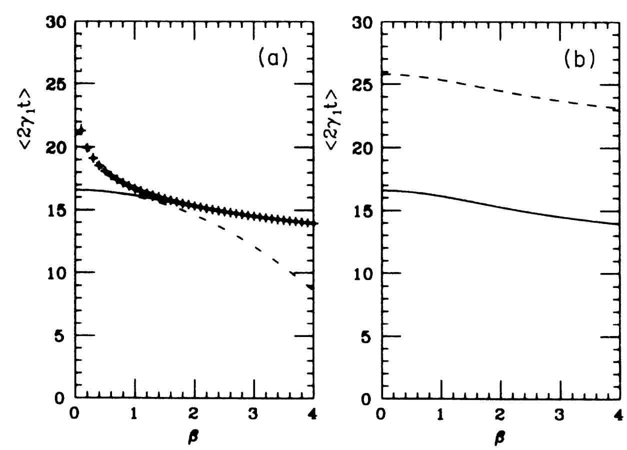

FIG. 1. (a) Results for the MPT as a function of β obtained from Eq. (20) (solid curve) and from the deterministic solution [Eq. (19), crosses] for $\alpha=1.7945\times10^7$. The dot-dashed curve corresponds to the small-β expansion [Eq. (22)]. (b) Results for MPT obtained from eq. (20) corresponding to two different values of α, $\alpha=1.7945\times10^7$ (solid curve) and $\alpha=1.7945\times10^{11}$ (dashed curve).

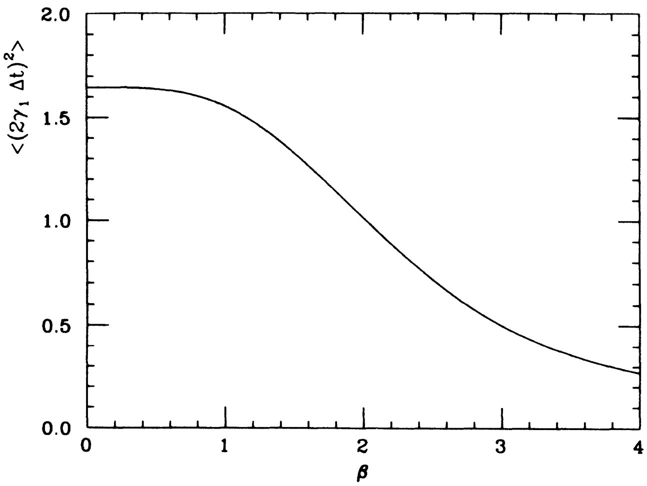

FIG. 2. Second moment of the PT distribution as a function of β.

$$(2\gamma_1 \Delta t)^2 = \frac{4}{\beta^2} \ . \tag{27}$$

The combination of our results for $\langle t \rangle$ and $(\Delta t)^2$ permits us to define a critical value of β, β_c, below which the injected signal cannot be detected. This limit of sensitivity of the detection method can be defined by the value of β for which the reduction of $\langle t \rangle$ is of the order of the maximum variance

$$(\langle t \rangle_{\beta_c} - \langle t \rangle_{\beta=0})^2 = (\Delta t)^2_{\beta=0} \ . \tag{28}$$

With this definition β_c is independent of α and from (22) and (26) we obtain

$$\beta_c \approx [4\psi'(1)]^{1/4} = 1.6 \ . \tag{29}$$

From Fig. 1 we see that this value of β corresponds to the matching between the two approximations (22) and (24). It can be understood as the lower bound for the validity of the deterministic approximation ($\epsilon = 0$) given by (19). For $\beta > \beta_c$ the switch-on process is dominated by the injected signal and not be spontaneous emission, so that E_e can be efficiently detected.

The dependence of $\langle t \rangle$ on the detuning γ_2 can also be analyzed from our results. Since $\langle t \rangle$ is a decreasing function of β, $\langle t \rangle$ has, as expected[3], a minimum for $\gamma_2 = 0$ and grows with γ_2 [see (17)]. The bandwidth resolution can be obtained from the general result (20). A particular simple result can be obtained in the limit of efficient detection $\beta \gg 1$. In this case $\langle t \rangle = T_{\epsilon=0}$, (24), and the value of $\gamma_2 = \gamma_2^0$ for which $\langle t \rangle$ is twice its value at $\gamma_2 = 0$ is given by

$$\gamma_2^0 = \gamma_1 (e^{2\gamma_1 T_{\epsilon=0}} - 1)^{1/2} = \gamma_1 \left[\left(\frac{I_r \gamma_1^2}{\kappa_e^2 E_e^2} \right) - 1 \right]^{1/2} \ . \tag{30}$$

Equation (30) gives an estimate of the bandwidth resolution as a function of the gain parameter and intensity of the injected signal.

In summary, the scaling result (15) gives a complete characterization of the PT distribution as needed to detect weak signals via the laser switch-on process.

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*Permanent address: Physics Department, Università di l'Aquila, Italy.

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