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Eigenenergies and oscillator strengths for the Hulthén potential

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A detailed study of the bound-state properties of the Hulthén potential is presented. Accurate eigenenergies are obtained for the Hulthén potential by numerical integration of the Schrödinger equation. One-parameter variational calculations are carried out. The variational results are practically identical to the exact energies, except in the high-screening region. The critical screening parameter is calculated for various values of l for $n \leq 10$ by a numerical solution of the wave equation. The energy eigenvalues obtained by a variety of methods are compared and discussed. The variational wave functions are employed to calculate absorption oscillator strengths for $1s \rightarrow 2p$, $1s \rightarrow 3p$, and $2p \rightarrow 3d$ transitions.

I. INTRODUCTION

The Hulthén potential[1] is one of the important short-range potentials in physics. The potential (in atomic units) is given by

$$V(r) = -Z\delta e^{-\delta r}/(1-e^{-\delta r}) \; . \qquad (1)$$

where Z is a constant and δ is the screening parameter. If the potential is used for atoms, the Z is identified with the atomic number. This potential is a special case of the Eckart potential.[2]

The Hulthén potential has been used in nuclear and particle physics,[3] atomic physics,[4–6] solid-state physics,[7,8] and chemical physics,[9] and its bound-state and scattering properties have been investigated by a variety of techniques.[10–31] Hulthén wave functions have been used in solid-state physics problems.[7,32] Hulthén-like wave functions have found use in investigations on atomic problems.[33,34] The Eckart potential has been investigated,[35] and other extended versions of the Hulthén potential have also been proposed.[36,37]

A great advantage with the Hulthén potential is the fact that the Schrödinger equation for this potential can be solved in a closed form for s waves.[1,5,14] For $l \neq 0$, a number of methods have been employed to evaluate bound-state energies.[24–31] However, no "exact" values obtained from a numerical integration of the Schrödinger equation have been available to assess the accuracy of the various methods. We present such values in Sec. II of this paper and compare them with those obtained by several other methods. For a number of purposes it is convenient to have the wave function in an analytical form. Greene and Aldrich[33] have given a method for generating pseudo-Hulthén wave functions for $l \neq 0$ states. In Sec. III we apply the variational method, using these wave functions to obtain energies for $l \neq 0$ states. Using only a single parameter in the wave functions, we are able to obtain energies that are almost equal to the exact energies, except in the high-screening region. We also suggest a modification for these wave functions and present the results for the $2p$ level obtained from the modified wave function. An important quantity of interest for the Hulthén potential (and for other similar screened potentials) is the critical screening parameter δ_c, which is that value of δ for which the binding energy of the level in question becomes zero. In Sec. IV we present values of δ_c for $n \leq 10$ obtained from a numerical solution of the Schrödinger equation. The eigenenergies obtained by various methods are compared in Sec. V. Using the analytical wave functions of Sec. II, we obtain expressions for the absorption oscillator strength for three transitions in Sec. VI and numerical results are presented for a few values of δ. Throughout this paper we shall take $Z = 1$ and shall use atomic units in which $\hbar = m = e = 1$.

II. NUMERICALLY DETERMINED EIGENENERGIES

Eigenenergies for the Hulthén potential were determined for certain values of δ for $n \leq 6$ and $l \neq 0$ by numerical integration of the Schrödinger equation using Numerov's method and a logarithmic mesh. The results are presented in column 3 of Table I, along with values obtained by several other methods.

The normalized probability density distribution $[r^2R^2(r)]$ for the $3p$ state for $\delta = 0.025$ (low screening) and 0.15 (high screening) are shown in Fig. 1. It will be noticed that as δ increases, the density distribution tends to spread out to larger values of r, and the peak values are reduced. Analogous results were found for the static screened Coulomb potential by Rogers *et al.*[38]

It will be noticed from Table I that as δ increases the splitting between different l levels (for a given n) increases. For low values of the principal quantum number, the ordering of the levels is the same as in the case of the Coulomb potential, but as one goes to higher values of n, it is found that there is a considerable amount of level crossing in the vicinity of $E = 0$. This effect is shown for $n = 7$ and 8 levels in Fig. 2. At high values of n, the level ordering is quite complex. We shall return to this point when we consider δ_c in Sec. IV.

Patil[26] has analytically investigated the behavior of the energy levels for the Hulthén potential near the critical screening parameter and found that for $l \geq 2$,

$$E = \text{const} \times (1/\delta - 1/\delta_c) + O[(1/\delta - 1/\delta_c)]^2 \qquad (2)$$

in the vicinity of δ_c. It was of interest to examine the range of δ in which the above relation is applicable. Equation (2) indicates that a plot of $E/(1/\delta - 1/\delta_c)$ versus $(1/\delta - 1/\delta_c)$ should be linear over the range of applicability. We made such plots for $n = 7$ and $l = 2$, 3, 4, 5, and 6. It was found that the linearity holds only over a very small interval of δ. We estimate that this range is from δ_c to approximately $0.99\delta_c$.

For s states, from the solution of the Schrödinger equation one can obtain

$$\frac{E}{E_0} = \left[1 - \frac{\delta}{\delta_c} \right]^2 , \qquad (3)$$

where E_0 is the energy for $\delta = 0$. Thus there is a univer-

TABLE I. Eigenenergies as obtained by various methods. A negative sign before the energy has been omitted everywhere.

State	δ	Numerical integration	Variational [Eq. (7)]	Variational [Eq. (9)]	Lai and Lin (Ref. 24)	Patil (Ref. 26)	Tang and Chan (Ref. 29)	Matthys and De Meyer (Ref. 31)
2*p*	0.025	0.112 760 5	0.112 760 5	0.112 760 5		0.112 76		0.112 760 4
	0.050	0.101 042 5	0.101 042 5	0.101 042 5	0.101 043	0.101 04	0.101 042 4	0.101 042 5
	0.075	0.089 847 8	0.089 847 8	0.089 847 8		0.089 85		
	0.100	0.079 179 4	0.079 179 4	0.079 179 4	0.079 179	0.079 18	0.079 179 4	0.079 179 4
	0.150	0.059 441 5	0.059 441 5	0.059 441 5		0.059 445		0.059 441 5
	0.200	0.041 886 0	0.041 886 0	0.041 885 9	0.041 886	0.041 895	0.041 885 7	0.041 886 0
	0.250	0.026 611 1	0.026 610 8	0.026 610 0				
	0.300	0.013 790 0	0.013 787 8	0.013 783 8	0.013 790			0.013 790 0
	0.350	0.003 793 1	0.003 773 4	0.003 753 4	0.003 779	0.038 375		
3*p*	0.025	0.043 706 9	0.043 706 9		0.043 707	0.043 708 5		0.043 707 1
	0.050	0.033 164 5	0.033 164 5		0.033 165	0.033 185	0.033 165 18	0.033 165 0
	0.075	0.023 939 7	0.023 939 7			0.024 016 5		
	0.100	0.016 053 7	0.016 053 7		0.016 054	0.016 22	0.016 067 72	0.016 053 7
	0.150	0.004 466 3	0.004 466 0		0.004 466	0.046 995		0.004 466 4
3*d*	0.025	0.043 603 0	0.043 603 0		0.043 603	0.043 602 5		0.043 603 0
	0.050	0.032 753 2	0.032 753 2		0.032 753	0.0327 45	0.032 753 2	0.032 753 2
	0.075	0.023 030 7	0.023 030 7			0.022 99		
	0.100	0.014 484 2	0.014 484 2		0.014 484	0.014 39	0.014 484 2	0.014 484 2
	0.150	0.001 396 6	0.001 389 4		0.001 391	0.001 375 5		0.001 396 5
4*p*	0.025	0.019 948 9	0.019 948 9		0.019 949	0.019 95		0.019 949 0
	0.050	0.011 058 2	0.011 058 2		0.011 058	0.011 075	0.011 072 5	0.011 058 3
	0.075	0.004 621 9	0.004 621 9		0.004 622	0.004 658 5		0.004 622 4
	0.100	0.000 755 0	0.000 753 2		0.000 754	0.000 752		
4*d*	0.025	0.019 846 2	0.019 846 2		0.019 846	0.019 845		0.019 846 2
	0.050	0.010 667 4	0.010 667 4		0.010 667	0.010 68	0.010 669 0	0.010 667 4
	0.075	0.003 834 5	0.003 834 4		0.003 834	0.003 875		0.003 834 6
4*f*	0.025	0.019 691 1	0.019 691 1		0.019 691	0.019 69		0.019 691 1
	0.050	0.010 062 0	0.010 062 0		0.010 062	0.010 045	0.010 062 0	0.010 061 9
	0.075	0.002 556 3	0.002 555 7		0.002 556	0.002 557		0.002 556 3
5*p*	0.025	0.009 403 6					0.009 408 7	
	0.050	0.002 649 0						
5*d*	0.025	0.009 303 7					0.009 305 0	
	0.050	0.002 313 1						
5*f*	0.025	0.009 152 1					0.009 152 3	
	0.050	0.001 783 5						
5*g*	0.025	0.008 946 5					0.008 946 5	
	0.050	0.001 015 9						
6*p*	0.025	0.004 154 8						
6*d*	0.025	0.004 060 6						
6*f*	0.025	0.003 916 8						
6*g*	0.025	0.003 720 1						

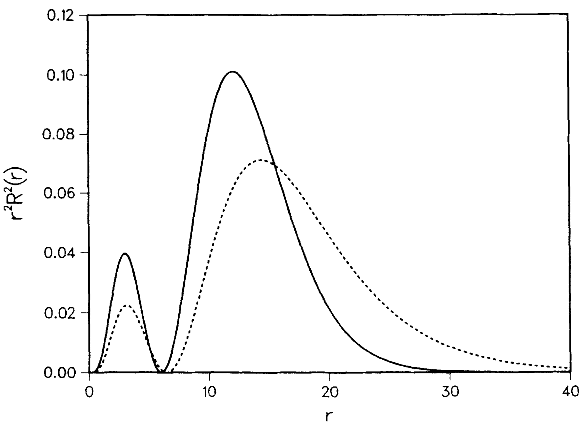

FIG. 1. $r^2R^2(r)$ vs r for the $3p$ state for two values of the screening parameter. The solid-line curve is for $\delta=0.025$ (low screening) and the dashed-line curve is for $\delta=0.15$ (high screening).

sal relation between the reduced energy and the reduced screening parameter for s states. It was of interest to examine whether such a relationship is also valid for $l\neq0$ states. In Fig. 3 we show E/E_0 versus δ/δ_c for p states for $n=2$ to 8, except $n=6$. It will be noted that while there is no universal curve for all n, the results for higher values of n rapidly approach an asymptotic curve.

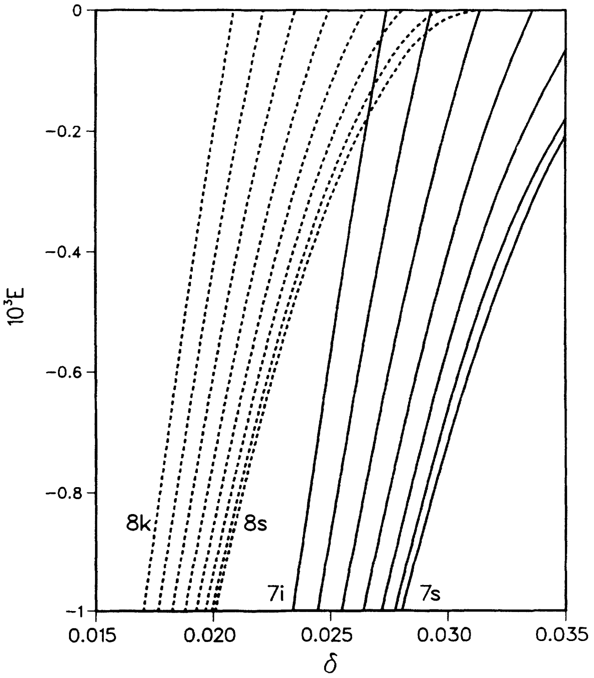

FIG. 2. The energy eigenvalue as a function of δ for various values of l for $n=7$ (solid line curves) and $n=8$ (dashed-line curves) in the vicinity of $E=0$.

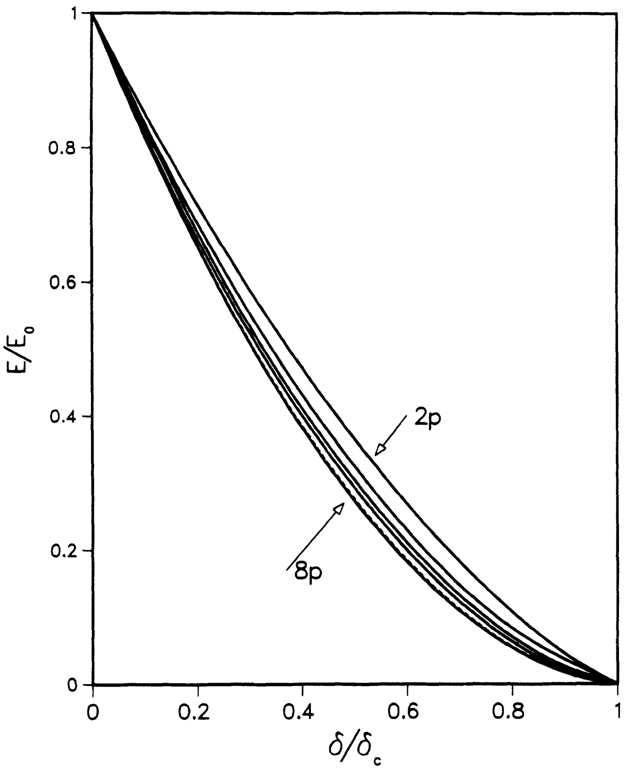

FIG. 3. The ratio E/E_0 as a function of δ/δ_c for $2p$, $3p$, $4p$, $5p$, $7p$, and $8p$ states. To avoid overlapping, $6p$ state has not been shown and $7p$ has been shown by a dashed-line curve.

III. VARIATIONAL CALCULATION

For $l\neq0$ states, Greene and Aldrich[33] have developed a method to generate pseudo-Hulthén wave functions. They replace

$$V_H(r)=-\frac{\delta e^{-\delta r}}{1-e^{-\delta r}}+\frac{l(l+1)}{2r^2} \tag{4}$$

by an effective potential

$$V_H^{\text{eff}}(r)=-\frac{\delta e^{-\delta r}}{1-e^{-\delta r}}+\frac{l(l+1)}{2}\left[\frac{\delta}{1-e^{-\delta r}}\right]^2 e^{-\delta r}\ . \tag{5}$$

For small δr, Eq. (5) is a good approximation to Eq. (4), and it has the advantage that the Schrödinger equation for this potential is solvable analytically. The resulting wave functions can be written in the form

$$\chi(r)=e^{-a\delta r}\sum_{i=0}^{\infty}C_i(1-e^{-\delta r})^{i+l+1}\ , \tag{6}$$

where a and the C_i's depend on the state. If δ is replaced by a variational parameter μ, wave functions of the form (6) can be expected to be a reasonable approximation for $l\neq0$ states for the Hulthén potential. Greene and Aldrich[33] used such wave functions for $l\neq0$ states of the static screened Coulomb potential. Here we shall employ them for the Hulthén potential itself.

The unnormalized wave functions for the various states considered here are as follows:

$$
\begin{aligned}
\chi_{2p}(r)&=e^{-a_2\mu r}S^2\ ,\\
\chi_{3p}(r)&=e^{-a_3\mu r}\left[S^2-\left[\frac{1}{2}+\frac{1}{6\mu}\right]S^3\right]\ ,\\
\chi_{3d}(r)&=e^{-a_3\mu r}S^3\ ,\\
\chi_{4p}(r)&=e^{-a_4\mu r}\left[S^2-\left[1+\frac{1}{4\mu}\right]S^3\ ,\right.\\
&\qquad\left.+\left[\frac{3}{10}+\frac{1}{20\mu}\right]\left[1+\frac{1}{4\mu}\right]S^4\right]\ ,\\
\chi_{4d}(r)&=e^{-a_4\mu r}\left[S^3-\left[\frac{1}{2}+\frac{1}{12\mu}\right]S^4\right]\ ,\\
\chi_{4f}(r)&=e^{-a_4\mu r}S^4\ .
\end{aligned}
\tag{7}
$$

Here

$$S=(1-e^{-\mu r})\ ,$$

$$a_2=\frac{1}{2\mu}-1\ ,$$

$$a_3=\frac{1}{3\mu}-\frac{3}{2}\ ,$$

and

$$a_4=\frac{1}{4\mu}-2\ .$$

The variational energy is given by

$$E=\frac{\int_0^\infty \chi\left[-\frac{1}{2}\frac{d^2}{dr^2}-\frac{\delta e^{-\delta r}}{1-e^{-\delta r}}+\frac{l(l+1)}{2r^2}\right]\chi\, dr}{\int_0^\infty \chi^2 dr} \tag{8}$$

The integration of the second term in the numerator of (8) is, however, analytically not tractable. Hence the second derivative was obtained analytically and the integration was carried out numerically. The energies thus obtained are shown in column 4 of Table I and the best values of μ in Table II. The wave functions for the $3p$ and $4p$ states with the given values of μ are not exactly orthogonal to the wave function of the corresponding lower states, but the deviation is quite small except when δ is close to δ_c.

TABLE II. Best values of the parameters μ and β.

State	δ	μ	β
$2p$	0.025	0.025 004	0.024 979
	0.050	0.050 040	0.050 001
	0.075	0.075 140	0.075 028
	0.10	0.100 337	0.100 092
	0.15	0.151 192	0.150 468
	0.20	0.203 037	0.201 518
	0.25	0.256 595	0.253 946
	0.30	0.313 364	0.309 228
	0.35	0.377 907	0.371 766
$3p$	0.025	0.025 011	
	0.050	0.050 094	
	0.075	0.075 333	
	0.100	0.100 851	
	0.150	0.153 820	
$3d$	0.025	0.025 030	
	0.050	0.050 251	
	0.075	0.075 901	
	0.100	0.102 364	
	0.150	0.162 638	
$4p$	0.025	0.025 021	
	0.050	0.050 180	
	0.075	0.075 726	
	0.100	0.102 906	
$4d$	0.025	0.025 057	
	0.050	0.050 438	
	0.075	0.077 150	
$4f$	0.025	0.025 109	
	0.050	0.050 924	
	0.075	0.079 170	

The difficulty concerning the analytic integration of the second term in the numerator of Eq. (8) can be avoided if we slightly modify the assumed wave function. This is achieved by multiplying the wave functions given in (7) by the factor $(1-e^{-\delta r})/(1-e^{-\mu r})$. This factor is almost equal to 1 and has only a small effect on the shape of the wave function. To examine whether the modified wave function gives satisfactory results, calculations were carried out on the $2p$ state. Thus the unnormalized modified χ is

$$\chi(r)=e^{-a_2\beta r}(1-e^{-\beta r})(1-e^{-\delta r})\ , \tag{9}$$

where β is a variational parameter, similar to μ. The corresponding expression for the energy can be evaluated to be

$$
\begin{aligned}
E=&[(1-2\beta)(1-2\beta+\delta)(1-2\beta+2\delta)(1-\beta)(1-\beta+\delta)(1-\beta+2\delta)(1+\delta)(1+2\delta)]\\
&\times\{4\delta^2\beta^2[4\delta^4+(-18\beta+18)\delta^3+(-66\beta+28\beta^2+33)\delta^2+(-72\beta+66\beta^2+24-18\beta^3)\delta\\
&\quad+33\beta^3-18\beta^3+6-24\beta+4\beta^4]\}^{-1}-(\tfrac{1}{2})\delta^2\beta^2[-4\delta^4+(20\beta-20)\delta^3+(22\beta-20\beta^2+3)\delta^2\\
&\qquad\qquad+(24+8\beta^2+4\beta^3-36\beta)\delta+23\beta^2-28\beta-6\beta^3+10]\\
&\times[(1-\beta)(1-2\beta+\delta)(1-\beta+\delta)(1-\beta+2\delta)(1+\delta)(1-2\beta+2\delta)(1+2\delta)]^{-1}\\
&\quad+(4\beta-2\delta-2)\ln(1-2\beta+\delta)+(2\beta-4\delta-2)\ln(1-\beta+2\delta)-(2\delta+2)\ln(1+\delta)\\
&\quad+(4\delta-4\beta+4)\ln(1-\beta+\delta)+(2\delta-2\beta+1)\ln(1-2\beta+2\delta)+(2\delta+1)\ln(1+2\delta)\\
&\quad+(1-2\beta)\ln(1-2\beta)+(2\beta-2)\ln(1-\beta)\ .
\end{aligned}
\tag{10}
$$

The energies obtained by minimizing this expression with respect to β are shown in the fifth column of Table I, and the corresponding values of β in Table II.

TABLE III. Critical screening parameter for various states of the Hulthén potential.

n	l	δ_c (Numerical)	δ_c [Patil, Eq. (12)]
1	0	2.000 000	2.000 000
2	0	0.500 000	0.500 000
2	1	0.376 936	0.377 367
3	0	0.222 222	0.222 222
3	1	0.186 486	0.186 017
3	2	0.157 662	0.157 990
4	0	0.125 000	0.125 000
4	1	0.110 491	0.109 826
4	2	0.097 564	0.097 256
4	3	0.086 405	0.086 727
5	0	0.080 000	0.080 000
5	1	0.072 863	0.072 275
5	2	0.066 108	0.065 616
5	3	0.059 973	0.059 838
5	4	0.054 505	0.054 790
6	0	0.055 556	0.055 556
6	1	0.051 579	0.051 105
6	2	0.047 661	0.047 169
6	3	0.043 975	0.043 670
6	4	0.040 585	0.040 547
6	5	0.037 504	0.037 747
7	0	0.040 816	0.040 816
7	1	0.038 398	0.038 024
7	2	0.035 948	0.035 508
7	3	0.033 581	0.033 234
7	4	0.031 353	0.031 172
7	5	0.029 284	0.029 296
7	6	0.027 379	0.027 584
8	0	0.031 250	0.031 250
8	1	0.029 681	0.029 385
8	2	0.028 058	0.027 681
8	3	0.026 459	0.026 122
8	4	0.024 926	0.024 691
8	5	0.023 478	0.023 374
8	6	0.022 124	0.022 160
8	7	0.020 864	0.021 038
9	0	0.024 691	0.024 691
9	1	0.023 621	0.023 384
9	2	0.022 497	0.022 179
9	3	0.021 372	0.021 064
9	4	0.020 278	0.020 031
9	5	0.019 230	0.019 072
9	6	0.018 237	0.018 180
9	7	0.017 303	0.017 350
9	8	0.016 427	0.016 575
10	0	0.020 000	0.020 000
10	1	0.019 240	0.019 049
10	2	0.018 433	0.018 165
10	3	0.017 615	0.017 340
10	4	0.016 810	0.016 571
10	5	0.016 030	0.015 851
10	6	0.015 283	0.015 178
10	7	0.014 573	0.014 546
10	8	0.013 902	0.013 953
10	9	0.013 268	0.013 396

IV. CRITICAL SCREENING PARAMETER

For the s states, δ_c is readily obtained from the expression for the energy to be

$$\delta_c = 2/n^2 \,. \tag{11}$$

For $l \neq 0$ states, δ_c was calculated by numerical solution of the Schrödinger equation and the results for $n \leq 10$ are shown in Table III. For the sake of completeness, the results for $l = 0$ are also shown, obtained from Eq. (11). Patil[26] has obtained an approximate expression for the critical strength Z_c. In terms of δ_c, his equation can be written as

$$\delta_c = 1/(n\sqrt{2} + 0.1645 l + 0.0983 l/n)^2 \,. \tag{12}$$

Values of δ_c obtained from this expression are also shown in Table III. It will be noticed that the results obtained from Patil's equation are in good agreement with the exact values—which is rather remarkable for such a simple equation.

The only other calculation of δ_c for the Hulthén potential is that of Popov and Weinberg,[27] who have calculated $n^2\delta_c$ for a few states which have $l = n - 1$ by perturbation series summation. Their values are in good agreement with the exact ones calculated here.

For s states, as Eq. (11) shows, $n^2\delta_c = 2$. For $l \neq 0$ states, we show the product $n^2\delta_c$ as a function of n in Fig. 4. Points for the same l have been joined by lines. It

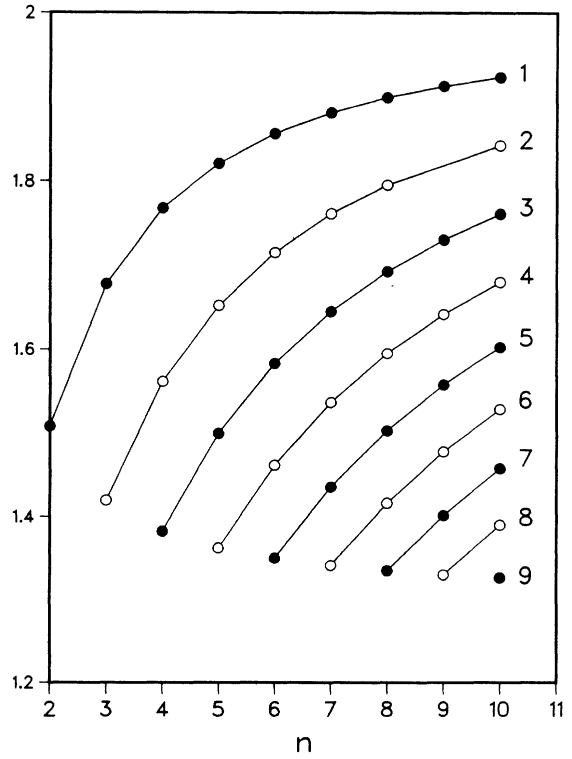

FIG. 4. $n^2\delta_c$ as a function of n. Points for the same l have been joined by lines. Tbe numbers on the right of the curves represent l values.

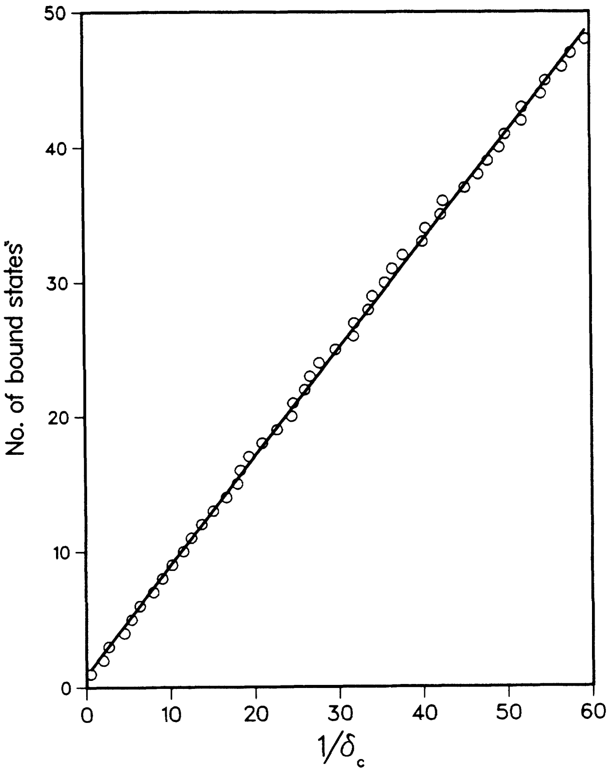

FIG. 5. Number of bound states n^* as a function of $1/\delta_c$. The straight line represents a least-squares fit, Eq. (13).

will be noticed that in each case $n^2\delta_c$ increases with n and appears to approach an asymptotic value close to 2.

In the order of decreasing δ_c values, the level order is as follows: 1s,2s,2p,3s,3p,3d,4s,4p,4d,4f,5s,5p,5d,5f, 6s,5g,6p,6d,6f,7s,6g,7p,6h,7d,7f,7g,8s,8p,7h,8d,7i, 8f, 8g,9s,9p,8h,8i,9f,8k,9g,10s,10p,9h,10d,9i, 10f,. . . . It will be noticed that as one goes to higher levels there is an increasing departure from the ordering of levels in a Coulomb potential. This is a consequence of an increasing degree of crossover of levels, which was discussed in Sec. II.

The total number of states n^* which have negative energy at a given δ has been studied for the static screened Coulomb potential by Rogers *et al.*[38] In Fig. 5 we show n^* as a function of $1/\delta_c$ for the Hulthén potential. The distribution of points indicates a linear relationship and a least-squares treatment yields the following relation:

$$n^*=0.8029/\delta_c+0.9018 . \tag{13}$$

Equation (13) can be used to calculate the number of states which are bound at a given δ, or one can calculate δ_c for a given n^*.

V. COMPARISON OF EIGENENERGIES

In this section we compare the results obtained by the various methods for the eigenenergies of $l\neq0$ levels. Besides the results obtained in this paper, we shall consider the results obtained by four other methods. We briefly describe these four other calculations.

Lai and Lin[24] expanded the Hulthén potential in a series in the form

$$V(r)=-(Z/r)\sum_{k=0}^{\infty} V_k(\delta r)^k . \tag{14}$$

By employing the Hellmann-Feynman theorem and the hypervirial theorems they were able to express the bound-state energies in a power series of the screening parameter δ. To improve the accuracy of the results, especially when δ is close to the critical screening parameter δ_c, they formed Padé approximants to the energy series. The results of Lai and Lin are shown in column 6 of Table I.

Patil's[26] treatment is based on the fact that the Hulthén potential allows a strong-coupling expansion. He uses an interpolation for the bound-state energies which incorporates the required behavior near the critical strength Z_c and also has a strong-coupling expansion in conformity with the first few known terms of the strong-coupling expansion. Patil's values are given in column 7 of Table I.

Roy and Roychoudhury[28] and Tang and Chan[29] have applied the shifted $1/N$ method of Sukhatme and collaborators[39] to calculate energies of $l\neq0$ states of the Hulthén potential. Roy and Roychoudhury[28] have used two terms in their calculations, while Tang and Chan[29] have used three terms. The results of the latter authors are shown in column 8 of Table I.

Matthys and De Meyer[31] have studied the $l\neq0$ states in the context of algebraic perturbation theory using an approach which is based on the Lie algebra of the group SO(2,1). They introduce a so-called tilting transformation that relates between physical states and the group states which constitute a basis of the relevant unitary irreducible representations of SO(2,1). It is shown that all the matrix elements of the energy functional can be expressed in closed form. Then the energy matrix, in which the unknown bound-state energies occur on diagonal as well as on off-diagonal positions, is numerically diagonalized by means of a Gauss-Seidel iteration scheme. The tilting parameter is adjusted in order to accelerate the convergence of the perturbation expansions. Column 9 of Table I shows the eigenenergies obtained by Matthys and De Meyer.[31]

Next we compare the various sets of results. A comparison of the eigenenergies obtained from the wave functions of Eq. (7) with the "exact" values shows that the two are almost the same except in the high-screening region. For the 2p state the values obtained by the modified wave function, Eq. (9), are the same as those obtained from the unmodified one at low screening, but slightly inferior in the high-screening region. Amongst the four sets (columns 6–9 of Table I) of other workers, the results of Matthys and De Meyer[31] are practically identical to the "exact" results. The next best results are those of Lai and Lin,[24] which agree with the exact results to the number of significant figures quoted by these authors, except in 3 cases in the high-screening region. Next in accuracy in the low- and medium-screening regions are the shifted $1/N$ expansion results of Tang and Chan,[29] but no conclusion is possible for the high-screening region as Tang and Chan have not given any results for such values δ. It has been found[40] that the shifted $1/N$ expansion gives poor results for the static screened Coulomb potential in the high-screening region and it is entirely possible that a similar situation may hold for the Hulthén potential.

TABLE IV. Absorption oscillator strengths in atomic units for the Hulthén potential.

Transition	δ	f
1s-2p	0.025	0.4155
	0.050	0.4134
	0.075	0.4098
	0.10	0.4047
	0.15	0.3897
	0.20	0.3671
	0.25	0.3348
	0.30	0.2886
	0.35	0.2169
1s-3p	0.025	0.078 34
	0.050	0.076 00
	0.075	0.071 96
	0.10	0.066 00
	0.15	0.046 33
2p-3d	0.025	0.6916
	0.050	0.6785
	0.075	0.6547
	0.10	0.6161
	0.15	0.4400

VI. OSCILLATOR STRENGTHS

There has been no previous calculation of the oscillator strengths for the Hulthén potential. In this section we obtain results for the absorption oscillator strength for $1s \rightarrow 2p$, $1s \rightarrow 3p$, and $2p \rightarrow 3d$ transitions by using the analytical wave functions of Sec. III. The absorption oscillator strength for the $nl \rightarrow n'l'$ transition is given by[41]

$$f = \frac{2}{3}\frac{\max(l,l')}{(2l+1)}(E_{n'l'} - E_{nl})\left|\int_0^\infty \chi_{nl} r \chi_{n'l'} dr\right|^2 . \qquad (15)$$

The normalization constants for the four required wave functions are as follows:

$$N_{1s} = (4/\delta^2 - 1)^{1/2} \qquad (16a)$$

$$N_{2p} = \left[\frac{(1-\mu^2)(1-4\mu^2)}{24\mu^4}\right]^{1/2} \qquad (16b)$$

$$N_{3p} = \left[\frac{2(4-9\mu^2)(4-81\mu^2)}{2187\mu^4}\right]^{1/2} \qquad (16c)$$

$$N_{3d} = \left[\frac{(1-9\mu_2^2)(4-9\mu_2^2)(4-81\mu_2^2)}{196\,830\mu_2^6}\right]^{1/2} . \qquad (16d)$$

In Eq. (16d), μ_2 has been used for μ to distinguish it from the μ for the $2p$ state in the expression for f for the $2p \rightarrow 3d$ transition. Following Shore and Menzel[42] we represent the transition integral by $I(nl,n'l')$,

$$I(nl,n'l') = \int_0^\infty \chi_{nl} r \chi_{n'l'} dr . \qquad (17)$$

Using wave functions for the $2p$, $3p$, and $3d$ states from Eq. (7) and the exact wave function for the $1s$ state, the transition integrals were evaluated and the results are as follows:

$$\begin{aligned} I(1s,2p) = {} & 768 N_{1s} N_{2p} \mu^2 \delta \\ & \times(-243\delta^2 - 1458\mu^2 - 27\delta^4 + 2187 + 3\delta^6 + 32\mu^4\delta^2 - 32\mu^6 - 18\delta^4\mu^2 + 288\mu^4 + 36\mu^2\delta^2) \\ & \times\{(9-\delta^2)^2[(3-\delta)^2 - 4\mu^2]^2[(3+\delta)^2 - 4\mu^2]^2\}^{-1} , \end{aligned} \qquad (18)$$

$$I(1s,3p) = N_{1s} N_{3p} \left[\frac{p^2 + 9\mu^2/4 + p}{(p^2 - 9\mu^2/4)^2} - \frac{p^2 + \mu^2/4 + p}{(p^2 - \mu^2/4)^2} - \frac{q^2 + 9\mu^2/4 + q}{(q^2 - 9\mu^2/4)^2} + \frac{q^2 + \mu^2/4 + q}{(q^2 - \mu^2/4)^2}\right] , \qquad (19)$$

where

$$p = -\tfrac{4}{3} + \delta/2, \quad q = -\tfrac{4}{3} - \delta/2 ,$$

and

$$\begin{aligned} I(2p,3d) = N_{2p} N_{3d} \Bigg[& -\frac{6p\mu_2}{(p^2 - 9\mu_2^2/4)^2} + \frac{6p\mu_2}{(p^2 - \mu_2^2/4)^2} - \frac{10\mu_2}{(\frac{25}{36} - 9\mu_2^2/4)^2} + \frac{10\mu_2}{(\frac{25}{36} - \mu_2^2/4)^2} \\ & - \frac{6q\mu_2}{(q^2 - 9\mu_2^2/4)^2} + \frac{6q\mu_2}{(q^2 - \mu_2^2/4)^2} \Bigg] , \end{aligned} \qquad (20)$$

where $p = -\frac{5}{6} + \mu$ and $q = -\frac{5}{6} - \mu$. In Eq. (20), μ refers to the $2p$ wave function and μ_2 to the $3d$ function.

The calculated oscillator strengths are shown in Table IV. It will be noticed that for each of the transitions, as δ increases, at first there is a slow decrease, but near δ_c there is a rapid drop in the value of the oscillator strength.

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