Casimir effect for dielectric plates

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A critical review of Casimir forces and their various interpretations is given. The Casimir effect for the case of dielectric slabs is studied in detail. The electromagnetic field is quantized in the presence of dielectric material and the notion of physical photons is introduced. The Casimir force originates from the pressure of the electromagnetic field in the vacuum state on the slab and is calculated with the help of the Maxwell stress tensor. It is shown that the Casimir force depends in a simple way on the mode density of the electromagnetic field. The dependence of the Casimir force on the reflection coefficient and the plate thickness is found.

I. INTRODUCTION: EXAMPLES AND INTERPRETATIONS OF THE CASIMIR EFFECT

What is the Casimir effect? One of several possible answers is that it is an attractive interaction between two neutral perfectly conducting parallel plates placed in the vacuum. The expression for this force was first given in 1948 by Casimir:[1]

$$F_C = \frac{\hbar c \pi^2}{240 a^4} , \qquad (1)$$

where a is the distance between the plates. The effect is very small: for a distance of 1 micrometer ($a = 10^{-6}$ m) it is about 1.3×10^{-3} N/m^2. Nevertheless, the effect was confirmed experimentally.[2]

Speaking more generally, Casimir forces form a class of long-range retarded interactions between particles, atoms, molecules, and macroscopic bodies. Long-range forces between neutral objects with zero dipole moment are known as van der Waals forces. The first explanation of such forces, based on quantum mechanics, was given for two atoms by London in 1930.[3] From second-order perturbation theory with the interaction given by the electrostatic dipole-dipole potential, he got the familiar R^{-6} distance dependence of the van der Waals potential. As it is now well understood, the atom-atom interaction may be explained on the basis of the semiclassical theory by the correlations of the fluctuating atomic dipole moments.

London's expression for the interaction potential is not valid if R is larger than c/ω_0 (ω_0 denotes a typical atomic frequency), as retardation effects connected with the finite velocity of light must be taken into account. This was done in 1948 by Casimir and Polder[4] in fourth-order perturbation theory. For large distances their result for the interaction potential can be written as

$$V = -\frac{23}{4\pi}\frac{\alpha_1\alpha_2}{R^7}\hbar c ,$$

where α_1 and α_2 are the polarizabilities of the atoms.

A simplified derivation of this result has been presented; it goes roughly as follows.[5] The electric field of the vacuum induces in the first atom the dipole moment $\mathbf{d}_1=\alpha_1\mathbf{E}_0(x_1)$. The electric field produced by $\mathbf{d}_1$ at the position of the second atom is $\mathbf{E}_{1\to 2}\propto \ddot{\mathbf{d}}_1/(c^2R)$. The time-averaged interaction of two atoms is thus:

$$V=\tfrac{1}{2}\mathbf{d}_2\mathbf{E}_{1\to 2}\propto \int_0^\infty d\omega\, \alpha_2(\omega)\mathbf{E}_0(\omega,x_2) \times \frac{\omega^2}{c^2R}\alpha_1(\omega)\mathbf{E}_0(\omega,x_1)N(\omega) ,$$

where $N(\omega)$ is the mode density function. For large frequencies (as compared to c/R) the integrand is a rapidly oscillating function, and the contributions from different frequencies cancel. The integral may be roughly evaluated as

$$V\propto \int_0^{c/R}\alpha_1(\omega)\alpha_2(\omega)\frac{\omega^2}{c^2R}|\mathbf{E}_0(\omega)|^2N(\omega)d\omega ,$$

with $|\mathbf{E}_0(\omega)|^2N(\omega)\propto \hbar\omega^3/c^3$ and $\alpha(\omega)\approx$const for $\omega < c/R$ (according to the assumption $c/R \ll \omega_0$), we get

$$V\propto \frac{\hbar}{c^5R}\alpha_1\alpha_2\int_0^{c/R}\omega^5 d\omega \approx \frac{\hbar c\alpha_1\alpha_2}{R^7} .$$

A similar approach may be used to calculate the interaction between any two polarizable bodies.

The derivation presented above suggests that the Casimir force may be interpreted as the direct effect of vacuum fluctuations. This is even better seen in the Casimir derivation of the wall-wall interaction, based on the calculation of the zero-point energy shift of the electromagnetic field in the presence of the plates.[1] In this case the Casimir force also has a simple interpretation as the vacuum pressure, as was recently stressed by Milonni *et al.*[6]

An alternative approach exists in which Casimir potentials arise not from vacuum fluctuations, but rather from the radiation reaction field.[7] It is just the same situation that is met in the calculations of the Lamb shift, where the possible interpretation depends upon the ordering of creation and annihilation operators: if we choose the symmetric ordering, the Lamb shift may be explained by vacuum fluctuations, but with normal ordering all the effect seems to come from the radiation reaction (Ref. 7

and references therein). So, the Lamb shift and the Casimir effect have the same origin, and, furthermore, there are situations where one may be calculated from the other. This is, for example, the case of an atom located near a perfectly conducting wall. The energy shift for such an atom depends upon the distance from the wall, and the force is given by the derivative of this shift with respect to the distance.[7] On the other hand, Power calculated the Lamb shift from the change of the vacuum energy when a small number of hydrogen atoms is put into the quantization box,[8] which is from the Casimir energy.

As a final example of Casimir forces it is worth mentioning that such a force was supposed to play the role of a Poincaré stress which stabilizes the electron.[9] However, calculations for the model of electron as a perfectly conducting charged shell gave a positive value of the Casimir energy, so the force was repulsive rather than attractive.[10] This showed that Casimir forces are in a sense a more general notion than the "usual" van der Waals attraction forces.

It should be also noticed that the concept of Casimir energy refers not only to the electromagnetic interactions. It was applied among other areas in the quantum-chromodynamics (QCD) bag model and in gravitation theory (Ref. 11 and references therein).

II. THE CONCEPT OF CASIMIR ENERGY

In the preceding section we presented various interpretations of the Casimir effect. However, the most popular one is based upon the electromagnetic vacuum fluctuations and the shift of the zero-point energy. In this section we will review some problems connected with this approach. As an example we will present the derivation of the Casimir energy for the case of metallic plates.

While calculating the vacuum energy, one faces the problem of infinities and the fundamental question—if such divergent quantities connected with the vacuum have any physical meaning. Casimir's concept for energy renormalization is as follows. In any real situation we do not deal with free space, but there are always certain boundaries present, for example conducting walls, some material objects, or fields in general. Then, the physical vacuum energy (Casimir energy) should be understood as the difference between the zero-point energy in the presence of boundaries and that of the free vacuum:

$$E_{\rm phys}=E_C=\langle\Omega|H|\Omega\rangle_{\rm bound}-\langle\Omega|H|\Omega\rangle_{\rm free}\ . \qquad (2)$$

Any change of boundaries is connected with a change of the Casimir energy, which is a measurable quantity—as for example the force between two conducting plates.

There are two methods of calculating the Casimir energy: (i) the direct mode summation, with the proper schemes of regularization (that is just what Casimir has done), and (ii) calculation of the vacuum energy-momentum tensor $T^{\mu\nu}_{\rm va}(\mathbf{r})$ from the difference of photon propagators for free space and that with boundaries. This local formulation was introduced in 1969 by Brown and Macley.[12]

In the case of perfectly conducting boundaries both methods give the same result, that is the quantities

$$E^{\rm mod}_{\rm ren}=\left[\int d^3r T^{00}(\mathbf{r})\right]_{\rm ren}$$

and

$$E_{\rm loc}=\int d^3r\,[T^{00}(\mathbf{r})_{\rm ren}]$$

are equal [the subscript ren refers to the procedure of renormalization given by (2)]. However, as pointed out by Deutsch and Candelas,[13] the above identity need not be the rule.

It may seem that in order to get the nonzero vacuum energy the normal ordering cannot be used. In fact, it is so for the free vacuum, but not for the physical vacuum. In the latter case it would probably be better not to speak about the "vacuum," but about the ground state of the combined system: electromagnetic field plus boundaries, as when the interaction is switched on, all the photonic states combine into the resulting ground state. The physical vacuum energy, given by formula (2), may be understood as the expectation value of the normally ordered (with respect to the free-space photon creation and annihilation operators) Hamiltonian in the ground state:

$$E_{\rm phys}=\langle g|{:}H(a,a^\dagger){:}|g\rangle=\langle g|H|g\rangle-\langle\Omega|H|\Omega\rangle\ .$$

However, this last definition is not convenient, as the ground state is expressed by free-space photonic states and the Hamiltonian by free-space operators. In most practical situations it is more natural to introduce "physical photons" and creation and annihilation operators for these photons. We will do this for calculating the Casimir force between dielectric plates, but first we will show the derivation for metallic plates.

As we aim only at presentation of the physical idea we will use a simplified, one-dimensional configuration (Fig. 1), following Pleunien *et al.*[11] To calculate the Casimir energy it is convenient to introduce the spatial cutoff. The Casimir energy is

$$E_C=\frac{2L}{\pi}\int_0^\infty dk\,\hbar\omega+\sum_{k=0}^{\infty}\hbar\omega-\frac{2L+a}{\pi}\int_0^\infty dk\,\hbar\omega\ , \qquad (3)$$

where $\omega=kc$ and the summation runs over $k=l\pi/a$, $(l=0,1,2,\ldots)$. The first term is the sum of the zero-point energies in regions I and III, where for large L the sum over modes can be replaced by an integral. The second term gives the energy in region II and the last term is the zero-point energy in the absence of the plates.

As it stands the expression for the Casimir energy is still divergent due to the infinite summation over k. However, in the real physical situation the walls are

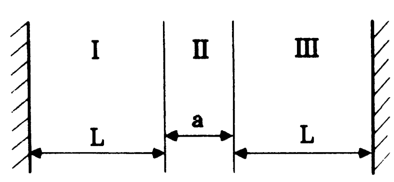

FIG. 1. Position of metallic plates and boundaries.

transparent for waves with high frequencies, that is to say the modes with frequencies $\omega > \omega_c$ are not altered by the boundaries and their contribution to the Casimir energy cancels. According to that, the regularization procedure may consist in the introduction of the cutoff function $\chi(\omega)=1$ for $\omega \lesssim \omega_c$ and $\chi(\omega)=0$ for $\omega >> \omega_c$. Introducing such a function in the expression (3) one gets

$$E_C=\sum_k \hbar\omega\,\chi(\omega)-\frac{a}{\pi}\int dk\,\omega\,\chi(\omega)$$
$$=\frac{\hbar c\pi}{a}\left[\sum l\,\chi(l\pi c/a)-\int dl\,l\,\chi(l\pi c/a)\right]. \quad (4)$$

The simplest choice for the function $\chi(\omega)$ is the exponential one: $\chi(\omega)=\exp(-\lambda\omega/c)$. It leads to

$$\int_0^\infty dl\,l\exp(-\lambda l\pi/a)=(\lambda\pi/a)^{-2}$$

and

$$\sum_{l=1}^{\infty} l\exp(-\lambda l\pi/a)=-\frac{a}{\pi}\frac{\partial}{\partial\lambda}\sum_{l=1}^{\infty}\exp(-\lambda l\pi/a)$$
$$=\frac{\exp(-\lambda\pi/a)}{[1-\exp(-\lambda\pi/a)]^2}$$
$$\underset{\lambda\to 0}{\sim}(\lambda\pi/a)^{-2}-1/12+O(\lambda^2).$$

In the limit $\lambda\to 0$ we get the final result:

$$E_C=-\frac{\hbar c\pi}{12a}$$

which leads to the attractive force $F_C=\hbar c\pi/12a^2$.

As previously mentioned, for the case of macroscopic bodies the Casimir force may be regarded as the pressure of the electromagnetic vacuum. We consider this approach in more detail later, for dielectric plates.

III. CASIMIR EFFECT FOR DIELECTRICS

Finally, we have arrived at the problem of the Casimir effect for the case when dielectrics are present. The literature on this subject comprises over a dozen articles, most of them concerning semi-infinite walls or spheres (Refs. 14–19 and references therein). However, they give no common, generally approved solution. The first calculation of the force between semi-infinite dielectric walls was given by Lifschitz.[14] He based his result on the theory of the fluctuating electromagnetic field, where the dielectric polarization of the medium was a stochastic variable. However, several authors stated that Lifshitz's results are not correct in the case of finite temperature, in the limit $\epsilon\to\infty$.[15] Schwinger *et al.* gave their own calculations based on the source theory.[15,16] Later their results were questioned by Candelas,[17] who pointed out several mistakes in Schwinger's and Milton's calculations and, furthermore, some contradictions in their different papers. The basic disagreement refers to the proper form of the energy density. Instead of the incorrect expression

$$e(x)=\int d\omega\frac{1}{4\pi}[\epsilon(\omega)\langle E^2(x)\rangle_\omega+\mu(\omega)\langle H^2(x)\rangle_\omega]$$

used by Milton, Candelas derived the expression

$$e(x)=\int d\omega\frac{1}{4\pi}\left[\frac{d}{d\omega}[\omega\,\epsilon(\omega)]\langle E^2(x)\rangle_\omega+\frac{d}{d\omega}[\omega\,\mu(\omega)]\langle H^2(x)\rangle_\omega\right].$$

The second disagreement concerns the cutoff dependent, infinite terms in the energy. One concept is to disregard such terms as unphysical, the other is to choose the proper cutoff.

The Casimir forces for the temperature $T=0$ are of purely quantum origin which suggests that one look for their source in the theory of quantum electrodynamics in the presence of dielectrics. The microscopic approach was applied by Renne,[18] who derived the formula for the retarded van der Waals potential in the system of atoms represented by harmonic oscillators, interacting with the electromagnetic field, from the zero-point energy of the coupled system.

We have briefly reviewed various approaches to the Casimir force in case of dielectric. In the following part we give our own interpretation of the effect. In our formulation the Casimir effect originates from the vacuum pressure. As opposed to previous treatments, we introduce physical photons which are the basis of the electromagnetic field quantization in the presence of dielectrics. As we will show the physical photons are a very convenient and natural tool for studying various quantum effects, with the presence of macroscopic dielectric bodies taken into account in the most natural way. The Casimir force is derived from simple arguments regarding the Maxwell stress tensor.

We will study the dependence of the Casimir force not only upon the distance between plates, but also upon the value of the refraction index and the thickness of the plates.

IV. QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN THE PRESENCE OF DIELECTRICS

We will assume that the electromagnetic field in the presence of a dielectric medium may be described phenomenologically, by introducing the dielectric constant ϵ. The Maxwell equations without sources take the following form:

$$\text{curl}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=0,$$
$$\text{curl}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0,$$
$$\text{div}\mathbf{B}=0,$$
$$\text{div}\mathbf{D}=0.$$

We assume the material relations to be linear: $\mathbf{D}=\epsilon(\mathbf{r})\mathbf{E}$ and $\mathbf{B}=\mathbf{H}$. We may introduce vector and scalar potentials

$$\mathbf{B}=\text{curl}\,\mathbf{A},\quad \mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\text{grad}\phi$$

and choose the gauge as $\phi=0$, $\mathrm{div}[\epsilon(\mathbf{r})\mathbf{A}]=0$. With such potentials and the gauge, the last three Maxwell equations are automatically fulfilled. The equation of motion for the vector potential is

$$\mathrm{curl\,curl}\,\mathbf{A}+\frac{\epsilon(\mathbf{r})}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=\mathbf{0}\ . \qquad (5)$$

If ϵ is constant in the region of the dielectric, then the potential $\mathbf{A}$ satisfies the wave equation:

$$-\nabla^2\mathbf{A}+\frac{\epsilon(\mathbf{r})}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=\mathbf{0}\ .$$

Boundary conditions are such that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous at the interface of the two media.

The Lagrangian which implies the equation of motion (5) is

$$L=\frac{1}{8\pi}\int d^3r\left[\frac{\epsilon(\mathbf{r})}{c^2}\dot{\mathbf{A}}^2-(\mathrm{curl}\,\mathbf{A})^2\right]$$
$$=\frac{1}{8\pi}\int d^3r(\epsilon\mathbf{E}^2-\mathbf{B}^2)\ .$$

From that we get the generalized momentum as the following:

$$\frac{\partial\mathcal{L}}{\delta\dot{\mathbf{A}}}=\frac{\epsilon(\mathbf{r})\dot{\mathbf{A}}}{4\pi c^2}=-\frac{\mathbf{D}}{4\pi c}\ .$$

Then the Hamiltonian reads

$$H=\int d^3r\frac{\partial\mathcal{L}}{\delta\dot{\mathbf{A}}}\cdot\dot{\mathbf{A}}-L=\frac{1}{8\pi}\int d^3r\left[\frac{\epsilon(\mathbf{r})}{c^2}\dot{\mathbf{A}}^2+(\mathrm{curl}\,\mathbf{A})^2\right]$$
$$=\frac{1}{8\pi}\int d^3r\left[\frac{\mathbf{D}^2}{\epsilon}+\mathbf{B}^2\right]\ .$$

The solution of equation (5) may be presented in the following form:

$$\mathbf{A}(\mathbf{r},t)=\sum_\mu c\left[\frac{2\pi\hbar}{\omega_\mu}\right]^{1/2}[\beta_\mu(t)\mathbf{f}_\mu(\mathbf{r})+\beta^*_\mu(t)\mathbf{f}^*_\mu(\mathbf{r})]\ ,$$

where the factor before the brackets was added for the convenience of future quantization, and the parameter μ may take discrete or continuous values. The amplitudes $\beta_\mu(t)$ and the mode functions $\mathbf{f}_\mu(\mathbf{r})$ obey the equations

$$\frac{\partial^2\beta_\mu(t)}{\partial t^2}+\omega^2_\mu\beta_\mu(t)=0$$

and (6)

$$\mathrm{curl\,curl}\,\mathbf{f}_\mu(\mathbf{r})-\frac{\omega^2_\mu\epsilon(\mathbf{r})}{c^2}\mathbf{f}_\mu(\mathbf{r})=\mathbf{0}$$

together with the appropriate boundary conditions and the gauge condition

$$\mathrm{div}[\epsilon(\mathbf{r})\mathbf{f}_\mu(\mathbf{r})]=0\ . \qquad (7)$$

Solutions of Eq. (6) form a complete set of functions in the space of functions satisfying the condition (7). They may be chosen to fulfill the orthonormality conditions:

$$\int d^3r\,\epsilon(\mathbf{r})\mathbf{f}_\mu(\mathbf{r})\cdot\mathbf{f}_{\mu'}(\mathbf{r})=\delta_{\mu\mu'}$$

and

$$\int d^3r\,\mathrm{curl}\mathbf{f}_\mu(\mathbf{r})\cdot\mathrm{curl}\mathbf{f}_{\mu'}(\mathbf{r})=\frac{\omega^2_\mu}{c^2}\delta_{\mu\mu'}\ .$$

The Hamiltonian when expressed in terms of β_μ takes the form:

$$H=\tfrac{1}{2}\sum_\mu(\beta_\mu\beta^*_\mu+\beta^*_\mu\beta_\mu)\hbar\omega_\mu\ .$$

The quantization will be achieved in the same way as in the case of free space. The classical amplitudes β_μ and β^*_μ are replaced by the operators b_μ and $b^\dagger_\mu$, which are postulated to fulfill the commutation relations:

$$[b_\mu,b^\dagger_{\mu'}]=\delta_{\mu\mu'}\ ,$$
$$[b_\mu,b_{\mu'}]=0\ .$$

The time evolution of the creation and the annihilation operators b_μ and $b^\dagger_\mu$ is

$$\dot{b}_\mu=\frac{1}{i\hbar}[b_\mu,H]=-i\omega_\mu b_\mu\ .$$

The vector potential also becomes an operator and takes the form:

$$\mathbf{A}(\mathbf{r},t)=\sum_\mu c\left[\frac{2\pi\hbar}{\omega_\mu}\right]^{1/2}[b_\mu(t)\mathbf{f}_\mu(\mathbf{r})+b^\dagger_\mu(t)\mathbf{f}^*_\mu(\mathbf{r})]\ .$$

The operators b_μ and $b^\dagger_\mu$ act in the Fock space, whose construction is again analogous to that for the free space. Of course the photon created by $b^\dagger_\mu$ is different from the photon created by the free-space operator $a^\dagger_\mu$. These new photons ("physical photons") are associated not with plane waves, but with the mode functions $\mathbf{f}_\mu(\mathbf{r})$. The "physical vacuum" is given by $b_\mu|\Omega_{\mathrm{ph}}\rangle=0$.

It should be stressed that the spatial dependence of the field, described by the mode functions $\mathbf{f}_\mu(\mathbf{r})$, is not derived from quantum mechanics, but from the classical equation (6). The electric polarization of the medium $\mathbf{P}$ does not enter as an additional variable but is assumed to depend on the electric field according to $\mathbf{P}=[(\epsilon-1)/4\pi]\mathbf{E}$. Physically it means that the dielectric is a passive agent, without its own dynamics.

Let us apply the above considerations to the configuration with two parallel infinite dielectric plates each of thickness d, a distance a apart, in free space. The refractive index is assumed to be constant and equal to n. As in the case of perfectly conducting plates, we will confine ourselves to the one-dimensional configuration, that is, to modes with the wave vector perpendicular to the plate's surface (Fig. 2). Then the vector potential may be written as

$$\mathbf{A}(x,t)=\sum_\lambda\left[\int_0^\infty+\int_{-\infty}^0\right]dk\frac{1}{2\pi}c\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}$$
$$\times\mathbf{e}_\lambda[b_{k\lambda}e^{-i\omega_k t}f_k(x)+b^\dagger_{k\lambda}e^{i\omega_k t}f^*_k(x)]\ . \qquad (8)$$

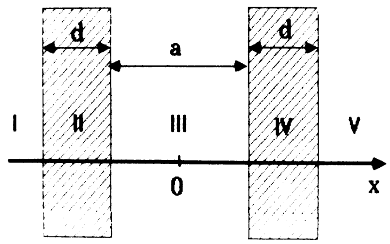

FIG. 2. Configuration of dielectric plates.

The first integral comprises the waves going from left to right, and the second one the waves going from right to left. The parameter $\lambda=1,2$ labels the two possible polarizations. The mode functions satisfying the equation of motion

$$\frac{d^2}{dx^2}f_k(x)-\frac{\omega_k^2\epsilon(x)}{c^2}f_k(x)=0$$

with appropriate boundary conditions, and normalized according to

$$\int dx\,\epsilon(x)f_k(x)f_{k'}(x)=2\pi\,\delta(k-k')$$

are given in Table I. The coefficients R_k, T_k, A_k, B_k, C_k, D_k, E_k, and F_k are

$$R_k=\left[r+\frac{t^2re^{2ika}}{1-r^2e^{2ika}}\right]e^{-ik(a+2d)}\,,$$

$$A_k=\frac{t(1+rr_de^{2ika})}{t_d(1-r^2e^{2ika})}e^{-ik[(a/2)+d]}e^{ikna/2}\,,$$

$$B_k=\frac{t(r_d+re^{2ika})}{t_d(1-r^2e^{2ika})}e^{-ik[(a/2)+d]}e^{-ikna/2}\,,$$

$$C_k=\frac{te^{-ikd}}{1-r^2e^{2ika}}\,,$$

$$D_k=\frac{rte^{ik(a-d)}}{1-r^2e^{2ika}}\,, \tag{9}$$

$$E_k=\frac{t}{(1-r^2e^{2ika})t_d}e^{ik[(a/2)-d]}e^{-ikn[(a/2)+d]}\,,$$

$$F_k=\frac{t^2r_d}{(1-r^2e^{2ika})t_d}e^{ik[(a/2)-d]}e^{ikn[(a/2)+d]}\,,$$

$$T_k=\frac{t^2}{1-r^2e^{2ika}}e^{-2ikd}\,,$$

with r and t the reflection and transmission coefficients for one plate

$$r=\frac{r_d(e^{2iknd}-1)}{1-r_d^2e^{2iknd}}\,,\quad t=\frac{4n}{(1+n)^2}\frac{e^{iknd}}{1-r_d^2e^{2iknd}}\,,$$

$$r_d=\frac{n-1}{n+1}\,,\quad t_d=\frac{2n}{n+1}\,.$$

The expressions for the electric and magnetic field operators follow from (8) on using $\mathbf{E}=-(1/c)(\partial\mathbf{A}/\partial t)$ and $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$. The Hamiltonian may be written as

TABLE I. Mode functions $f_k(x)$.

$k>0$	$k<0$	
$e^{ikx}+R_ke^{-ikx}$	$T_k^*e^{ikx}$	$x\in\left[-\infty,-\frac{a}{2}-d\right]$
$A_ke^{iknx}+B_ke^{-iknx}$	$E_k^*e^{iknx}+F_k^*e^{-iknx}$	$x\in\left[-\frac{a}{2}-d,-\frac{a}{2}\right]$
$C_ke^{ikx}+D_ke^{ikx}$	$C_k^*e^{ikx}+D_k^*e^{-ikx}$	$x\in\left[-\frac{a}{2},+\frac{a}{2}\right]$
$E_ke^{iknx}+F_ke^{-iknx}$	$A_k^*e^{iknx}+B_k^*e^{-iknx}$	$x\in\left[+\frac{a}{2},+\frac{a}{2}+d\right]$
T_ke^{ikx}	$e^{ikx}+R_k^*e^{-ikx}$	$x\in\left[+\frac{a}{2}+d,+\infty\right]$

$$H=\tfrac{1}{2}\sum_\lambda\int_{-\infty}^{\infty}dk\frac{1}{2\pi}(b_{k\lambda}b_{k\lambda}^\dagger+b_{k\lambda}^\dagger b_{k\lambda})\hbar\omega_k\,.$$

The commutation relations are

$$[b_{k\lambda},b_{k'\lambda'}^\dagger]=2\pi\,\delta(k-k')\delta_{\lambda\lambda'}\,,$$

$$[b_{k\lambda},b_{k'\lambda'}]=0\,.$$

V. CLASSICAL EXPRESSION FOR THE LIGHT PRESSURE

With the explicit form of the mode functions given, we are almost ready to calculate the Casimir force. As we are going to derive it as the effect of the vacuum pressure, let us look for a moment at the problem of the radiation pressure in classical electrodynamics.

The first results connected with the problem of the mechanical effect of light are due to Maxwell who predicted in 1873 that when light impinges on a wall it exerts a pressure proportional to the energy density of the wave and to the quantity $1+R$, where R is the reflection coefficient. Maxwell also gave the conservation law for the system composed of the electromagnetic field and free particles:

$$\mathrm{div}T+\frac{\partial\mathbf{S}}{\partial t}=-(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})\,,$$

where $\mathbf{S}=(\mathbf{E}\times\mathbf{B}/4\pi c)$ is the momentum density of the electromagnetic field, and

$$T^{ij}=-\frac{1}{8\pi}[2E^iE^j+2B^iB^j-(\mathbf{E}^2+\mathbf{B}^2)\delta_{ij}]$$

is the Maxwell stress tensor, whose components give the flow of the momentum density.

The correct form of the analogous law for the more general case when material bodies are present was for a long time controversial. For example the momentum density vector given by the T^{j0} components of the

Minkowski's energy-momentum density tensor is proportional to $\mathbf{D}\times\mathbf{B}$, while in Abraham's tensor the same role is played by $\mathbf{E}\times\mathbf{H}$. The discussion of such controversies together with a very careful derivation of the expression for the force density in the material body in the presence of electromagnetic field may be found in Refs. 20 and 21. Here we need not go into detail as we are not interested in the force densities inside the plates, but in the overall force acting on it. All we need is the general law for the closed system:

$$\operatorname{div}T+\frac{\partial\mathbf{S}}{\partial t}=\mathbf{0}\ ,$$

where T is the properly defined total stress tensor, and $\mathbf{S}$ is the total momentum density, composed of the field and material momentum densities. In the stationary case where the momentum of the electromagnetic field inside the plates does not change, the force acting on the body will be given by

$$F^i=-\int T^{ij}n^j dS\ ,$$

where the integration is taken over the surrounding surface and may be calculated from the value of the stress tensor outside the dielectric. In our case of two parallel plates the pressure exerted by the light will be the difference between the T_{zz} components of the stress tensor outside the plates and that between the plates, calculated on the surface:

$$F=T_{zz}^{(\mathrm{I})}[x=-(a/2)-d]-T_{zz}^{(\mathrm{III})}(x=-d)\ . \qquad (10)$$

For the one-dimensional configuration, the value of the T^{zz} component of the stress tensor is equal to the energy density:

$$T_{zz}(x)=T_{00}(x)=\frac{1}{8\pi}[\mathbf{E}^2(x)+\mathbf{B}^2(x)]\ .$$

VI. CASIMIR FORCE FOR DIELECTRIC PLATES

The expression for the Casimir force will be given if we replace classical quantities by quantum operators and evaluate their value for the vacuum state. Of course the vacuum value of T_{zz} is divergent, however, the difference (10), after a proper regularization, turns out to be finite.

This approach yields an intuitive, very simple explanation of the attractive force between metallic plates.[6] The modes of the electromagnetic field are discrete between the plates and form a continuum outside the plates. The energy density is then bigger outside, and the plates are pushed together.

In the one-dimensional case of dielectric plates the situation is somehow different, as there are no discrete modes. However, the energy is not distributed uniformly. As an example let us calculate the energy density in region I (it is of course equal to that in region V). From (8) we get

$$\begin{aligned}
\mathbf{E}(x)=&\sum_\lambda\int_0^\infty dk\frac{1}{2\pi}\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_\lambda[b_{k\lambda}i\omega_k e^{-i\omega_k t}(e^{ikx}+R_k e^{-ikx})+\mathrm{H.c.}]\\
&+\sum_\lambda\int_{-\infty}^0 dk\frac{1}{2\pi}\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_\lambda(b_{k\lambda}i\omega_k e^{-i\omega_k t}T_k^* e^{-ikx}+\mathrm{H.c.})\ ,\\
\mathbf{B}(x)=&\int_0^\infty dk\frac{1}{2\pi}c\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_2[b_{k1}e^{-i\omega_k t}(ike^{ikx}-ikR_k e^{-ikx})+\mathrm{H.c.}]\\
&+\int_0^\infty dk\frac{1}{2\pi}c\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_1[b_{k2}e^{-i\omega_k t}(-ike^{ikx}+ikR_k e^{-ikx})+\mathrm{H.c.}]\\
&+\int_{-\infty}^0 dk\frac{1}{2\pi}c\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_2(b_{k1}e^{-i\omega_k t}ikT_k^* e^{ikx}+\mathrm{H.c.})\\
&+\int_{-\infty}^0 dk\frac{1}{2\pi}c\left[\frac{2\pi\hbar}{\omega_k}\right]^{1/2}\mathbf{e}_1[b_{k2}e^{-i\omega_k t}(-ik)T_k^* e^{ikx}+\mathrm{H.c.}]\ ,
\end{aligned}$$

where we have chosen vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to point, respectively, in the y and z directions.

The time-averaged vacuum value of the energy density outside the plates will be

$$\langle\Omega|T^{00}|\Omega\rangle_{\mathrm{I}}=\tfrac{1}{2}\sum_\lambda\int_0^\infty dk\frac{1}{2\pi}(1+|R_k|^2)\hbar\omega_k+\tfrac{1}{2}\sum_\lambda\int_{-\infty}^0 dk\frac{1}{2\pi}|T_k^*|^2\hbar\omega_k\ .$$

Taking into account that $T_k^*=T_{-k}$ and $|R_k|^2+|T_k|^2=1$, we can transform the above into

$$\begin{aligned}
\langle\Omega|T^{00}|\Omega\rangle_{\mathrm{I}}&=\tfrac{1}{2}\sum_\lambda\int_0^\infty dk\frac{1}{2\pi}\hbar\omega_k(1+|R_k|^2+|T_k|^2)\\
&=2\hbar c\int_0^\infty dk\frac{1}{2\pi}k\ .
\end{aligned}$$

Similarly for the region between the plates

$$\langle\Omega|T^{00}|\Omega\rangle_{\rm III}=\tfrac{1}{2}\sum_\lambda\int_0^\infty dk\frac{1}{2\pi}\hbar\omega_k(|C_k|^2+|D_k|^2)$$
$$+\tfrac{1}{2}\sum_\lambda\int_{-\infty}^0 dk\frac{1}{2\pi}\hbar\omega_k(|C_k^*|^2+|D_k^*|^2)$$
$$=2\hbar c\int_0^\infty dk\frac{1}{2\pi}k(|C_k|^2+|D_k|^2)\ .$$

Finally we get the expression for the Casimir force in the form:

$$F_C=\frac{\hbar c}{\pi}\int_0^\infty dk[1-(|C_k|^2+|D_k|^2)]k$$

or when the explicit form of the coefficients C_k and D_k is used

$$F_C=\frac{\hbar c}{\pi}\int_0^\infty dk\left[1-\frac{1-|r|^4}{|1-r^2e^{2ika}|^2}\right]k\ . \tag{11}$$

As it stands, the expression is infinite. We will regularize it, as in the case of metallic plates, by means of an exponential cutoff function.

The expression in the brackets is a rapidly oscillating function of k and is determined by two characteristic frequencies: one connected with the resonances of the coefficient r and determined by the effective thickness nd, the other connected with the resonances inside the cavity formed by the two plates and determined by the distance a (Fig. 3). If $d\gg a$ (to get a measurable effect the plates should be sufficiently thick or their separation sufficiently small) the oscillations of r are much faster, and we may replace r by its mean value. Now we get, after a change of variable,

$$F_C=\frac{\hbar c}{4\pi a^2}\int_0^\infty du\, f(u)ue^{-\lambda u}\ ,$$

where

$$f(u)=1-\frac{1-r_d^4}{|1-r_d^2e^{iu}|^2}=1-\frac{1-r_d^4}{1+r_d^4-2r_d^2\cos u}\ .$$

Depending on the value of r_d, the function $f(u)$ is composed of broader or narrower resonances (Fig. 4) and gives rise to the repulsive force for u near a resonance and the attractive force outside a resonance. Let us evaluate the Casimir force for $r_d\to1$. The second term in the brackets is nonzero only for $u_n=2\pi n$, $n=0,1,2,\ldots$. For u near resonances we may write

$$\lim_{r_d\to1}\frac{1-r_d^4}{1+r_d^4-2r_d^2\cos u}\approx\lim_{r_d\to1}\frac{1-r_d^4}{1+r_d^4-2r_d^2[1-\frac{1}{2}(u-u_n)^2]}$$
$$=\lim_{r_d\to1}\frac{1+r_d^2}{r_d}\frac{(1-r_d^2)/r_d}{\dfrac{(1-r_d^2)^2}{r_d^2}+(u-u_n)^2}=2\pi\delta(u-u_n)\ .$$

In this limit the Casimir force reads

$$F_C=\frac{\hbar c}{4\pi a^2}\int_0^\infty du\left[1-2\pi\sum_{n=0}^\infty\delta(u-u_n)\right]ue^{-\lambda u}$$
$$=\frac{\hbar c}{4a^2}\left[\frac{1}{\pi}\int_0^\infty du\, ue^{-\lambda u}-2\sum_{n=0}^\infty u_ne^{-\lambda un}\right]$$
$$=\frac{\hbar c\pi}{a^2}\left[\int_0^\infty dn\, ne^{-\lambda n}-\sum_{n=0}^\infty ne^{-\lambda n}\right]\ .$$

The structure of that expression is just the same as for metallic plates. Using the results of earlier calculations, at once we get

$$F_C=\frac{\hbar c\pi}{12a^2}\ .$$

Having checked that the expression (11) gives a correct value for $r_d\to1$, let us calculate how the Casimir force depends on r_d. The function $f(u)$ is periodic and symmetric so it may be expressed as

$$f(u)=\sum_{m=-\infty}^\infty a_me^{-imu}=a_0+2\sum_{m=1}^\infty a_m\cos(mu)\ ,$$

where

$$a_m=a_{-m}=\frac{1}{2\pi}\int_0^{2\pi}f(u)e^{imu}du$$
$$=\frac{1}{2\pi}\int_0^{2\pi}e^{imu}du-\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r_d^4)e^{imu}}{(1-r_d^2e^{iu})(1-r_d^2e^{-iu})}du\ .$$

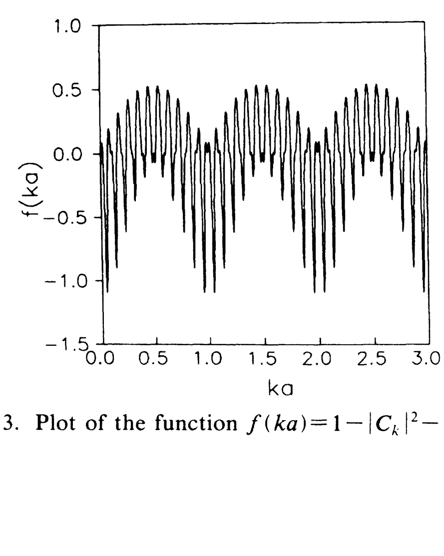

FIG. 3. Plot of the function $f(ka)=1-|C_k|^2-|D_k|^2$, $n=2$, $d/a=5$.

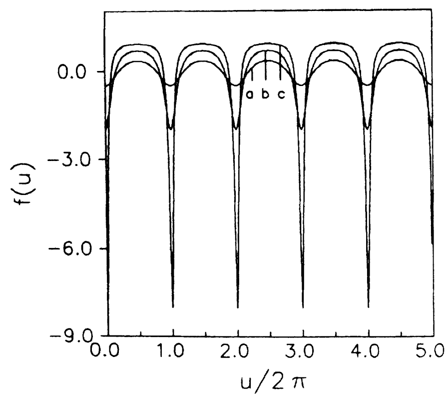

FIG. 4. Plot of the function $f(u)$: (a) $r_d^2=0.2$, (b) $r_d^2=0.5$, and (c) $r_d^2=0.8$.

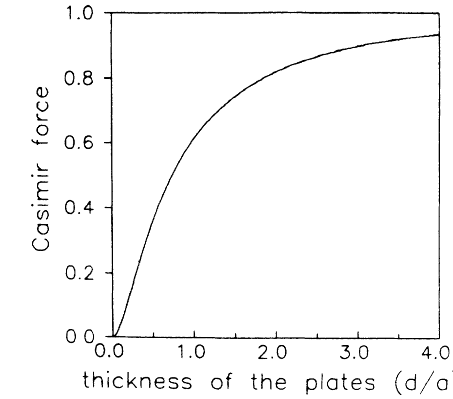

FIG. 6. Dependence of the Casimir force (in units $\hbar c r_d^2/2\pi a^2$) on the thickness of the plates.

The first integral is equal $\delta_{m,0}$. The second may be easily calculated putting $z=e^{iu}$ and integrating in the complex plane over the unit circle. One thereby obtains

$$a_m=\delta_{m0}+\frac{1}{2\pi i}\frac{1-r_d^4}{r_d^2}\oint dz\frac{z^m}{(z-r_d^{-2})(z-r_d^2)}a_m$$
$$=\delta_{m0}-r_d^{2m}\ .$$

Finally $f(u)$ may be written as

$$f(u)=-2\sum_{m=1}^{\infty} r_d^{2m}\cos(mu)\ .$$

Now the Casimir force takes the form

$$F_C=-\frac{\hbar c}{2\pi a^2}\sum_{m=1}^{\infty} r_d^{2m}\int_0^{\infty} u\cos(mu)e^{-\lambda u}du$$
$$=-\frac{\hbar c}{2\pi a^2}\sum_{m=1}^{\infty} r_d^{2m}\frac{\lambda^2-m^2}{(m^2+\lambda^2)^2}\ .$$

In the limit $\lambda\to 0$ we get

$$F_C=\frac{\hbar c}{2\pi a^2}\sum_{m=1}^{\infty}\frac{r_d^{2m}}{m^2}\ . \tag{12}$$

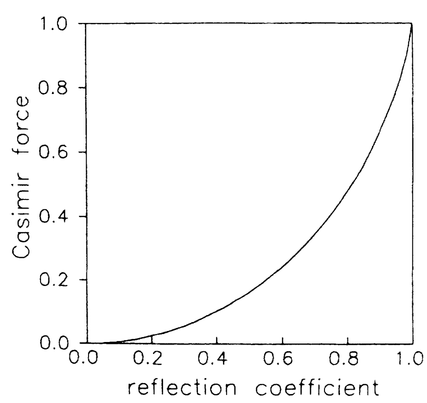

FIG. 5. Dependence of the Casimir force (in units $\hbar c/2\pi a^2$) on the reflection coefficient.

For $r_d=1$ it gives the familiar result

$$F_C=\frac{\hbar c}{2\pi a^2}\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\hbar c\pi}{12a^2}\ .$$

For $r_d<1$ the formula (12) gives the result that agrees with our intuition, that for dielectric plates the force should be smaller than for metallic plates (Fig. 5). The Casimir force decreases rapidly for smaller values of the reflection coefficient and may become undetectable in the framework of the present experimental possibilities.

The dependence of the Casimir force on the plate thickness d cannot be calculated analytically in the general case. However, it may be done for small r, that is, for $n-1$ small. We then have

$$r\approx r_d(e^{2iknd}-1)$$

and, by Eq. (11),

$$F_C\approx\frac{\hbar c}{\pi}\int_0^{\infty}dk\left[1-\frac{1}{1-r^2e^{2ika}-r^{*2}e^{-2ika}}\right]ke^{-\lambda k}$$
$$\approx\frac{\hbar c}{\pi}\int_0^{\infty}dk\,(-r^2e^{2ika}-r^{*2}e^{-2ika})ke^{-\lambda k}\ .$$

In the limit $\lambda\to 0$ we get (Fig. 6)

$$F_C=\frac{\hbar c r_d^2}{2\pi a^2}[(1+2nd/a)^{-2}-2(1+nd/a)^{-2}+1]\ .$$

The increase of the Casimir force with the increase of the thickness d can easily be understood if one remembers that the attraction between the plates may be regarded as the macroscopic manifestation of the microscopic interaction between the atoms. In the present case, that is the case of rarefied media, F_C is the sum of atom-atom interactions so it must increase with the number of interacting atoms.

VII. FINAL REMARKS

In the preceding section we gave a discussion of the Casimir force in the case of dielectric plates. Here we would like to make additional comments on the validity of our approach.

(1) All of the above calculations are valid for the temperature $T=0$. Higher temperatures lead to incoherent excitation of dielectric media which in turn emit blackbody radiation. This radiation is a source of additional pressure, which is of classical rather than quantum nature.

(2) We have assumed magnetic permeability $\mu=1$, which means that the media are nonmagnetic and the energy connected with the spontaneous and induced magnetic dipole moment may be neglected. This assumption may not be valid in some special cases.

(3) The results derived above for dielectric plates have qualitative character only. To be compared with any real experiment, they should be repeated in three dimensions. The mode functions may be easily written in analogy to (9). The basic modification refers to the coefficients which now depend on the polarization of the wave and the angle between the wave vector **k** and the dielectric surface. Moreover, additional kinds of modes exist, which may be called "evanescent modes," as they behave like $e^{-\chi|x|}$ for $x \to \infty$. For these modes the inequality $\omega^2/c^2 < (k_x^2+k_y^2)=k_\parallel^2$ holds, and $k_\parallel$ may take discrete values only.

(4) Our approach neglects the absorption of energy inside the dielectric. This may be taken into account by using the complex, frequency-dependent refractive index. Then there will be no need to introduce a cutoff function, as it will be introduced naturally by $n(\omega)\to 1$ for $\omega \to \infty$. The exact theory should be based on the quantum Hamiltonian containing the field, the atoms, the "heat bath," and their interaction.

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