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Current flow under anomalous-diffusion conditions: Lévy walks

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We investigate the current flow under anomalous-diffusion conditions (disordered media and also turbulent motion) in the limit of strong-bias fields. A unified picture for anomalous behavior is provided by Lévy walks. These are continuous-time random walks with coupled spatial and temporal memories. We find that the long-time asymptotic behavior of the current adheres to the following power laws: dispersive transport decreasing and enhanced diffusion increasing with time. In both cases the characteristic exponents depend (in a complex way) on the memory terms. We corroborate the values of the exponents by Monte Carlo simulations.

For large classes of materials, transport does not follow diffusive (Brownian) behavior. Examples are diffusion in porous media (dispersive behavior) on the one hand and enhanced diffusion, such as that found in turbulent motion, on the other hand. The unique shape of the diffusive character is given by the asymptotic long-time behavior of the mean-squared displacement $\langle r^2(t)\rangle$ that usually obeys the form

$$\langle r^2(t)\rangle \sim t^{\alpha} . \tag{1}$$

For simple Brownian motion one has $\alpha=1$. The case $\alpha<1$ characterizes the sublinear dispersive behavior typical for transport in tortuous systems.[1-3] In chaotic dynamics and in turbulent motion one finds $\alpha>1$, i.e., superlinear enhanced diffusional behavior.[4-8]

What is now the influence of a strong external field? The question has been investigated intensively for the case of dispersive motion in disordered media: important fields of research are the photoconductivity in xerographic materials[1,9] and material flow in percolating systems (such as porous rocks). In general one finds for the current $I(t)$

$$I(t)\sim t^{\epsilon} . \tag{2}$$

Whereas for materials that show diffusive behavior the current is, in general, constant in time, $\epsilon=0$, for dispersive motion one finds $\epsilon<0$, i.e., a decay with time. Such a situation was studied for geometrical disorder by Scher and Montroll[1] in terms of the continuous-time random-walk (CTRW) approach. Similar results are also found for energetical disorder in multiple-trapping models.[9]

What happens now with the currents under conditions of enhanced motion? Such questions have only recently begun to emerge; a recent experimental work on anomalous diffusion in a linear array of vortices[10] mentions as examples the diffusion of a magnetic field in convective cells in stars and dispersion of pollutants by atmospheric turbulence. Evidently, one still expects behavior as that in Eq. (2), now with $\epsilon>0$.

Theoretically, nowadays one is in the fortunate position of being able to describe, using Lévy walks, both dispersive and also enhanced motion in a unified way. This method opens the possibility—given the microscopic waiting-time distribution—to compute the parameter ϵ.

In the following we perform this task for a CTRW memory function of widespread use; we start, however, by recalling several basic CTRW features. For the theoretical description of anomalous diffusion CTRW models have proven to be a very adequate tool.[1,4-8,11] The basic ingredients of such models are the waiting time distributions $\psi(\mathbf{r},t)$; $\psi(\mathbf{r},t)$ gives the probability distribution to step a distance $\mathbf{r}$ in the time interval from t to $t+dt$.

The general CTRW procedure is complex. However, a considerable simplification is achieved by taking for $\psi(\mathbf{r},t)$ decoupled forms[11]

$$\psi(\mathbf{r},t)=\lambda(\mathbf{r})\psi(t) . \tag{3}$$

In this case the temporal behavior of several basic quantities such as $\langle r^2(t)\rangle$ may depend on the waiting-time distribution $\psi(t)$ only. Exemplarily, for waiting-time distributions which asymptotically obey

$$\psi(t)\sim t^{-1-\gamma},\ \gamma<1 , \tag{4}$$

one finds that $\langle r^2(t)\rangle \sim t^{\gamma}$, i.e., $\alpha=\gamma$. Furthermore, for γ values well below the critical value $\gamma_c=1$, one has in the pretransit region $I(t)\sim t^{\gamma-1}$,[12] which reproduces a classical result of Scher and Montroll.[1] Here one may note that $\epsilon=\gamma-1$; i.e., $\epsilon<0$.

Unfortunately, decoupled memories cannot be used to model enhanced motion; as previously stressed,[6] only dispersive or regular behavior may result from Eq. (3). The description of enhanced diffusion in terms of the CTRW approach requires coupled memory kernels.[6]

Most technical difficulties which arise from coupled memories can, however, be alleviated by choosing as waiting-time distribution the form[5-8]

$$\psi(\mathbf{r},t)=Ar^{-\mu}\delta(r-t^{\nu}) . \quad (5)$$

This expression provides a very good means to describe anomalous diffusion. Through the δ function the position $\mathbf{r}$ and the time t are coupled; at a given time t positions far away from the occupied site are not yet accessible, and nearby positions are no more accessible. Moreover, from a technical point of view, the δ function renders some quite cumbersome integrals manageable.

In former works we used the coupled memory approach in our study of the mean-squared displacement $\langle r^2(t)\rangle$ (Refs. 5-7) and of $S(t)$, the mean number of distinct sites visited.[8] The connection of coupled memories to experimental situations was made for superlinear phase diffusion in Josephson junctions[5] and for turbulent diffusion[13] using Lévy walks. In this work we apply the coupled memory approach to the analysis of current flow under dispersive and enhanced diffusion conditions.

We first note that the current $\mathbf{I}(t)$ in a biasing field can be evaluated as being the time derivative of the mean position $\langle\mathbf{r}(t)\rangle$ of a charge carrier, $\mathbf{I}(t)=(d/dt)\langle\mathbf{r}(t)\rangle$. On the other hand, $\langle\mathbf{r}(t)\rangle$ follows directly from the knowledge of $P(\mathbf{r},t)$, the probability distribution to be at $\mathbf{r}$ at time t, provided that the particle started at $t=0$ from $\mathbf{r}=0$; thus one has

$$\langle\mathbf{r}(t)\rangle=\sum_{\mathbf{r}}\mathbf{r}P(\mathbf{r},t) . \quad (6)$$

Now $P(\mathbf{r},t)$ obeys the following generalized Master equation:[1,7]

$$P(\mathbf{r},t)=\sum_{\mathbf{r}'}\int_0^t P(\mathbf{r}',\tau)\psi(\mathbf{r}-\mathbf{r}',t-\tau)d\tau+\Phi(t)\delta_{\mathbf{r},0} . \quad (7)$$

In Eq. (7), $\Phi(t)$ is the probability of not having until t left the starting site of the walk. One can express $\Phi(t)$ as a function of $\psi(t)=\sum_{\mathbf{r}}\psi(\mathbf{r},t)$:

$$\Phi(t)=1-\int_0^t\psi(\tau)d\tau . \quad (8)$$

For the analytical calculation of $\mathbf{I}(t)$ it is advisable to use the Laplace-Fourier transform ($t\rightarrow u$, $\mathbf{r}\rightarrow\mathbf{k}$) of Eq. (7), which results in

$$P(\mathbf{k},u)=\frac{1-\psi(u)}{u}\frac{1}{1-\psi(\mathbf{k},u)} . \quad (9)$$

The sum in Eq. (6) can be expressed by means of the derivative of the Fourier transform, and one finds for the current:

$$\mathbf{I}(u)=-iu\frac{\partial}{\partial\mathbf{k}}P(\mathbf{k},u)\bigg|_{\mathbf{k}=0}=\frac{-i}{1-\psi(u)}\frac{\partial\psi(\mathbf{k},u)}{\partial\mathbf{k}}\bigg|_{\mathbf{k}=0} . \quad (10)$$

Equation (10) may be compared to the expression for the mean-squared displacement of a particle in a CTRW without bias:[6]

$$\langle r^2(t)\rangle=-\frac{\partial^2}{\partial\mathbf{k}^2}P(\mathbf{k},t)\bigg|_{\mathbf{k}=0} . \quad (11)$$

We now utilize Eq. (10) and, for strong biasing fields, restrict ourselves to evaluating the current in the direction of the field. This corresponds to a one-dimensional picture, in which all steps are taken in the same direction: formally one considers only $\psi(r,t)$ with $r>0$. (See Refs. 2 and 12 for a detailed justification of the procedure.)

Let us start our evaluation of the current from decoupled memories, Eq. (3). Here the calculation of the current is straightforward and one obtains:

$$I(u)=L\frac{\psi(u)}{1-\psi(u)} , \quad (12)$$

with $L=-i\lambda'(\mathbf{k}=0)$, where L is the mean distance traveled per step in the direction of the bias.[2,12,14]

For coupled memories the procedure is more subtle. We have to compute the large-t (i.e., small u) behavior of $I(u)$ from Eq. (10). Note, however, that the limit $k\rightarrow 0$ is to be taken in Eq. (10) for all u. Hence the analysis of the asymptotic behavior of $I(t)$ requires the evaluation of $\psi(k,u)$ in the limit of small k and u, where $k\ll u$. The situation is somewhat analogous to the calculation of $\langle r^2(t)\rangle$ in the absence of external fields.[8]

Let us focus first on the space-averaged waiting-time distribution $\psi(t)$, whose Laplace transform is

$$\begin{aligned}\psi(u)&=\int_0^\infty dt\int_0^\infty\psi(\mathbf{r},t)e^{-ut}d\mathbf{r}\\&=A\int_0^\infty\int_{\theta_1}^\infty r^{-\mu}\delta(r-t^{\nu})e^{-ut}drdt\\&=A\int_{\theta_2}^\infty t^{-\nu\mu}e^{-ut}dt .\end{aligned} \quad (13)$$

In Eq. (13) we indicated the lower-bound cutoffs by θ_i. From the normalization condition $\psi(u=0)=1$, one has as a condition on the product $\nu\mu$ of the exponents $\nu\mu>1$. Moreover, for $\nu\mu>2$ a finite mean waiting time τ_1 is obtained. Hence, for the asymptotic expansion of $\psi(u)$ it follows that

$$1-\psi(u)\sim u^{\gamma},\ \text{where}\ \gamma=\min(\mu\nu-1,1) . \quad (14)$$

In order to evaluate $I(u)$ according to Eqs. (9) and (10), we still need to establish the asymptotic behavior of $\psi(k,u)-\psi(u)$. The procedure is as follows:[8] one computes

$$\begin{aligned}\psi(k,u)-\psi(u)&=\int_0^\infty dt\int_0^\infty(e^{ikr}-1)\psi(r,t)e^{-ut}dr\\&=A\int_0^\infty e^{-ut}dt\int_{\theta_1}^\infty r^{-\mu}(e^{ikr}-1)\\&\qquad\times\delta(r-t^{\nu})dr\\&=A\int_{\theta_2}^\infty e^{-ut}t^{-\nu\mu}(e^{ikt^{\nu}}-1)dt .\end{aligned} \quad (15)$$

For $kt^{\nu}\ll ut$ the term in parenthesis changes little during the decay of $\exp(-ut)$ and may be expanded in powers of k. Thus for $k\ll u$ one has

$$\psi(\mathbf{k},u)-\psi(u)\sim ik\int_{\theta_2}^\infty t^{-\nu(\mu-1)}e^{-ut}dt . \quad (16)$$

Now, for $\nu(\mu-1)>1$, the integral Eq. (16) is finite for all u, even for $u=0$. On the other hand, for $\nu(\mu-1)<1$ the integral diverges for $u=0$; only $u>0$ values give a finite integral, whose leading $u^{\nu\mu-\nu-1}$ dependence can be established by a variable transformation. Both cases can be summarized through

$$\psi(k,u)-\psi(u)\sim iku^{\delta}, \text{ where } \delta=\min(\mu\nu-\nu-1,0). \tag{17}$$

Inserting Eqs. (14) and (17) into Eq. (10) we find for the current

$$I(u)\sim u^{\delta-\gamma}. \tag{18}$$

Now, due to the combination of the two exponents δ and γ, Eqs. (14) and (17), four different cases arise. Reverting Eq. (18) to the time domain we obtain the final forms

$$I(t)\sim t^{\nu-1} \text{ for } 1<\nu\mu<\min(2,1+\nu), \tag{19a}$$

$$I(t)\sim t^{\nu\mu-2} \text{ for } 1+\nu<\nu\mu<2, \tag{19b}$$

$$I(t)\sim t^{1-(\mu-1)\nu} \text{ for } 2<\nu\mu<1+\nu, \tag{19c}$$

$$I(t)\sim \text{const for } \nu\mu>\max(2,1+\nu). \tag{19d}$$

Equations (19) now provide a unified description for the current, whose interpretation is as follows: In the dispersive regime ($\nu<1$) we find for small μ values that the current decreases with time, $I(t)\sim t^{\epsilon}$, $\epsilon<0$, and that ϵ is independent of μ, Eq. (19a); this is the dispersive Scher-Montroll behavior.[1] For large μ values the current shows regular diffusive behavior, Eq. (19d). For intermediate μ values the current is dispersive with ϵ being linearly dependent on μ, Eq. (19b).

In the enhanced anomalous regime ($\nu>1$) and for small μ values the current is enhanced, Eq. (19a). For large μ the current is again regular, Eq. (19d). For intermediate μ values one finds enhanced current with the exponent depending linearly on μ, Eq. (19c). Both for the dispersive and for the enhanced current regimes the (μ,ν) dependence of ϵ shows two marginal crossovers: $\mu\nu=1+\nu$ and $\mu\nu=2$.

Now, one may object that our analytical results are, after all, only asymptotically valid. Thus, in order to establish the validity range of Eqs. (19), we also performed Monte Carlo simulations. The procedure used is as follows:[7] For each step of a CTRW the distance r and the time t to reach the next location are determined according to the distribution given by Eq. (5). The total displacement $r(t)$ of the walker is then evaluated as a cumulated distance and the average is taken over typically 10^5 realizations. Then the current flow, $I(t)=(d/dt)\langle r(t)\rangle$, is determined using a standard procedure for the numerical calculation of the time derivative. Finally, from a linear least-squares fit to the $\log I(t)$-$\log t$ representation, the exponents ϵ are obtained for the time interval $10^3<t<10^4$.

In Fig. 1 we show the typical behavior for $I(t)$ as a function of time. We use log-log scales and present the findings for several μ values. The upper part of the figure gives the current for an enhanced situation, $\nu=2$. The lower part of the figure depicts the current under a dispersive condition, $\nu=\frac{1}{2}$. We note that at long times the numerical results follow straight lines satisfactorily close, a fact which validates our asymptotic analysis; deviations from linear behavior are only visible for short times.

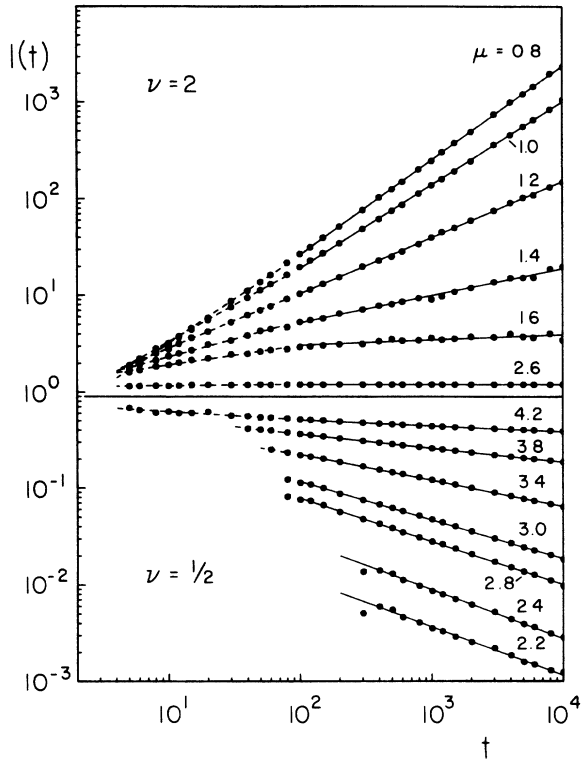

FIG. 1. The current flow $I(t)$ plotted as a function of time on log-log scales. In the upper part of the figure an enhanced situation, $\nu=2$, is shown; in the lower part a dispersive situation, $\nu=\frac{1}{2}$, is depicted. The values of μ are given parametrically, as indicated. The dots are simulation results, the solid lines denote linear least-squares fits, and the dashed lines are guides to the eye.

The numerical results for $\nu=\frac{1}{2}$, $\frac{3}{2}$, and 2 are summarized in Fig. 2. The fitted slopes agree reasonably well with the analytical results, Eqs. (19), which are also given in the figure as solid lines. Here the transition regimes $3<\mu<4$ for $\nu=\frac{1}{2}$, $\frac{4}{3}<\mu<\frac{5}{3}$ for $\nu=\frac{3}{2}$, and $1<\mu<\frac{3}{2}$ for $\nu=2$ are clearly visible. In all three cases, for

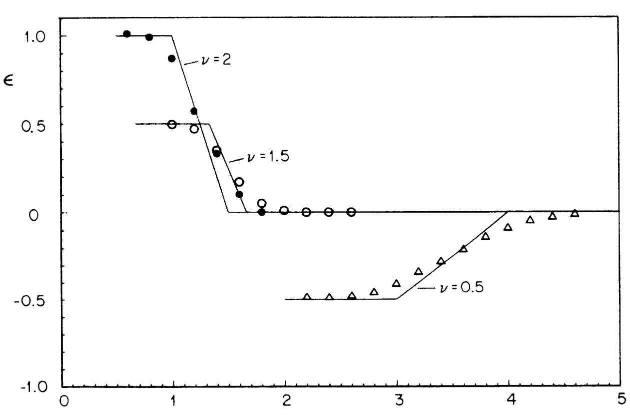

FIG. 2. Asymptotic behavior of the current; the symbols denote the numerically determined exponents for $\nu=\frac{1}{2}$, $\frac{3}{2}$, and 2. The solid lines give the analytically expected behavior, Eqs. (19).

small μ, the limiting ϵ values, $\epsilon = -\frac{1}{2}$ for dispersive current and $\epsilon = \frac{1}{2}$, $\epsilon = 1$ for enhanced current, are reached within the time of the numerical experiment. Whereas in general the numerical results follow the theoretical predictions, the crossover behavior is considerably smoother than expected theoretically. We attribute this finding to logarithmic corrections to scaling and to the very long time which is required to reach the asymptotic regime of $I(t)$ for μ close to the marginal μ values.

Concluding this study, we have evaluated the currents under anomalous diffusion conditions, both for dispersive and for enhanced motion. For this we had to use the full CTRW formalism, with coupled space-time memories. Our task was rendered easier by using a very convenient waiting-time distribution form and by an analytically compact expression for the current in the Fourier-Laplace domain. The determined asymptotic forms agree well with the simulations and show a very rich behavior. We hope that our findings will further enhance the interest in experimentally investigating currents in turbulent motion.

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