Multimode squeeze operators and squeezed states

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Multimode squeeze and rotation operators are defined such that they have extremely similar algebraic properties as those of their single-mode counterparts. It is shown that the introduction of N-mode squeeze operators provides a convenient set of parameters to describe the variances of the quadrature amplitudes in multimode Gaussian squeezed states. Some important properties of these N-mode unitary operators are investigated. It is also shown that the time-evolution operator for a general N-mode quadratic Hamiltonian can be conveniently expressed as an operator product containing an N-mode squeeze operator, an N-mode rotation operator, and an N-mode displacement operator.

I. INTRODUCTION

Single-mode Gaussian squeezed states (GSS's) of quantum harmonic oscillators have been studied extensively in recent years.[1–10] However, multimode GSS's have not been studied as much. Some time ago Schumaker[6] investigated the properties of the most general two-mode GSS's ("two-mode Gaussian pure states") defined by

$$|z_1,z_2,z_{12},\alpha_1,\alpha_2\rangle \equiv \hat{S}_1(z_1)\hat{S}_2(z_2)\hat{S}_{12}(z_{12})|\alpha_1,\alpha_2\rangle \ , \tag{1.1}$$

where $\hat{S}_i(z_i)$ is the single-mode squeeze operator[2] for the ith mode ($i=1,2$),

$$\hat{S}_i(z_i) \equiv \exp\left[\frac{z_i}{2}\hat{a}_i^{\dagger 2} - \frac{z_i^*}{2}\hat{a}_i^2\right] , \tag{1.2}$$

$\hat{S}_{12}(z_{12})$ is called the two-mode squeeze operator[4–6] defined by

$$\hat{S}_{12}(z_{12}) \equiv \exp(z_{12}\hat{a}_1^\dagger\hat{a}_2^\dagger - z_{12}^*\hat{a}_1\hat{a}_2) , \tag{1.3}$$

$\hat{a}_i^\dagger$ and $\hat{a}_i$ are boson creation and annihilation operators for the ith mode, and $|\alpha_1,\alpha_2\rangle$ is a two-mode coherent state which is a direct product of single-mode coherent states defined by[11]

$$\hat{a}_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle \quad i=1,2 \ . \tag{1.4}$$

It should be noted that a general two-mode GSS is not simply a direct product of two single-mode GSS's $|z_1,\alpha_1\rangle|z_2,\alpha_2\rangle$, where

$$|z_i,\alpha_i\rangle \equiv \hat{S}_i(z_i)|\alpha_i\rangle \ . \tag{1.5}$$

A possible definition of N-mode GSS's could be

$$|z_1,z_2,\ldots,z_N,z_{12},\ldots,z_{N-1N},\alpha_1,\alpha_2,\ldots,\alpha_N\rangle \equiv \prod_{i=1}^N \hat{S}_i(z_i)\prod_{j<k}^N \hat{S}_{jk}(z_{jk})|\alpha_1\rangle|\alpha_2\rangle,\ldots,|\alpha_N\rangle \ . \tag{1.6}$$

However, the above definition of N-mode GSS's is not convenient because the operator product in (1.6) is often difficult to manipulate when N is large. Another kind of operator product that will be frequently encountered in describing N-mode coherent states and N-mode GSS's is

$$\prod_{i=1}^N \hat{R}_i(\phi_i)\prod_{j<k}^N \hat{R}_{jk}(\phi_{jk}) , \tag{1.7}$$

where $\hat{R}_i(\phi_i)$ are single-mode rotation operators[4–6] defined by

$$\hat{R}_i(\phi_i) \equiv \exp(i\phi_i\hat{a}_i^\dagger\hat{a}_i) \tag{1.8}$$

with real parameters ϕ_i, and $\hat{R}_{jk}(\phi_{jk})$ are called mixing operators[6] defined by

$$\hat{R}_{jk}(\phi_{jk}) \equiv \exp(\phi_{jk}\hat{a}_j^\dagger\hat{a}_k - \phi_{jk}^*\hat{a}_k^\dagger\hat{a}_j) \ . \tag{1.9}$$

The properties of these operator products are often difficult to see. For example, the unitary transformation of the annihilation or creation operators by these operator products are generally hard to find because repeated applications of these single-mode and two-mode operators many times involve tedious algebras that eventually lead to messy results. As a consequence, it is inconvenient and awkward to describe the variances of the quadrature amplitudes in multimode GSS's in terms of the parameters z_i and z_{jk} ($i,j,k=1,2,\ldots,N$, $j<k$) in (1.6).

In this paper we define N-mode squeeze and rotation operators to eliminate the difficulties mentioned above. These N-mode operators turn out to have extremely similar algebraic properties as those of single-mode operators and are thus easy to handle. It will be seen that the introduction of the so-defined N-mode squeeze operators also gives a convenient set of parameters for the description of all possible second-order moments of the quadrature amplitudes. The two-mode squeeze operator (1.3) defined by Caves and Schumaker[4–6] is a special case of our general two-mode squeeze operators, and the mixing operator (1.9) is included into our definition of multimode rotation operators. In Sec. II, we give the definition of the most general multimode GSS which naturally follows when multimode squeeze and rotation operators are defined. Some important properties of these multimode operators are investigated in Sec. III. It is shown in Sec. IV that the time evolution of the systems driven by multimode quadratic Hamiltonians can be succinctly described by these multimode operators.

II. N-MODE SQUEEZE AND ROTATION OPERATORS

We now consider an N-mode case. Let $\hat{a}$ and $\hat{a}^\dagger$ be column matrices as defined below with $\tilde{\hat{a}}$ and $\widetilde{\hat{a}^\dagger}$ denoting their transposes:

$$\hat{a} \equiv (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)^T , \tag{2.1}$$

$$\hat{a}^\dagger \equiv (\hat{a}_1^\dagger, \hat{a}_2^\dagger, \ldots, \hat{a}_N^\dagger)^T \tag{2.2}$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ $(i=1,2,\ldots,N)$ are the annihilation and creation operators for the ith mode. For convenience, some shorthand notations will be used in the formulations below, such as

$$\hat{a}|\psi\rangle \equiv (\hat{a}_1|\psi\rangle, \hat{a}_2|\psi\rangle, \ldots, \hat{a}_N|\psi\rangle)^T , \tag{2.3}$$

$$\hat{S}^\dagger \hat{a} \hat{S} \equiv (\hat{S}^\dagger \hat{a}_1 \hat{S}, \hat{S}^\dagger \hat{a}_2 \hat{S}, \ldots, \hat{S}^\dagger \hat{a}_N \hat{S})^T , \tag{2.4}$$

$$\frac{\partial}{\partial \hat{a}} \equiv \left[\frac{\partial}{\partial \hat{a}_1}, \frac{\partial}{\partial \hat{a}_2}, \ldots, \frac{\partial}{\partial \hat{a}_N} \right]^T , \tag{2.5}$$

and

$$\langle (\hat{M}_{ij})_{N\times N} \rangle \equiv (\langle \psi | \hat{M}_{ij} | \psi \rangle)_{N\times N} .$$

We define N-mode squeeze operators in a compact form as

$$\hat{S}_N(z) \equiv \exp \left[\frac{\widetilde{\hat{a}^\dagger} z \hat{a}^\dagger}{2} - \frac{\tilde{\hat{a}} z^\dagger \hat{a}}{2} \right] , \tag{2.6}$$

where z is an $N \times N$ matrix which is assumed to be symmetric for convenience. This definition of N-mode squeeze operators has the advantage that $\hat{S}_N(z)$ behaves very much like a single-mode squeeze operator as we will see in a moment. We may also define N-mode rotation operators in a similar way as

$$\hat{R}_N(\Phi) \equiv \exp(i \widetilde{\hat{a}^\dagger} \Phi \hat{a}) , \tag{2.7}$$

where $\Phi = \Phi^\dagger$ is an $N \times N$ Hermitian matrix. It is clear that $\hat{S}_N(z)$ and $\hat{R}_N(\Phi)$ are unitary operators with

$$\hat{S}_N^\dagger(z) = \hat{S}_N^{-1}(z) = \hat{S}_N(-z) , \tag{2.8}$$

$$\hat{R}_N^\dagger(\Phi) = \hat{R}_N^{-1}(\Phi) = \hat{R}_N(-\Phi) . \tag{2.9}$$

N-mode displacement operators are commonly defined as

$$\hat{D}_N(\alpha) \equiv \exp(\tilde{\alpha} \hat{a}^\dagger - \alpha^\dagger \hat{a}) , \tag{2.10}$$

where $\alpha^\dagger \equiv (\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*)$.

Using the operator identity[12]

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots \tag{2.11}$$

we find that

$$\hat{S}_N^\dagger(z) \hat{a} \hat{S}_N(z) = \hat{a} + z\hat{a}^\dagger + \frac{(zz^\dagger)}{2!}\hat{a} + \frac{(zz^\dagger)z}{3!}\hat{a}^\dagger + \frac{(zz^\dagger)^2}{4!}\hat{a} + \cdots . \tag{2.12}$$

It is well known that any matrix of finite dimension may be decomposed into a product of a Hermitian matrix and a unitary matrix. In particular, for a symmetric matrix z, we have

$$z = r e^{i\theta} = e^{i\tilde{\theta}} \tilde{r} , \tag{2.13}$$

where r and θ are Hermitian and the matrix r is positive semidefinite or positive definite, which can be written as $r \geq 0$, depending on whether $|z| \equiv \det(z) = 0$ or not, θ can be chosen such that $2\pi I > \theta \geq 0$. The matrix r is always uniquely determined but θ is not when $|z| = 0$. It should be noted that r and θ generally do not commute. From (2.13) it follows that

$$f(r) e^{i\theta} = \begin{cases} e^{i\tilde{\theta}} f(\tilde{r}) & \text{if } f(-r) = f(r), \\ e^{i\tilde{\theta}} f(\tilde{r}) & \text{if } f(-r) = -f(r), \end{cases} \tag{2.14}$$

where $f(r)$ is assumed to be expandable in a power series of r. Substitution of (2.13) into (2.12) gives

$$\hat{S}_N^\dagger(z) \hat{a} \hat{S}_N(z) = \cosh(r)\hat{a} + \sinh(r) e^{i\theta} \hat{a}^\dagger . \tag{2.15}$$

One can similarly obtain the unitary transformations of the annihilation operator matrix $\hat{a}$ by the N-mode rotation and displacement operators

$$\hat{R}_N^\dagger(\Phi) \hat{a} \hat{R}_N(\Phi) = e^{i\Phi} \hat{a} , \tag{2.16}$$

$$\hat{D}_N^\dagger(\alpha) \hat{a} \hat{D}_N(\alpha) = \hat{a} + \alpha . \tag{2.17}$$

Notice that Eqs. (2.15)–(2.17) are of exactly the same forms as their single-mode counterparts.[9] From (2.15)–(2.17) one immediately obtains the transformed creation operator matrix,

$$\hat{S}_N^\dagger(z) \hat{a}^\dagger \hat{S}_N(z) = \cosh(\tilde{r}) \hat{a}^\dagger + \sinh(\tilde{r}) e^{-i\tilde{\theta}} \hat{a} , \tag{2.18}$$

$$\hat{R}_N^\dagger(\Phi) \hat{a}^\dagger \hat{R}_N(\Phi) = e^{-i\tilde{\Phi}} \hat{a}^\dagger , \tag{2.19}$$

$$\hat{D}_N^\dagger(\alpha) \hat{a}^\dagger \hat{D}_N(\alpha) = \hat{a}^\dagger + \alpha^* . \tag{2.20}$$

A straightforward application of Eqs. (2.15)–(2.20) gives the following transformation relations:

$$\hat{S}_N^\dagger(z) \hat{D}_N(\alpha) \hat{S}_N(z) = \hat{D}_N[\cosh(r)\alpha - \sinh(r) e^{i\theta} \alpha^*] , \tag{2.21}$$

$$\hat{R}_N^\dagger(\Phi) \hat{S}_N(z) \hat{R}_N(\Phi) = \hat{S}_N(e^{-i\Phi} z e^{-i\tilde{\Phi}}) , \tag{2.22}$$

$$\hat{R}_N^\dagger(\Phi) \hat{D}_N(\alpha) \hat{R}_N(\Phi) = \hat{D}_N(e^{-i\Phi}\alpha) , \tag{2.23}$$

which show that the orders of these N-mode operators can be switched at will with appropriate changes in the parameters.

N-mode GSS's may be naturally defined as

$$|z,\alpha\rangle \equiv \hat{S}_N(z)|\alpha\rangle , \tag{2.24}$$

where $|\alpha\rangle \equiv |\alpha_1\rangle|\alpha_2\rangle, \ldots, |\alpha_N\rangle$ is an N-mode coherent state, which is simply a direct product of N single-mode coherent states, satisfying

$$\hat{a}_k|\alpha\rangle = \alpha_k|\alpha\rangle, \quad k=1,2,\ldots,N \tag{2.25}$$

or

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle . \tag{2.26}$$

Since N-mode coherent states $|\alpha\rangle$ can be written as

$$|\alpha\rangle=\hat{D}_N(\alpha)|0\rangle \tag{2.27}$$

where $|0\rangle$ is the vacuum state, we may write

$$|z,\alpha\rangle=\hat{S}_N(z)\hat{D}_N(\alpha)|0\rangle \ . \tag{2.28}$$

The operator $\hat{R}_N(\Phi)$ is called an N-mode rotation operator simply because an N-mode coherent state $|\alpha\rangle$ acted on by $\hat{R}_N(\Phi)$ is still an N-mode coherent state but is "rotated" such that

$$\hat{R}_N(\Phi)|\alpha\rangle=|e^{i\Phi}\alpha\rangle \ . \tag{2.29}$$

Now let us examine how all the variances of the quadrature amplitudes

$$\hat{a}_+\equiv\tfrac{1}{2}(\hat{a}+\hat{a}^\dagger), \quad \hat{a}_-\equiv\frac{1}{2i}(\hat{a}-\hat{a}^\dagger) \tag{2.30}$$

described by the variance matrices

$$\sigma_+^2\equiv\langle\Delta\hat{a}_+\widetilde{\Delta\hat{a}_+}\rangle, \quad \Delta\hat{a}_+\equiv\hat{a}_+-\langle\hat{a}_+\rangle \ , \tag{2.31}$$

$$\sigma_-^2\equiv\langle\Delta\hat{a}_-\widetilde{\Delta\hat{a}_-}\rangle, \quad \Delta\hat{a}_-\equiv\hat{a}_--\langle\hat{a}_-\rangle \ , \tag{2.32}$$

$$\sigma_{+-}^2\equiv\tfrac{1}{2}(\langle\Delta\hat{a}_+\widetilde{\Delta\hat{a}_-}\rangle+\langle\Delta\hat{a}_-\widetilde{\Delta\hat{a}_+}\rangle^T) \tag{2.33}$$

are related to the squeeze matrix z. It is straightforward to show that in an N-mode GSS $|z,\alpha\rangle$, where $z\equiv re^{i\theta}=e^{i\tilde{\theta}}\tilde{r}$,

$$\sigma_+^2=\tfrac{1}{8}[\cosh(2r)+\sinh(2r)e^{i\theta}+\cosh(2\tilde{r})+\sinh(2\tilde{r})e^{-i\tilde{\theta}}] \ , \tag{2.34}$$

$$\sigma_-^2=\tfrac{1}{8}[\cosh(2r)-\sinh(2r)e^{i\theta}+\cosh(2\tilde{r})-\sinh(2\tilde{r})e^{-i\tilde{\theta}}] \ , \tag{2.35}$$

$$\sigma_{+-}^2=\frac{1}{8i}[-\cosh(2r)+\sinh(2r)e^{i\theta}+\cosh(2\tilde{r})-\sinh(2\tilde{r})e^{-i\tilde{\theta}}] \ , \tag{2.36}$$

satisfying

$$\sigma_+^2\sigma_-^2=\frac{1}{16}I+(\sigma_{+-}^2)^2 \ . \tag{2.37}$$

When $z=z^\dagger$, that is, $e^{i\theta}=e^{-i\tilde{\theta}}$ and $r=\tilde{r}=(z^2)^{1/2}$, we have

$$\sigma_{+-}^2=0 \ , \tag{2.38}$$

$$\sigma_+\sigma_-=\sigma_-\sigma_+=\tfrac{1}{4}I \ . \tag{2.39}$$

Therefore, an N-mode GSS is an N-mode minimum-uncertainty state in a sense of (2.39) if the squeeze matrix z is real and symmetric. In particular, when $z=z^\dagger\geq 0$ or $z=z^\dagger\leq 0$, we have $e^{i\theta}=\pm I$ and $r=\tilde{r}=\pm z$ which gives

$$\sigma_+=\tfrac{1}{2}e^{\pm r}, \quad \sigma_-=\tfrac{1}{2}e^{\mp r} \ . \tag{2.40}$$

Equations (2.34)–(2.40) show that variances can be described very conveniently by the squeeze matrix z.

It should be pointed out that our definition of N-mode GSS's (2.24) and the definition (1.6) define the same set of states but they are characterized by different parameters. The difference between our general two-mode squeeze operator $\hat{S}_2(z)$ and the two-mode squeeze operator of Caves and Schumaker (1.3) should also be noticed.

III. SOME PROPERTIES OF N-MODE SQUEEZE OPERATORS

In this section we investigate some important properties of N-mode squeeze operators $\hat{S}_N(z)$.

A. Disentangling of $\hat{S}_N(z)$

In the following we perform disentangling (factorization) of the N-mode squeeze operators with Lie algebra matrix techniques[13–15] to derive the Baker-Campbell-Hausdorff relation[14] for $\hat{S}_N(z)$.

Let us define the operators

$$\hat{A}(u)\equiv\frac{\tilde{\hat{a}}u^\dagger\hat{a}}{2} \ , \tag{3.1}$$

$$\hat{A}^\dagger(v)\equiv\frac{\widetilde{\hat{a}^\dagger}v\hat{a}^\dagger}{2} \ , \tag{3.2}$$

$$\hat{B}(w)\equiv\tfrac{1}{2}(\widetilde{\hat{a}^\dagger}w\hat{a}+\tilde{\hat{a}}\tilde{w}\hat{a}^\dagger)=\widetilde{\hat{a}^\dagger}w\hat{a}+\tfrac{1}{2}\mathrm{Tr}(w) \ , \tag{3.3}$$

where u, v, and w are $N\times N$ matrices. It is easy to show that

$$\exp[-\hat{B}(w)]\hat{A}(u)\exp[\hat{B}(w)]=\hat{A}(e^{w\dagger}ue^{\tilde{w}\dagger}) \ , \tag{3.4}$$

$$\exp[\hat{B}(w)]\hat{A}^\dagger(v)\exp[-\hat{B}(w)]=\hat{A}^\dagger(e^{w}ve^{\tilde{w}}) \ . \tag{3.5}$$

The operator set

$$\{\hat{A}(u),\hat{A}^\dagger(v),\hat{B}(w)|\ u,v=(zz^\dagger)^nz, \quad w=(zz^\dagger)^m \ , \quad n=0,1,\ldots, \quad m=1,2,\ldots\}$$

with $z=\tilde{z}$ forms a Lie algebra[16] $\mathcal{L}$ of infinite dimension,

$$[\hat{A}(u),\hat{A}(v)]=[\hat{A}^\dagger(u),\hat{A}^\dagger(v)]=[\hat{B}(w),\hat{B}(w')]=0 \ , \tag{3.6a}$$

$$[\hat{A}(u),\hat{A}^\dagger(v)]=\hat{B}(vu^\dagger) \ , \tag{3.6b}$$

$$[\hat{B}(w),\hat{A}^\dagger(v)]=\hat{A}^\dagger(wv)+\hat{A}^\dagger(w\tilde{v}) \ , \tag{3.6c}$$

$$[\hat{B}(w),\hat{A}(u)]=-\hat{A}(wu)-\hat{A}(w\tilde{u}) \ , \tag{3.6d}$$

which is homomorphic to the Lie algebra su(1,1) under the mapping ψ defined by

$$\psi[\hat{A}(u)]=L_-, \quad \psi[\hat{A}^\dagger(v)]=L_+, \quad \psi[\hat{B}(w)]=2L_0 \ , \quad \forall u,\ v,\text{ and } w \tag{3.7}$$

where L_-, L_+, and L_0 satisfy the following commutation relations

$$[L_-,L_+]=2L_0\ , \tag{3.8a}$$

$$[L_0,L_\pm]=\pm L_\pm\ . \tag{3.8b}$$

The Lie algebra $\mathcal{L}$ has a faithful matrix representation

$$\{A(u),A^\dagger(v),B(w)|\ u,v=(zz^\dagger)^n z,\quad w=(zz^\dagger)^m\ ,\quad n=0,1,\ldots,\quad m=1,2,\ldots\}\ ,$$

where

$$A(u)\equiv\begin{bmatrix}0 & 0\\ -u^\dagger & 0\end{bmatrix},\quad A^\dagger(v)\equiv\begin{bmatrix}0 & v\\ 0 & 0\end{bmatrix},\quad B(w)\equiv\begin{bmatrix}w & 0\\ 0 & -\tilde{w}\end{bmatrix} \tag{3.9}$$

are $2N\times 2N$ matrices which obey the same commutation rules as (3.6).

Using (2.14) we can factor the matrix $\exp[A^\dagger(z)-A(z)]$ that represents $\hat{S}_N(z)$ as

$$\begin{aligned}\exp[A^\dagger(z)-A(z)]=\exp\left[\begin{bmatrix}0 & z\\ z^\dagger & 0\end{bmatrix}\right]&=\begin{bmatrix}\cosh(r) & \sinh(r)e^{i\theta}\\ e^{-i\theta}\sinh(r) & \cosh(\tilde{r})\end{bmatrix}\\ &=\begin{bmatrix}I & T\\ 0 & I\end{bmatrix}\begin{bmatrix}S & 0\\ 0 & \tilde{S}^{-1}\end{bmatrix}\begin{bmatrix}I & 0\\ T^\dagger & I\end{bmatrix}\\ &=\exp[A^\dagger(T)]\exp[B(\ln S)]\exp[-A(T)]\ ,\end{aligned} \tag{3.10}$$

where $z=re^{i\theta}$, $T\equiv\tanh(r)e^{i\theta}$, and $S\equiv\text{sech}(r)$. Since the disentangling of $\hat{S}_N(z)$ is uniquely determined by the structure of the Lie algebra $\mathcal{L}$, we can obtain simply from (3.10) the disentangled form of the N-mode squeeze operators

$$\begin{aligned}\hat{S}_N(z)&=\exp[\hat{A}^\dagger(T)]\exp[\hat{B}(\ln S)]\exp[-\hat{A}(T)]\\ &=|S|^{1/2}\exp(\tfrac{1}{2}\widetilde{\hat{a}^\dagger}T\hat{a}^\dagger)\exp[\widetilde{\hat{a}^\dagger}(\ln S)\hat{a}]\\ &\quad\times\exp(-\tfrac{1}{2}\tilde{\hat{a}}T^\dagger\hat{a})\ ,\end{aligned} \tag{3.11}$$

where we have used

$$\exp[\text{Tr}(\ln S)]=|S|\ . \tag{3.12}$$

The exponential operators in (3.11) can be put in any other orders using Eqs. (3.4) and (3.5) and the following identity:

$$\begin{aligned}&\exp[-\hat{A}(T_1)]\exp[\hat{A}^\dagger(T_2)]\\ &\quad=\exp[\hat{A}^\dagger(T_2\tilde{P})]\exp[\hat{B}(\ln P)]\exp[-\hat{A}(T_1\tilde{P}^\dagger)]\ ,\end{aligned} \tag{3.13}$$

where T_1 and T_2 are symmetric matrices and $P\equiv(I+T_2T_1^\dagger)^{-1}$.

B. Normal ordering of $\hat{S}_N(z)$

In order to cast $\hat{S}_N(z)$ into the normal-ordered form, all we need to do is to find the normal-ordered form for the exponential operator $\exp[\widetilde{\hat{a}^\dagger}(\ln S)\hat{a}]$ in (3.11). In the following we show that for an arbitrary matrix M,

$$\begin{aligned}\exp(\widetilde{\hat{a}^\dagger}M\hat{a})&=\sum_{n=0}^{\infty}\frac{:[\widetilde{\hat{a}^\dagger}(e^M-I)\hat{a}]^n:}{n!}\\ &=\sum_{n=0}^{\infty}\sum_{\{n_{ij}\}}\prod_{i,j=1}^{N}\frac{(e^M-I)_{ij}^{n_{ij}}}{n_{ij}!}\\ &\quad\times\prod_{k=1}^{N}\hat{a}_k^{\dagger q_k}\prod_{k=1}^{N}\hat{a}_k^{p_k}\end{aligned} \tag{3.14}$$

where $q_k=\sum_m n_{km}$, $p_k=\sum_m n_{mk}$, :: denotes the normal ordering and $\sum_{\{n_{ij}\}}$ means a summation over all partitions of $n=\sum_{i,j=1}^{N}n_{ij}$.

From (2.11) we have

$$\exp(-\widetilde{\hat{a}^\dagger}M\hat{a})\hat{a}\exp(\widetilde{\hat{a}^\dagger}M\hat{a})=e^M\hat{a}\ , \tag{3.15}$$

which may be rewritten as

$$[\hat{a},\exp(\widetilde{\hat{a}^\dagger}M\hat{a})]=(e^M-I)\exp(\widetilde{\hat{a}^\dagger}M\hat{a})\hat{a}\ . \tag{3.16}$$

Now let $F(\hat{a}^\dagger,\hat{a})$ be the normal-ordered form of the operator $\exp(\widetilde{\hat{a}^\dagger}M\hat{a})$, then from (2.26) we have

$$\langle\alpha|\exp(\widetilde{\hat{a}^\dagger}M\hat{a})|\beta\rangle=\langle\alpha|\beta\rangle F(\alpha^*,\beta)\ , \tag{3.17}$$

$$\langle\alpha|[\hat{a},\exp(\widetilde{\hat{a}^\dagger}M\hat{a})]|\beta\rangle=\langle\alpha|\beta\rangle\frac{\partial}{\partial\alpha^*}F(\alpha^*,\beta)\ . \tag{3.18}$$

From (3.16)–(3.18) we obtain a differential equation

$$\frac{\partial}{\partial\alpha^*}F(\alpha^*,\beta)=(e^M-I)F(\alpha^*,\beta)\beta \tag{3.19a}$$

with the boundary condition

$$F(0,\beta)=1\ . \tag{3.19b}$$

Integration of (3.19) yields

$$F(\alpha^*,\beta)=\exp[\alpha^\dagger(e^M-I)\beta]$$
$$=\sum_{n=0}^{\infty}\frac{[\alpha^\dagger(e^M-I)\beta]^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{\{n_{ij}\}}\prod_{i,j=1}^{N}\frac{(e^M-I)_{ij}^{n_{ij}}}{n_{ij}!}\prod_{k=1}^{N}\alpha_k^{*q_k}\prod_{k=1}^{N}\beta_k^{p_k}, \quad (3.20)$$

which immediately leads to (3.14).

Application of (3.14) to (3.11) gives the normal-ordered form of $\hat{S}_N(z)$:

$$\hat{S}_N(z)=|S|^{1/2}\exp(\tfrac{1}{2}\widetilde{\hat{a}^\dagger}T\hat{a}^\dagger)\left[\sum_{n=0}^{\infty}\frac{:[\widetilde{\hat{a}^\dagger}(S-I)\hat{a}]^n:}{n!}\right]\times\exp(-\tfrac{1}{2}\tilde{\hat{a}}T^\dagger\hat{a}) . \quad (3.21)$$

Equation (3.20) also enables us to find the normal-ordered form of the operator product $\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)$:

$$\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)=|S|^{1/2}\exp[-\tfrac{1}{2}(\alpha^\dagger\alpha+\tilde{\alpha}T^\dagger\alpha)]\exp(\tilde{\alpha}\tilde{S}\hat{a}^\dagger+\tfrac{1}{2}\widetilde{\hat{a}^\dagger}T\hat{a}^\dagger)\left[\sum_{n=0}^{\infty}\frac{:[\widetilde{\hat{a}^\dagger}(Se^{i\Phi}-I)\hat{a}]^n:}{n!}\right]$$
$$\times\exp[-(\tilde{\alpha}T^\dagger+\alpha^\dagger)e^{i\Phi}\hat{a}-\tfrac{1}{2}\tilde{\hat{a}}e^{i\Phi}T^\dagger e^{i\Phi}\hat{a}] . \quad (3.22)$$

C. Products of $\hat{S}_N(z)$

With the help of Eqs. (3.4), (3.5), and (3.13), the product of two N-mode squeeze operators can be easily found to be

$$\hat{S}_N(z_1)\hat{S}_N(z_2)=\exp\left[\frac{i}{2}\mathrm{Tr}(\Phi)\right]\hat{S}_N(z_3)\hat{R}_N(\Phi) , \quad (3.23)$$

where z_3 and Φ are given by

$$T_3=S_1^{-1}(T_1+T_2)(I+T_1^\dagger T_2)^{-1}\tilde{S}_1 , \quad (3.24)$$
$$e^{i\Phi}=S_3^{-1}S_1(I+T_2T_1^\dagger)^{-1}S_2 \quad (3.25)$$

with $z_k\equiv r_ke^{i\theta_k}$, $T_k\equiv\tanh(r_k)e^{i\theta_k}$, $S_k\equiv\mathrm{sech}(r_k)$, and $k=1,2,3$.

The products of N-mode rotation and displacement operators are also straightforward to get

$$\hat{R}_N(\Phi_1)\hat{R}_N(\Phi_2)=\hat{R}_N(\Phi_3) , \quad (3.26)$$
$$\hat{D}_N(\alpha)\hat{D}_N(\beta)=\exp[\tfrac{1}{2}(\beta^\dagger\alpha-\alpha^\dagger\beta)]\hat{D}_N(\alpha+\beta) , \quad (3.27)$$

where Φ_3 is given by

$$e^{i\Phi_3}=e^{i\Phi_1}e^{i\Phi_2} . \quad (3.28)$$

Using Eqs. (3.22) and (3.23) one can easily find overlaps of N-mode GSS's:

$$\langle -z_1,\alpha_1|z_2,\alpha_2\rangle=\exp\left[\frac{i}{2}\mathrm{Tr}(\Phi)\right]\langle\alpha_1|\hat{S}_N(z_3)\hat{R}_N(\Phi)|\alpha_2\rangle$$
$$=|S_3|^{1/2}\exp\left[\frac{i}{2}\mathrm{Tr}(\Phi)-\tfrac{1}{2}(\alpha_1^\dagger\alpha_1+\alpha_2^\dagger\alpha_2)\right]\exp(\tfrac{1}{2}\alpha_1^\dagger T_3\alpha_1^*)\exp(\alpha_1^\dagger S_3e^{i\Phi}\alpha_2)\exp(-\tfrac{1}{2}\tilde{\alpha}_2e^{i\Phi}T_3^\dagger e^{i\Phi}\alpha_2) , \quad (3.29)$$

where T_3, S_3, and $e^{i\Phi}$ are given by Eqs. (3.24) and (3.25).

D. Nonexistence of proper eigenstates of $\hat{S}_N(z)$

Recently we proved that single-mode squeeze operators and two-mode squeeze operators of Caves and Schumaker do not have proper eigenstates, the eigenstates that can be normalized to unity.[17] This conclusion is also true for our general N-mode squeeze operators $\hat{S}_N(z)$ as is shown below.

Let us first assume that $\hat{S}_N(z)$ $(z\neq 0)$ has a proper eigenstate $|\lambda(z)\rangle$ with the eigenvalue $e^{i\lambda(z)}$, i.e.,

$$\hat{S}_N(z)|\lambda(z)\rangle=e^{i\lambda(z)}|\lambda(z)\rangle \quad (3.30)$$

with $\langle\lambda(z)|\lambda(z)\rangle=1$, where $\lambda(z)$ is real because $\hat{S}_N(z)$ is unitary.

Now consider the set of states in the column matrix $(\hat{a}\pm e^{i\theta}\hat{a}^\dagger)|\lambda(z)\rangle$, where $e^{i\theta}$ is given by $z=re^{i\theta}$. From (2.15) and (2.18), we have

$$\hat{S}_N(z)(\hat{a}\pm e^{i\theta}\hat{a}^\dagger)\hat{S}_N^\dagger(z)=e^{\mp r}(\hat{a}\pm e^{i\theta}\hat{a}^\dagger) . \quad (3.31)$$

It follows that

$$\hat{S}_N(z)(\hat{a}\pm e^{i\theta}\hat{a}^\dagger)|\lambda(z)\rangle=e^{i\lambda(z)}e^{\mp r}(\hat{a}\pm e^{i\theta}\hat{a}^\dagger)|\lambda(z)\rangle , \quad (3.32)$$

If we let R be the unitary matrix of dimension N that diagonalizes r, namely,

$$RrR^{\dagger}=r_d\equiv\begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_N \end{bmatrix}, \quad (3.33)$$

where r_k ($k=1,2,\ldots,N$) are nonnegative real numbers, then we have

$$\begin{aligned}\hat{S}_N(z)R(\hat{a}\pm e^{i\theta}\hat{a}^{\dagger})|\lambda(z)\rangle &= R\hat{S}_N(z)(\hat{a}\pm e^{i\theta}\hat{a}^{\dagger})|\lambda(z)\rangle \\ &= \exp[\mp r_d+i\lambda(z)I]R(\hat{a}\pm e^{i\theta}\hat{a}^{\dagger})|\lambda(z)\rangle \ . \quad (3.34)\end{aligned}$$

Since the states in $R(\hat{a}\pm e^{i\theta}\hat{a}^{\dagger})|\lambda(z)\rangle$ are not vanishing and can be normalized to 1, Eq. (3.34) implies that all the states in that column matrix are proper eigenstates (unnormalized) of $\hat{S}_N(z)$ with the eigenvalues given by the diagonal matrix $\exp[\mp r_d+i\lambda(z)I]$. However, the unitarity of $\hat{S}_N(z)$ demands that $r_d=0$, which leads to

$$r=z=0 \quad (3.35)$$

contradicting the assumption $z\neq 0$. Therefore we are forced to conclude that proper eigenstates of $\hat{S}_N(z)$ do not exist.

IV. TIME-EVOLUTION OPERATORS FOR N-MODE QUADRATIC HAMILTONIANS

For the Hamiltonians of the form

$$\hat{H}(t)=\sum_{i=1}^{n} f_i(t)\hat{H}_i \quad (4.1)$$

where $f_i(t)$ are c numbers and $\{\hat{H}_i, i=1,2,\ldots,n\}$ forms a Lie algebra of dimension n, the time-evolution operator $\hat{U}(t)$ can be expressed as[18]

$$\hat{U}(t)=\prod_{i=1}^{n}\exp[c_i(t)\hat{H}_i] \ . \quad (4.2)$$

For example, single-mode quadratic Hamiltonians

$$\begin{aligned}\hat{H}_1(t)=&\ \omega(t)\hat{a}^{\dagger}\hat{a}+f(t)\hat{a}^{\dagger 2}+f^*(t)\hat{a}^2 \\ &+g(t)\hat{a}^{\dagger}+g^*(t)\hat{a}+h(t) \quad (4.3)\end{aligned}$$

are associated with a six-dimensional Lie algebra $\{\frac{1}{2}\hat{a}^{\dagger 2},\frac{1}{2}\hat{a}^2,\hat{a}^{\dagger}\hat{a}+\frac{1}{2},\hat{a}^{\dagger},\hat{a},\hat{I}\}$. Thus $\hat{U}_1(t)$ may have six exponential factors. It has been shown[6,9] that these factors can be arranged in such an order that

$$\hat{U}_1(t)=e^{i\gamma_1(t)}\hat{S}_1(z(t))\hat{D}_1(\alpha(t))\hat{R}_1(\phi(t)) \ , \quad (4.4)$$

where $\hat{S}_1(z)$, $\hat{R}_1(\phi)$, and $\hat{D}_1(\alpha)$ are single-mode squeeze, rotation, and displacement operators and $e^{i\gamma_1}$ is a phase factor. Similarly, for general N-mode quadratic Hamiltonians

$$\begin{aligned}\hat{H}_N(t)=&\ \widetilde{\hat{a}^{\dagger}}\omega(t)\hat{a}+\widetilde{\hat{a}^{\dagger}}f(t)\hat{a}^{\dagger}+\tilde{\hat{a}}f^{\dagger}(t)\hat{a} \\ &+\tilde{g}(t)\hat{a}^{\dagger}+g^{\dagger}(t)\hat{a}+h(t) \quad (4.5)\end{aligned}$$

where $\omega(t)$ is an $N\times N$ Hermitian matrix, $f(t)$ is an $N\times N$ symmetric matrix, $g(t)$ is a column matrix of dimension N, and $h(t)$ is an arbitrary real function, the time-evolution operator $\hat{U}_N(t)$ can be expressed as a product of $\frac{1}{2}(3N^2+7N+2)$ exponential factors because of the structure of Lie algebra associated with $\hat{H}_N(t)$. We now show that these exponential operators can be arranged in such an order that $\hat{U}_N(t)$ is simply a product of an N-mode squeeze operator, an N-mode rotation operator, and an N-mode displacement operator multiplied by an overall phase factor, namely,

$$\hat{U}_N(t)=e^{i\gamma_N(t)}\hat{S}_N(z(t))\hat{D}_N(\alpha(t))\hat{R}_N(\Phi(t)) \ . \quad (4.6)$$

Using normal ordering techniques[19] one may find the time-evolution operator $\hat{U}_N(t)$ in the normal-ordered form,

$$\begin{aligned}\hat{U}_N(t)=&\exp[A(t)]\exp[\tilde{B}(t)\hat{a}^{\dagger}+\widetilde{\hat{a}^{\dagger}}C(t)\hat{a}^{\dagger}] \\ &\times\sum_{n=0}^{\infty}\frac{:[\widetilde{\hat{a}^{\dagger}}D(t)\hat{a}]^n:}{n!}\exp[\tilde{E}(t)\hat{a}+\tilde{\hat{a}}F(t)\hat{a}] \quad (4.7)\end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ are given by the following equations:

$$i\dot{A}=\mathrm{Tr}[f^{\dagger}(2C+B\tilde{B})]+g^{\dagger}B+h(t) \ , \quad (4.8a)$$

$$i\dot{B}=(4Cf^{\dagger}+\omega)B+2Cg^*+g \ , \quad (4.8b)$$

$$i\dot{C}=4Cf^{\dagger}C+2\omega C+f \ , \quad (4.8c)$$

$$i\dot{D}=(4Cf^{\dagger}+\omega)(D+I) \ , \quad (4.8d)$$

$$i\dot{E}=(\tilde{D}+I)(2f^{\dagger}B+g^*) \ , \quad (4.8e)$$

$$i\dot{F}=(\tilde{D}+I)f^{\dagger}(D+I) \quad (4.8f)$$

with the initial condition

$$A(0)=B(0)=C(0)=D(0)=E(0)=F(0)=0 \ . \quad (4.9)$$

$E(t)$ and $F(t)$ are related to $B(t)$, $C(t)$, and $D(t)$ because of the unitarity of $\hat{U}_N(t)$. From Eq. (4.7) and the identities

$$[\hat{a},\hat{U}_N(t)]=\frac{\partial}{\partial\hat{a}^{\dagger}}\hat{U}_N(t), \quad [\hat{a}^{\dagger},\hat{U}_N(t)]=-\frac{\partial}{\partial\hat{a}}\hat{U}_N(t) \ , \quad (4.10)$$

we find

$$\begin{aligned}\hat{U}_N^{\dagger}(t)\hat{a}\hat{U}_N(t)=&(I-4CC^{\dagger})^{-1}[(D+I)\hat{a}+2C(\tilde{D}^{\dagger}+I)\hat{a}^{\dagger} \\ &+(2CB^*+B)] \quad (4.11)\end{aligned}$$

$$\hat{U}_N^{\dagger}(t)\hat{a}^{\dagger}\hat{U}_N(t)=(\tilde{D}+I)^{-1}(-2F\hat{a}+\hat{a}^{\dagger}-E) \ . \quad (4.12)$$

Comparison of (4.12) with the Hermitian conjugate of (4.11) gives

$$F=-(D^*+I)^{-1}C^{\dagger}(D+I) \ , \quad (4.13)$$

$$E=-(D^*+I)^{-1}(2C^{\dagger}B+B^*) \ , \quad (4.14)$$

$$I-4CC^{\dagger}=(D+I)(D^{\dagger}+I) \ . \quad (4.15)$$

Equation (4.15) implies that $I-4CC^{\dagger}\geq 0$. We may thus define

$$C\equiv\tfrac{1}{2}\tanh(r)e^{i\theta} \quad (4.16)$$

where $r \geq 0$, and let

$$B \equiv \text{sech}(r)\alpha \tag{4.17}$$

so that

$$I - 4CC^{\dagger} = \text{sech}^2(r) , \tag{4.18}$$

$$D + I = \text{sech}(r)e^{i\Phi} , \tag{4.19}$$

$$\begin{aligned} \hat{U}_N^{\dagger}(t)\hat{a}\hat{U}_N(t) &= [\cosh(r)e^{i\Phi}\hat{a} + \sinh(r)e^{i\theta}e^{-i\Phi}\hat{a}^{\dagger} \\ &\quad + \cosh(r)e^{i\Phi}\alpha + \sinh(r)e^{i\theta}e^{-i\Phi}\alpha^*] \\ &= \hat{R}_N^{\dagger}(\Phi)\hat{D}_N^{\dagger}(\alpha)\hat{S}_N^{\dagger}(z) \\ &\quad \times \hat{a}\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) , \end{aligned} \tag{4.20}$$

where we used Eqs. (2.15)–(2.17). Equation (4.20) implies that

$$\hat{U}_N(t) = e^{i\gamma_N}\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) . \tag{4.21}$$

Comparison of (4.21) with (3.22) and (4.7) gives the phase angle γ_N to be

$$\gamma_N = \text{Im}(A + \tilde{\alpha}C^{\dagger}\alpha) . \tag{4.22}$$

The existence and uniqueness of a solution to the Riccati equation (4.8c) guarantee that $\hat{U}_N(t)$ can be uniquely expressed in the form of (4.21). This result is very important in describing the time evolution of the systems governed by multimode quadratic Hamiltonians. Wave-packet propagations may be exactly calculated with ease using the properties of multimode squeeze, rotation, and displacement operators. Our multimode operator formulation can be applied to many problems such as theoretical calculation of molecular spectroscopies.[20–22]

From (2.21)–(2.23) and (4.21) it is clear that any N-mode quadratic Hamiltonian preserves all N-mode GSS's. On the other hand, it can be shown with exactly the same approach used in Ref. 10 that only N-mode quadratic Hamiltonians preserve all N-mode GSS's.

V. CONCLUSION

We have defined general multimode squeeze operators and rotation operators such that they have extremely similar algebraic properties as those of their single-mode counterparts. The definitions include two-mode squeeze and mixing operators of Caves and Schumaker as special cases. It has been shown that all the variances of the quadrature amplitudes in multimode Gaussian squeezed states can be conveniently described by the squeeze matrix introduced. The disentangling, normal ordering, and some other properties of N-mode squeeze operators have been formulated in matrix notations. We have also shown that the time-evolution operator for a general N-mode quadratic Hamiltonian can be expressed as an operator product containing an N-mode squeeze operator, an N-mode rotation operator and an N-mode displacement operator.

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