Thermal information-entropic uncertainty relation

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Within the framework of thermo-field-dynamics, we define the reduced probability densities, and investigate the sum of information-theoretic entropies in quantum mechanics associated with the measurements of the position and momentum of a particle surrounded by the thermal environment. It is found that this quantity cannot be made arbitrarily small but has a universal lower bound dependent on the temperature. We develop the finite-temperature variational technique for the reduced system, and show that this relation, termed the thermal information-entropic uncertainty relation, is exactly saturated by the thermal coherent state. We also show that the Heisenberg uncertainty relation at finite temperature can be derived from this relation.

I. INTRODUCTION

In the last few years, information-theoretic entropies in quantum mechanics[1–5] have been repeatedly studied in order to establish *stronger* relations than the standard Heisenberg uncertainty principle. Deutsch[1] and Partovi[2] discussed that the sum of entropies associated with the measurements of a generic noncommutative pair of observables (A,B) in a normalized state $|\psi\rangle$, that is,

$$U[A,B:\psi]=S_A[\psi]+S_B[\psi]\ , \tag{1}$$

cannot be made arbitrarily small but has an irreducible lower bound independent of the choice of $|\psi\rangle$, where the information entropy is defined by

$$S_A[\psi]=-\mathop{\mathsf{S}}_{\alpha}|\langle\alpha|\psi\rangle|^2\ln|\langle\alpha|\psi\rangle|^2\ , \tag{2}$$

with $\{|\alpha\rangle\}$, the eigenbasis of A. The symbol S_α stands for the summation (integration) over the discrete (continuous) spectra $\{\alpha\}$. Prior to their discussion, for a canonically conjugate pair of continuous observables, the position and momentum (X,P) with the commutation relation $[X,P]=i$ ($\hbar=1$), Bialynicki-Birula and Mycielski[3] proved the optimal relation

$$U[X,P:\psi]\geq 1+\ln\pi\ . \tag{3}$$

This is closely related to the logarithmic Sobolev inequality, and seems to be a purely mathematical result in Fourier analysis.[5] Indeed, in their proof, it is only assumed that the probability amplitudes $\langle x|\psi\rangle$ and $\langle p|\psi\rangle$ are connected with each other simply by the Fourier transformation.

The authors of Ref. 3 and, more recently, Aragone and Zypman[6] discussed that the information-entropic uncertainty relation of the form (3) is exactly saturated by the coherent state, which is known to also saturate the Heisenberg uncertainty relation: $\Delta X\Delta P=\frac{1}{2}$.

Our interest here concerns a more physically general situation where the system under consideration is open. A general formalism for such a nonidealized situation has been presented and analyzed by Blankenbecler and Partovi.[7]

In extending the relation like Eq. (3) to the case of an open system, a difficulty is caused by the fact that it might stay not in a pure state but in a mixed state. Technically, this means that the probability density $\rho(x)$ is not equal to $|\langle x|\psi\rangle|^2$ but may be expressed by the density operator ρ as the unfactorizable form $\langle x|\rho|x\rangle$, and, therefore, the method of the logarithmic Sobolev inequality might not be directly applicable any longer.

In this paper, we derive the uncertainty relation to the sum of the information entropies associated with the measurements of the position and momentum of a particle in equilibrium with the thermal reservoir. A universal nonnegative temperature correction to the right-hand side of Eq. (3) is determined. This has the significance of a precise realization of the statement in information theory[8] that the thermal disturbance leads to the loss of information in general. As an application of this thermal information-entropic uncertainty relation, we also show that the thermal Heisenberg uncertainty relation, the Heisenberg uncertainty relation at finite temperature, can be derived from it.

For this purpose, we employ thermo-field-dynamics (TFD) originally formulated and extensively developed by Takahashi and Umezawa.[9] This theory is equivalent to the standard density matrix formalism, at least, as long as equilibrium systems concern. It, however, supplies a very convenient tool in treating the thermal systems, since it enables us to discuss them in a manner analogous to the pure state quantum theory. A main feature of TFD is the basic requirements of the doubled Hilbert space $H\otimes\tilde{H}$, the *normal* operator (A) acting on the objective space H, and its corresponding *tildian* operator $(\tilde{A})$ on the fictitious space $\tilde{H}$. The necessity of these extra objects was investigated mathematically by Ojima[10] in the context of the operator (C^*) algebra. Their physical meanings, however, seem still not to have been satisfactorily clarified.

Our strategy is as follows. First we define the reduction of states in TFD which ensures that the measurements of physical quantities should be characterized by the coordinates of the representations in H alone. Next we perform this for the coherent state of a harmonic oscillator in TFD,[11] and calculate the sum of entropies for the measurements of (X,P) in the reduced thermal coherent state. Then, by developing a new finite-temperature variational technique for the reduced system, we prove that it is indeed the minimum value, and consequently establish the thermal information-entropic uncertainty relation.

We feel that, as we shall see, the questions concerning the reduction in TFD cannot be answered by mathematics alone, but requires some physical assumptions on dealing with the tildian degrees of freedom. In this sense, our discussion may also shed new light on their own attributes.

II. BASICS OF THERMO-FIELD-DYNAMICS AND THE THERMAL COHERENT STATE

In this section, we briefly recapitulate the basics of TFD and some properties of the thermal coherent state relevant to our discussion. (For further details and many applications of TFD, see Ref. 12.) The expectation value of an operator A in quantum-statistical physics is usually defined by

$$\langle A \rangle = Z^{-1}(\beta) T_r[A \exp(-\beta\mathcal{H})] , \tag{4}$$

where $Z(\beta)=T_r\exp(-\beta\mathcal{H})$ is the partition function for the system, $\mathcal{H}$ is the total Hamiltonian (including a possible chemical potential term), and $\beta=(k_B T)^{-1}$. One may say that this is not in the field theoretical fashion, since an expectation value in quantum field theory is customarily expressed as a certain vacuum expectation value. In this respect, Takahashi and Umezawa[9] claimed that the expectation value should be written in terms of the thermal vacuum as follows:

$$\langle A \rangle = \langle O(\beta) | A | O(\beta) \rangle . \tag{5}$$

However, they immediately recognized that such a temperature-dependent vacuum cannot be written as a superposition of the basis in the ordinary Hilbert space H. To see this simply, let us examine its expansion in terms of the orthonormal complete energy eigenbasis $\{|n\rangle\}$:

$$|O(\beta)\rangle = \sum_n f_n(\beta)|n\rangle . \tag{6}$$

Then, from Eqs. (4) and (5), we find the condition

$$f_m^*(\beta) f_n(\beta) = \delta_{mn} Z^{-1}(\beta)\exp(-\beta E_n) . \tag{7}$$

It should be noted that this cannot be satisfied as long as $f_n(\beta)$'s are merely numbers. Consequently, the authors of Ref. 9 introduced another Hilbert space $\tilde{H}$ and constructed the normalized thermal vacuum as follows:

$$|O(\beta)\rangle = Z^{-1/2}(\beta) \sum_n \exp(-\beta E_n/2)|n,\tilde{n}\rangle , \tag{8}$$

where $|n,\tilde{n}\rangle \equiv |n\rangle \otimes |n\rangle$. Thus the dynamical degrees of freedom are doubled in this formalism. Accordingly, in addition to a normal operator A acting on the objective space H, one also has to work with an extra operator $\tilde{A}$ termed the tildian operator on the fictitious space $\tilde{H}$, which is assumed to (anti) commute with A. They are connected with each other by the so-called tilde conjugation rules

$$\begin{gathered}(A_1 A_2)^\sim = \tilde{A}_1 \tilde{A}_2, (c_1 A_1 + c_2 A_2)^\sim = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2 , \\ (\tilde{A})^\sim = \sigma A, (A^\dagger)^\sim = \tilde{A}^\dagger ,\end{gathered} \tag{9}$$

where c's are complex c numbers and $\sigma=+1(-1)$ for bosonic (fermionic) A.

Ojima[10] showed that this doubling property naturally appears in the C^* algebraic approach to quantum field theory combined with the Kubo-Martin-Schwinger condition and the Gelfand-Naimark-Segal representation theorem. This supplies a firm foundation to the investigations of systems of infinite degrees of freedom with continuous spectra, for example.

Let us consider here a harmonic oscillator with a frequency ω described by the Hamiltonian

$$\mathcal{H} = \omega a^\dagger a \tag{10}$$

where $a^\dagger$ and a are the usual creation and annihilation operators, respectively. Together with the associated tildian operators $\tilde{a}^\dagger$ and $\tilde{a}$, they obey the commutation relations

$$[a,a^\dagger]=[\tilde{a},\tilde{a}^\dagger]=1, \quad (\text{others})=0 . \tag{11}$$

In this case, the thermal vacuum (8) is related to the zero-temperature vacuum $|O,\tilde{O}\rangle \equiv |O\rangle \otimes |O\rangle$ by the following unitary transformation:

$$|O(\beta)\rangle = \exp(-iG)|O,\tilde{O}\rangle , \tag{12}$$

$$-iG = \theta(\beta)(a^\dagger \tilde{a}^\dagger - \tilde{a}a) . \tag{13}$$

Correspondingly, the operators undergo the Bogoliubov transformations

$$\begin{aligned} a(\beta) &= \exp(-iG)a\exp(iG) \\ &= a\cosh[\theta(\beta)] - \tilde{a}^\dagger \sinh[\theta(\beta)] , \end{aligned} \tag{14a}$$

$$\begin{aligned} \tilde{a}(\beta) &= \exp(-iG)\tilde{a}\exp(iG) \\ &= \tilde{a}\cosh[\theta(\beta)] - a^\dagger \sinh[\theta(\beta)] , \end{aligned} \tag{14b}$$

and so on, where $\theta(\beta)$ is given by

$$\begin{gathered}\cosh[\theta(\beta)] = [1-\exp(-\beta\omega)]^{-1/2} , \\ \sinh[\theta(\beta)] = [\exp(\beta\omega)-1]^{-1/2} .\end{gathered} \tag{15}$$

$a(\beta)$ and $\tilde{a}(\beta)$ are now the annihilation operators at finite temperature ($T\neq 0$) with respect to the thermal vacuum.

Recently, Mann and Revzen[11] gave a natural extension of the definition of the well-known coherent state[13] to the one at $T\neq 0$. The thermal coherent state (TCS) they constructed is the normalized eigenstate of both $a(\beta)$ and $\tilde{a}(\beta)$:

$$a(\beta)|z,\tilde{z};\beta\rangle = z|z,\tilde{z};\beta\rangle , \tag{16a}$$

$$\tilde{a}(\beta)|z,\tilde{z};\beta\rangle = \tilde{z}^*|z,\tilde{z};\beta\rangle , \tag{16b}$$

$$|z,\tilde{z};\beta\rangle=\exp[za^{\dagger}(\beta)-z^{*}a(\beta)+\tilde{z}^{*}\tilde{a}^{\dagger}(\beta)-\tilde{z}\,\tilde{a}(\beta)]|O(\beta)\rangle\ , \tag{17}$$

where z and $\tilde{z}^{*}$ are the complex eigenvalues. The choice of Eq. (16b) is consistent with the rules (9).

TCS was also investigated by Emch and Hegerfeldt[14] within the framework of the standard density matrix formalism, which can be completely characterized by the observable quantities alone in contrast to TFD.

In order to see the equivalence between these two approaches, the authors of Ref. 11 considered the quantum characteristic function labeled by the c-number phase-space variables $(x,p,\tilde{x},\tilde{p})$:

$$\begin{aligned}\chi(x,p,\tilde{x},\tilde{p})&\equiv\langle z,\tilde{z};\beta|\exp[-i(xX+pP+\tilde{x}\tilde{X}+\tilde{p}\tilde{P})]|z,\tilde{z};\beta\rangle\\&=\exp\{-\tfrac{1}{4}\cosh[2\theta(\beta)](p^{2}+x^{2}+\tilde{p}^{2}+\tilde{x}^{2})-\tfrac{1}{2}\sinh[2\theta(\beta)](p\tilde{p}+x\tilde{x})-i(x\langle X\rangle_c+p\langle P\rangle_c\\&\qquad+\tilde{x}\langle\tilde{X}\rangle_c+\tilde{p}\langle\tilde{P}\rangle_c)\}\ ,\end{aligned}\tag{18}$$

provided that the position and momentum operators ($\omega\equiv1$, henceforth) and their expectation values with respect to TCS (17) are, respectively, given by

$$X=\frac{1}{\sqrt{2}}(a+a^{\dagger}),\quad P=\frac{1}{i\sqrt{2}}(a-a^{\dagger})\ , \tag{19a}$$

$$\tilde{X}=\frac{1}{\sqrt{2}}(\tilde{a}+\tilde{a}^{\dagger}),\quad \tilde{P}=-\frac{1}{i\sqrt{2}}(\tilde{a}-\tilde{a}^{\dagger})\ , \tag{19b}$$

$$\begin{aligned}\langle X\rangle_c&=\frac{z+z^{*}}{\sqrt{2}}\cosh[\theta(\beta)]+\frac{\tilde{z}+\tilde{z}^{*}}{\sqrt{2}}\sinh[\theta(\beta)]\ ,\\\langle P\rangle_c&=\frac{z-z^{*}}{i\sqrt{2}}\cosh[\theta(\beta)]+\frac{\tilde{z}-\tilde{z}^{*}}{i\sqrt{2}}\sinh[\theta(\beta)]\ ,\end{aligned}\tag{20a}$$

$$\begin{aligned}\langle\tilde{X}\rangle_c&=\frac{\tilde{z}+\tilde{z}^{*}}{\sqrt{2}}\cosh[\theta(\beta)]+\frac{z+z^{*}}{\sqrt{2}}\sinh[\theta(\beta)]\ ,\\\langle\tilde{P}\rangle_c&=\frac{\tilde{z}-\tilde{z}^{*}}{i\sqrt{2}}\cosh[\theta(\beta)]+\frac{z-z^{*}}{i\sqrt{2}}\sinh[\theta(\beta)]\ .\end{aligned}\tag{20b}$$

This quantum characteristic function becomes identical with that in Ref. 14, if $\tilde{x}=\tilde{p}=0$.

For the later discussion, here we calculate the Heisenberg uncertainty in TCS. From Eq. (18) we immediately obtain the following second moments:

$$\begin{aligned}\langle X^{2}\rangle_c&=\left[i\frac{\partial}{\partial x}\right]^{2}\chi(x,p,\tilde{x},\tilde{p})|_{x=p=\tilde{x}=\tilde{p}=0}\\&=\langle X\rangle_c^{2}+\tfrac{1}{2}\cosh[2\theta(\beta)]\ ,\end{aligned}\tag{21a}$$

$$\begin{aligned}\langle P^{2}\rangle_c&=\left[i\frac{\partial}{\partial p}\right]^{2}\chi(x,p,\tilde{x},\tilde{p})|_{x=p=\tilde{x}=\tilde{p}=0}\\&=\langle P\rangle_c^{2}+\tfrac{1}{2}\cosh[2\theta(\beta)]\ .\end{aligned}\tag{21b}$$

Therefore we find

$$\Delta_c X=\Delta_c P=\{\tfrac{1}{2}\cosh[2\theta(\beta)]\}^{1/2}\ , \tag{22}$$

$$\Delta_c X\Delta_c P=\tfrac{1}{2}\cosh[2\theta(\beta)]\ . \tag{23}$$

In Sec. V, we shall show that Eq. (23) gives the minimum value at fixed T.

III. REDUCTION OF THERMAL STATES

We start with some heuristic discussions concerning the measurement of physical quantities in TFD.

First of all, we point out that we always measure the physical observable quantities in the coordinates of the representations in H alone, not in the whole including the tildian. Therefore a state in the full space $H\otimes\tilde{H}$ must be reduced to a suitable substate. This is a problem characteristic of the present discussion, *since the information entropy (2) depends on the choice of the basis of representation $|\alpha\rangle$ in general and is not expressed as a quantum-mechanical expectation value of a certain operator.* Let us discuss this by employing the probability amplitude associated with the measurement of a normal quantity A in a thermal state $|\psi,\tilde{\psi};\beta\rangle$. In accordance with the general quantum-mechanical framework, such an amplitude may be represented as $\langle\alpha,\tilde{\alpha}|\psi,\tilde{\psi};\beta\rangle$, where $|\alpha,\tilde{\alpha}\rangle\equiv|\alpha\rangle\otimes|\alpha\rangle$ is the complete eigenbasis of the operator A and its tilde conjugation $\tilde{A}$. A possible reduction procedure is tracing out the tildian coordinate in the probability density:

$$\rho_R(\alpha)=\underset{\tilde{\alpha}}{\mathrm{S}}\langle\alpha,\tilde{\alpha}|\psi,\tilde{\psi};\beta\rangle\langle\psi,\tilde{\psi};\beta|\alpha,\tilde{\alpha}\rangle\ . \tag{24}$$

This finds formal conformities with the standard reduction procedure in quantum theory of open systems,[15,16] when the density operator describing the total system and the reduced probability density of an open subsystem in that theory are compared to the present *density superoperator* $|\psi,\tilde{\psi};\beta\rangle\langle\psi,\tilde{\psi};\beta|$ and $\rho_R(\alpha)$, respectively. We note that the above prescription does not bring any difficulties to usual TFD calculations of expectation values of normal operators.

Next let us examine this concretely with respect to the measurements of the position X and momentum P of an oscillator in TCS defined by Eq. (17). The bases of the position and momentum representations are the eigenstates of the operators (19), and are, respectively, given by

$$\begin{aligned}|x,\tilde{x}\rangle=\frac{1}{\sqrt{\pi}}&\exp[-\tfrac{1}{2}(x^{2}+\tilde{x}^{2})]\\&\times\exp[-\tfrac{1}{2}a^{\dagger2}+\sqrt{2}xa^{\dagger}-\tfrac{1}{2}\tilde{a}^{\dagger2}+\sqrt{2}\tilde{x}\tilde{a}^{\dagger}]|O,\tilde{O}\rangle\ ,\end{aligned}\tag{25a}$$

$$\begin{aligned}|p,\tilde{p}\rangle=\frac{1}{\sqrt{\pi}}&\exp[-\tfrac{1}{2}(p^{2}+\tilde{p}^{2})]\\&\times\exp[\tfrac{1}{2}a^{\dagger2}+i\sqrt{2}pa^{\dagger}+\tfrac{1}{2}\tilde{a}^{\dagger2}-i\sqrt{2}\tilde{p}\tilde{a}^{\dagger}]|O,\tilde{O}\rangle\ .\end{aligned}\tag{25b}$$

Straightforward calculations lead to the following full probability amplitudes:

$$\langle x,\tilde{x}|z,\tilde{z};\beta\rangle=\frac{1}{\sqrt{\pi}}\exp\left[-\tfrac{1}{2}\cosh[2\theta(\beta)](x-\langle X\rangle_c)^2-\tfrac{1}{2}\cosh[2\theta(\beta)](\tilde{x}-\langle\tilde{X}\rangle_c)^2+\sinh[2\theta(\beta)](x-\langle X\rangle_c)(\tilde{x}-\langle\tilde{X}\rangle_c)\right.$$
$$\left.+i\langle P\rangle_c(x-\langle X\rangle_c)-i\langle\tilde{P}\rangle_c(\tilde{x}-\langle\tilde{X}\rangle_c)+\frac{i}{2}(\langle X\rangle_c\langle P\rangle_c-\langle\tilde{X}\rangle_c\langle\tilde{P}\rangle_c)\right], \tag{26a}$$

$$\langle p,\tilde{p}|z,\tilde{z};\beta\rangle=\frac{1}{\sqrt{\pi}}\exp\left[-\tfrac{1}{2}\cosh[2\theta(\beta)](p-\langle P\rangle_c)^2-\tfrac{1}{2}\cosh[2\theta(\beta)](\tilde{p}-\langle\tilde{P}\rangle_c)^2+\sinh[2\theta(\beta)](p-\langle P\rangle_c)(\tilde{p}-\langle\tilde{P}\rangle_c)\right.$$
$$\left.-i\langle X\rangle_c(p-\langle P\rangle_c)+i\langle\tilde{X}\rangle_c(\tilde{p}-\langle\tilde{P}\rangle_c)-\frac{i}{2}(\langle X\rangle_c\langle P\rangle_c-\langle\tilde{X}\rangle_c\langle\tilde{P}\rangle_c)\right]. \tag{26b}$$

Tracing out the tildian coordinates, we obtain the following reduced probability densities:

$$\rho_R^{\mathrm{TCS}}(x)=\int d\tilde{x}\,\langle x,\tilde{x}|z,\tilde{z};\beta\rangle\,\langle z,\tilde{z};\beta|x,\tilde{x}\rangle$$
$$=\{\pi\cosh[2\theta(\beta)]\}^{-1/2}\times\exp\left[-\frac{1}{\cosh[2\theta(\beta)]}(x-\langle X\rangle_c)^2\right], \tag{27a}$$

$$\rho_R^{\mathrm{TCS}}(p)=\int d\tilde{p}\,\langle p,\tilde{p}|z,\tilde{z};\beta\rangle\,\langle z,\tilde{z};\beta|p,\tilde{p}\rangle$$
$$=\{\pi\cosh[2\theta(\beta)]\}^{-1/2}\times\exp\left[-\frac{1}{\cosh[2\theta(\beta)]}(p-\langle P\rangle_c)^2\right]. \tag{27b}$$

Thus we verify the desirable property that the observable probability densities are characterized by the (expectation values of) normal operators alone.

IV. A NEW FINITE-TEMPERATURE VARIATIONAL TECHNIQUE FOR REDUCED SYSTEMS IN TFD AND THERMAL INFORMATION-ENTROPIC UNCERTAINTY RELATION

We define the entropy for the measurement of a normal observable A at given T in a normalized state $|\psi,\tilde{\psi};\beta\rangle$ as follows:

$$S_A[\psi,\tilde{\psi};\beta]=-\underset{\alpha}{\mathrm{S}}\rho_R(\alpha)\ln\rho_R(\alpha), \tag{28}$$

and so on, where ρ_R's are given in Eq. (24).

For the measurements of the position X and the momentum P in TCS [i.e., ρ_R^{TCS}'s in Eqs. (27)],

$$S_X[z,\tilde{z};\beta]=S_P[z,\tilde{z};\beta]=\tfrac{1}{2}(1+\ln\pi+\ln\{\cosh[2\theta(\beta)]\}), \tag{29}$$

and, therefore, the entropy for the measurements of a pair (X,P) is

$$U[X,P{:}z,\tilde{z};\beta]\equiv S_X[z,\tilde{z};\beta]+S_P[z,\tilde{z};\beta]=1+\ln\pi+\ln\{\cosh[2\theta(\beta)]\}. \tag{30}$$

In what follows, we develop the variational technique for the reduced system and prove that Eq. (30) gives the optimal value at given T.

As emphasized in Sec. III, we are not concerned with the whole system including the tildian but only with its reduced one. Such a situation should be also reflected in the procedure of variational calculus. So, it is quite reasonable to assume that the variation of a thermal state is undertaken by the H component alone. That is, the finite-temperature variational operation in the reduced system, proposed here, is expressed as

$$|\psi,\tilde{\psi};\beta\rangle\to|\psi,\tilde{\psi};\beta\rangle+\varepsilon|\xi,\tilde{\psi};\beta\rangle, \tag{31}$$

where ε and ξ denote an infinitesimal variation parameter and an arbitrary deformation of the H component, respectively. Here we note that the variation must be generated by the thermal state with β, since we are considering the variation at fixed T.

Now we would like to find the thermal state giving rise to the stationary value of the functional

$$U[X,P{:}\psi,\tilde{\psi};\beta]=S_X[\psi,\tilde{\psi};\beta]+S_P[\psi,\tilde{\psi};\beta], \tag{32}$$

where

$$S_X[\psi,\tilde{\psi};\beta]=-\int dx\frac{\rho_R(x)}{N^2}\ln\frac{\rho_R(x)}{N^2}, \tag{33}$$

$$\rho_R(x)=\int d\tilde{x}\,\langle x,\tilde{x}|\psi,\tilde{\psi};\beta\rangle\,\langle\psi,\tilde{\psi};\beta|x,\tilde{x}\rangle, \tag{34}$$

and so on, with

$$N^2=\langle\psi,\tilde{\psi};\beta|\psi,\tilde{\psi};\beta\rangle. \tag{35}$$

Owing to the operation (31), the functional varies as

$$U[X,P{:}\psi,\tilde{\psi};\beta]\to U[X,P{:}\psi,\tilde{\psi};\beta]+\varepsilon\Gamma+0(\varepsilon^2), \tag{36}$$

$$\Gamma\equiv\left[\int dx\,\rho_R(x)\ln\rho_R(x)+\int dp\rho_R(p)\ln\rho_R(p)\right]\langle\psi,\tilde{\psi};\beta|\xi,\tilde{\psi};\beta\rangle$$
$$-\int\int dx\,d\tilde{x}\,\ln[\rho_R(x)]\langle\psi,\tilde{\psi};\beta|x,\tilde{x}\rangle\,\langle x,\tilde{x}|\xi,\tilde{\psi};\beta\rangle-\int\int dp\,d\tilde{p}\,\ln[\rho_R(p)]\langle\psi,\tilde{\psi};\beta|p,\tilde{p}\rangle\,\langle p,\tilde{p}|\xi,\tilde{\psi};\beta\rangle, \tag{37}$$

where we have employed the normalization condition $N=1$. We do not know how to solve the equation $\Gamma=0$ generally with respect to the unknown state $|\psi,\tilde{\psi};\beta\rangle$. Here TCS (17) is examined, since, in the $T=0$ case, the coherent state saturates the entropic uncertainty relation (3).[3,6] For ρ_R^{TCS}'s in Eqs. (27), Γ becomes

$$\Gamma^{\rm TCS}=-\langle z,\tilde{z};\beta|\xi,\tilde{z};\beta\rangle-\frac{1}{\cosh[2\theta(\beta)]}\langle z,\tilde{z};\beta|\left[\int\int dx\,d\tilde{x}\,|x,\tilde{x}\rangle\langle x,\tilde{x}|(x-\langle X\rangle_c)^2+\int\int dp\,d\tilde{p}\,|p,\tilde{p}\rangle\langle p,\tilde{p}|(p-\langle P\rangle_c)^2\right]|\xi,\tilde{z};\beta\rangle\ . \tag{38}$$

The operator in the second term of $\Gamma^{\rm TCS}$ can be easily calculated by the so-called integration within ordered product technique.[17] By use of Eqs. (25) and the formula

$$|O,\tilde{O}\rangle\langle O,\tilde{O}|=:\exp(-a^\dagger a-\tilde{a}^\dagger\tilde{a}): \tag{39}$$

with the normal ordered product with respect to the $T=0$ vacuum, the integration is performed as

$$\int\int dx\,d\tilde{x}\,|x,\tilde{x}\rangle\langle x,\tilde{x}|(x-\langle X\rangle_c)^2+\int\int dp\,d\tilde{p}\,|p,\tilde{p}\rangle\langle p,\tilde{p}|(p-\langle P\rangle_c)^2=:(X-\langle X\rangle_c)^2+(P-\langle P\rangle_c)^2:\ . \tag{40}$$

Substituting this into Eq. (38) and doing some simple algebraic calculations with Eqs. (14), (16), and (19a), we find

$$\Gamma^{\rm TCS}=0\ , \tag{41}$$

which leads to

$$U[X,P:z,\tilde{z};\beta]\to 1+\ln\pi+\ln\{\cosh[2\theta(\beta)]\}+O(\varepsilon^2)\ . \tag{42}$$

This means that TCS yields the stationary value of the entropy functional (32).

Thus we have established the thermal information-entropic uncertainty relation

$$U[X,P:\psi,\tilde{\psi};\beta]\ge 1+\ln\pi+\ln\{\cosh[2\theta(\beta)]\}\ . \tag{43}$$

This is our main result. The third term on the right-hand side determines the minimum of the loss of information due to the thermal disturbance effects in the measurements of the position and momentum in finite-temperature quantum theory.

V. DERIVATION OF THE THERMAL HEISENBERG UNCERTAINTY RELATION FROM THE THERMAL INFORMATION-ENTROPIC UNCERTAINTY RELATION

It is also possible to derive the thermal Heisenberg uncertainty relation along the same line discussed so far. Here, instead, we derive it from the thermal information-entropic uncertainty relation (43). The method is essentially based on the discussion in Ref. 3.

Let us find the reduced probability density $\rho_R(x)$ that maximize the concave entropy functional (33) under the condition

$$\langle (X-\langle X\rangle)^2\rangle=(\Delta X)^2\ . \tag{44}$$

Here the brackets denote the expectation values with respect to $\rho_R(x)/N^2$. This is just the constrained variational problem characterized by the functional

$$\Phi[\psi,\tilde{\psi};\beta]\equiv S_X[\psi,\tilde{\psi};\beta]-\lambda[\langle (X-\langle X\rangle)^2\rangle-(\Delta X)^2]\ , \tag{45}$$

where λ is Lagrange's multiplier. Applying the variational operation (31) to Φ, the stationarity conditions are found to yield

$$\lambda=\frac{1}{2(\Delta X)^2}\ , \tag{46}$$

$$\rho_R^{\max}(x)=\frac{1}{[2\pi(\Delta X)^2]^{1/2}}\exp\left[-\frac{1}{2(\Delta X)^2}(x-\langle X\rangle)^2\right]\ , \tag{47}$$

provided that $\rho_R^{\max}$ has been normalized. The associated entropy is

$$S_X^{\max}[\psi,\tilde{\psi};\beta]=\tfrac{1}{2}\ln[2\pi e(\Delta X)^2]\ , \tag{48}$$

and therefore the following inequality holds:

$$S_X[\psi,\tilde{\psi};\beta]\le\tfrac{1}{2}\ln[2\pi e(\Delta X)^2]\ . \tag{49}$$

Repeating a similar calculation for the entropy functional for the momentum P, one can also obtain

$$S_P[\psi,\tilde{\psi};\beta]\le\tfrac{1}{2}\ln[2\pi e(\Delta P)^2]\ . \tag{50}$$

The combination of Eqs. (49) and (50) leads to

$$\begin{aligned}2(\Delta P)^2&\ge\exp\{-1-\ln\pi+2S_P[\psi,\tilde{\psi};\beta]\}\\&\ge\exp(1+\ln\pi+2\ln\{\cosh[2\theta(\beta)]\}-2S_X[\psi,\tilde{\psi};\beta])\\&\ge\tfrac{1}{2}\cosh^2[2\theta(\beta)](\Delta X)^{-2}\ ,\end{aligned} \tag{51}$$

where we have used Eq. (43) in the second inequality.

Thus we have shown that the thermal Heisenberg uncertainty relation

$$\Delta X\Delta P\ge\tfrac{1}{2}\cosh[2\theta(\beta)]\ , \tag{52}$$

can be derived from the thermal information-entropic uncertainty relation. From Eq. (23), TCS is also found to saturate this inequality, analogously to the $T=0$ case.

VI. REMARKS

As can be seen from Eqs. (22) and (23), the (thermal) coherent state makes the variances ΔX and ΔP equal to each other, keeping the (thermal) Heisenberg uncertainty in its minimum value. Also, from Eqs. (29) and (30), a similar situation is found in the information entropies.

Recently, much attention has been focused on the squeezed states in quantum optics both from experimental and theoretical aspects.[18] These states are known to reduce either ΔX or ΔP, also keeping $\Delta X\Delta P$ minimum. In this context, one may discuss squeezing the thermal information entropies.[19]

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