Analysis of the breakdown of continuum and semidiscrete approximation of a nonlinear model for the DNA double helix

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We show that the continuum and semidiscrete approximations break down for the propagation of pulses in a nonlinear model developed to mimic some dynamic aspects of the DNA double helix. The model assumes nonlinear behavior for the interbase hydrogen bonds. The effects of longer-range interactions and additional nonlinearities of the system are explored. Numerical solutions showing the effects of these additional factors are displayed.

The search for an energy transport mechanism in the DNA molecule has been the subject of intense research efforts, and, in particular, the exciting possibility that solitonlike excitations play a key role in energy transport has received much attention.[1] The importance of nonlinearities in the DNA molecule, and more precisely the nonlinearity of the hydrogen bonds between the nucleotides, was pointed out recently by Prohofsky.[2] In order to treat these nonlinearities, we studied a simple model of the DNA chain in which each nucleotide was approximated by a single point mass m, the nonlinearity of the hydrogen bonds between the nucleotides was modeled by a Morse potential, and a harmonic coupling corresponding to neighboring bases' stacking was added. The displacements along the direction of the hydrogen bonds of the two nucleotides in a base pair are denoted u_n and v_n. The hope was that the nonlinearity of the hydrogen bonds might give rise to solitonlike excitations. A numerical study of this model has been presented elsewhere.[3] The essential conclusions are that rather broad, sufficiently large-amplitude pulses propagate easily along the molecule, but in this case the effect is purely harmonic and has nothing to do with the nonlinearity of the hydrogen bonds; on the other hand, narrow, smaller-amplitude pulses are found to propagate, the effect being seemingly a nonlinear effect. For a large range of amplitudes and widths, we also noticed that a pulse can remain trapped (i.e., does not propagate).

In our previous paper, we discussed briefly the use of the multiple-scale expansion in the continuum limits to study the propagation of small-amplitude pulses.[3] However, the fact that only narrow pulses seems to propagate suggests that a semidiscrete or discrete analysis might be more appropriate. In this paper we present an analysis of the system in the semidiscrete approximation via the multiple-scale expansion. We find that even the semidiscrete approximation does not describe the dynamics of the system adequately, and a discrete approach seems to be in order. First, the simple model described in Ref. 3 is examined. The analysis is then repeated for a somewhat more complicated model including next-nearest-neighbor interaction as well as a cubic nearest-neighbor interaction. In this latter case, a brief numerical study of the dynamics is presented.

First of all, consider the following equation of motion:[3]

$$m\ddot{y}_n = k(y_{n+1}+y_{n-1}-2y_n) - \frac{\partial V}{\partial y_n} \tag{1}$$

where $y_n=(u_n-u_n)/\sqrt{2}$ and $V(y_n)$ is the Morse potential. The first step in the multiple-scale expansion method is to expand the potential $V(y_n)$ in a power series:

$$\begin{aligned} V(y_n) &= V_\infty[1-\exp(-a\sqrt{2}y_n)]^2 \\ &\simeq 4V_\infty a^2\left[y_n - \frac{3\sqrt{2}}{2}y_n^2+\tfrac{7}{3}a^2y_n^3\right] . \end{aligned} \tag{2}$$

The relative displacement y_m is expanded in the following way:

$$y_n = \sum_{\nu=0}^{+\infty}\sum_{m=0}^{+\infty}\epsilon^\nu F_{n,m}(\epsilon^\nu t,\epsilon^\nu x)\exp(im\theta_n)+\text{c.c.} , \tag{3}$$

where $\theta_m = nqd-\omega t$. ϵ is the scaling parameter of the multiple-scale expansion and d is the separation between base pairs. It can be shown[4,5] that only a few terms contribute, to order ϵ^2:

$$\begin{aligned} y_n &= \epsilon F_1(\epsilon nd,\epsilon t)\exp(i\theta_n)+\epsilon F_1^*(\epsilon nd,\epsilon t)\exp(-i\theta_n) \\ &\quad +\epsilon^2[F_0(\epsilon nd,\epsilon t)+F_2(\epsilon nd,\epsilon t)\exp(2i\theta_n) \\ &\quad +F_2^*\exp(-2i\theta_n)] . \end{aligned} \tag{4}$$

Upon substitution of this expansion into Eq. (1) and after collecting the terms corresponding to the different powers of $\exp(i\theta_n)$, one obtains three equations for $F_{1,n}$, $F_{2,n}$, and $F_{0,n}$. At this point a continuum approximation is made for these envelope functions and henceforth we shall drop the subscript n. Notice that we embrace the continuum approximation for the envelope function only, the "carrier" is treated exactly. $F_0(\epsilon t,\epsilon x)$ and $F_2(\epsilon t,\epsilon x)$ can then be expressed in terms of $F_1(\epsilon t,\epsilon x)$ and the equation governing the evolution of F_1 is found to be the nonlinear Schrödinger equation:

$$i\frac{\partial F_1}{\partial s}+P\frac{\partial^2 F_1}{\partial z^2}+Q|F_1|^2F_1=0 , \tag{5}$$

where

$$z=\epsilon(x-v_g t) , \tag{6}$$

$$s = \epsilon^2 t \ , \tag{7}$$

$$v_g = \frac{kd}{m\omega}\sin(qd) \ , \tag{8}$$

$$\omega^2 = \frac{4V_\infty a^2}{m} + \frac{4k}{m}\sin^2\left[\frac{qd}{2}\right] \ , \tag{9}$$

and the coefficients P and Q are given by

$$P = \frac{kd^2}{2m\omega}\cos(qd) - \frac{k^2 d^2}{2m^2\omega^3}\sin^2(qd) \ , \tag{10}$$

$$Q = \frac{8V_\infty a^4}{m\omega}\left[\frac{6+(11k/V_\infty a^2)\sin^4(qd/2)}{3+(4k/V_\infty a^2)\sin^4(qd/2)}\right] . \tag{11}$$

The solution of the nonlinear Schrödinger equation is of course well known;[6] in our case y can be written, after substitution of the general solution of the nonlinear Schrödinger equation in the first two terms of Eq. (4);

$$y = \epsilon A \ \mathrm{sech}[\epsilon L_x(x - v_e t)]\cos(q_c x - \omega_c t) + O(\epsilon^2) \ , \tag{12}$$

where

$$A = \left[\frac{u_e(u_e - 2u_c)}{2PQ}\right]^{1/2} \ , \tag{13}$$

$$L_e = \left[\frac{u_e(u_e - 2u_c)}{2P^2}\right]^{-1/2} \ , \tag{14}$$

are the amplitude and width of the envelope, respectively, and

$$v_e = v_g + \epsilon u_e \ , \tag{15}$$

$$q_c = q + \frac{\epsilon u_e}{2P} \ , \tag{16}$$

$$\omega_c = \omega + \epsilon\frac{u_e v_g}{2P} + \epsilon^2\frac{u_e u_c}{2P} \ . \tag{17}$$

u_e and u_c are arbitrary integration constants. q_c and ω_c are the wave number and frequency of the "carrier" wave and v_e the velocity of the "envelope."

We now proceed to a careful discussion of Eqs. (15)–(17). The constants v_e, q_c, and ω_c are first set equal to reasonable values (for the DNA molecule), for instance v_e is approximately the speed of sound.[7] Equation (16), together with Eqs. (15), (8), and (9) lead to an algebraic equation for q. Once q has been determined, Eqs. (15) and (17) can be used to obtain the constants u_e and u_c and finally, the amplitude A and the width L_e of the pulse [Eqs. (13) and (14)]. The equation for q, being a transcendental equation, has to be solved numerically. The graph shown in Fig. 1 is very suggestive; the line $q_c - q$ and the curve $\epsilon u_e(q)/2P(q)$ were plotted versus q. Their intersections correspond to solutions of the algebraic equation for q. Notice that only positive values of P should be considered (one has to impose this restriction in order to have solitons in the nonlinear Schrödinger equation). Hence, for a given q_c, there is always a finite number of solutions. Furthermore, there is generally always at least one solution, unless q_c is too small. Even more interesting is the fact that because of the requirement that P be positive, there are "bands" of allowed and forbidden values for q. There is an additional restriction, namely, the product $u_e(u_e - 2u_c)$ must be positive in order for the amplitude and the width of the soliton to be real numbers. If, for example, v_e is too small, or ω_c too large, this latter condition cannot be met and the problem has no solution at all. In Fig. 2, we show the value of the amplitude A and width L_e of the soliton in the first two

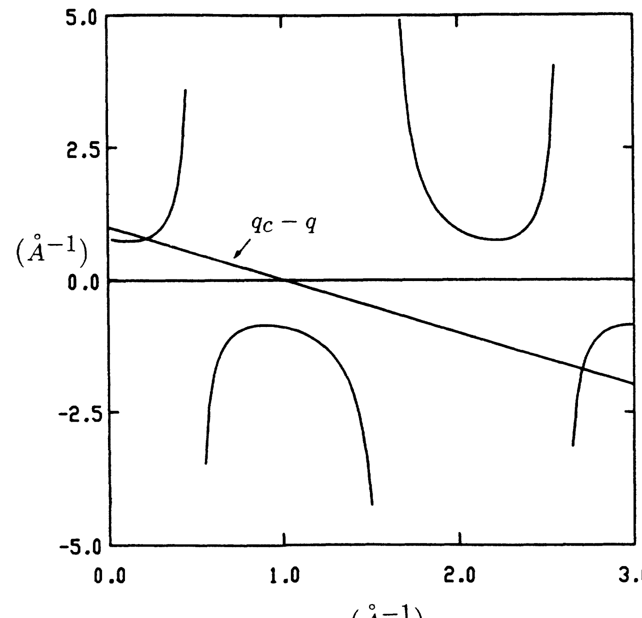

FIG. 1. Solution of the equation for q. The parameters used are $v_e = 10$ Å/ps, $\omega_c = 18$ rad/ps, $V_\infty = 0.4$ eV, $k = 0.88$ eV/Å^2, $d = 3$ Å, and $m = 0.04$ eV/Å^2 ps^2.

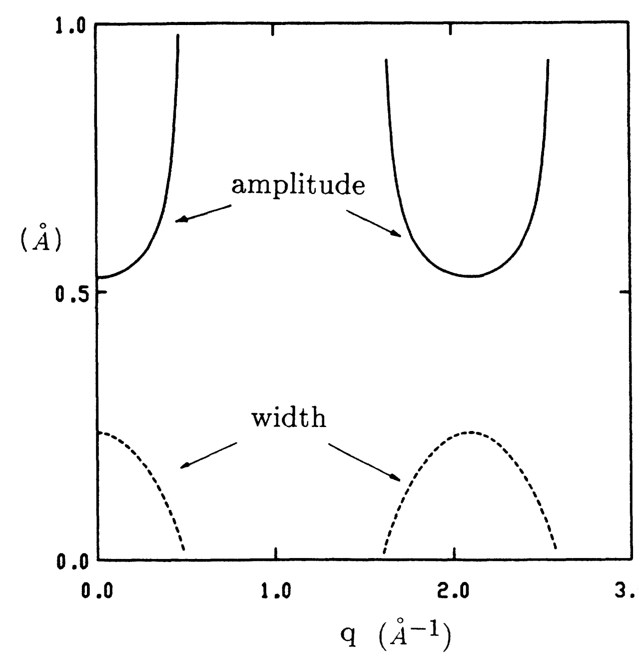

FIG. 2. The solid line is the amplitude (in Å) of the soliton vs wave number q. The dashed line is the soliton width (in Å). The various parameters are the same as for Fig. 1. The equation of motion is Eq. (1).

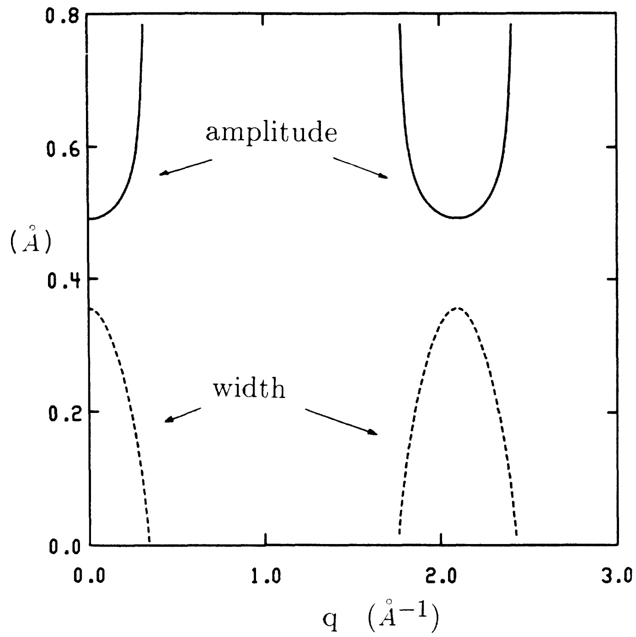

FIG. 3. The solid line is the amplitude (in Å) of the soliton vs wave number q. The dashed line is the soliton width (in Å). The various parameters are the same as for Fig. 1, $k_2=0.2$ eV/Å^2 and $\lambda=0.1$ eV/Å^4. The equation of motion is Eq. (18).

bands for given values of v_e and ω_c. The smallest amplitude is roughly 0.5 Å, far too large for the power expansion of the Morse potential, Eq. (2), to be valid. Even worse, the width of the soliton does not exceed a few tenths of an angstrom, smaller than the separation between base pairs ($\simeq 3$ Å). This, of course, invalidates *a posteriori* a semidiscrete analysis, and more precisely the approximation which consists in treating the envelope function $F_{i,n}$ in the continuum limit. Keeping more terms in the expansion of the Morse potential does improve the situation somewhat, as far as the amplitude restriction is concerned, but the soliton remains very narrow.

The model can be expanded to take into account next-nearest-neighbor interactions and cubic interaction terms between base pairs. Both of these factors are likely to be important in modeling DNA. Although DNA is called a nucleic acid, it is in its biologically active state a salt. Net charge is distributed over the atoms of the helix. The shielding counterions are restricted to regions on the outer edge of the helix. Longer-range electromagnetic interactions can be shown to be important.[8] The linear interaction term k_1, is in the double helix a shear modulus between stacked bases along the helix. These bases are finite in size and one expects that the actual area of overlap between the bases will change with relative displacement between adjacent planar bases. In the large-amplitude solutions discussed earlier, the displacement can become an appreciable fraction of the length over which the shear interaction acts. One would then expect nonlinearities in the shear interaction. More precisely, the equation of motion considered now is

$$m\ddot{y}_n = k_1(y_{n+1}+y_{n-1}-2y_n)+k_2(y_{n+2}+y_{n-2}-2y_n)+\lambda[(y_{n+1}-y_n)^3+(y_{n-1}-y_n)^3]-\frac{\partial V}{\partial y_n} . \quad (18)$$

At this point, it is worth stressing that the cubic interaction term in the equation of motion was chosen deliberately to be of the form indicated in Eq. (18) in order to make subsequent calculations easier. The essential motivation for including this term was to have nonlinear interactions acting between nearest neighbors. In the simpler version of the model, Eq. (1), the nonlinear term was "local" (i.e., depended on n only). The dispersion relation for this model is

$$\omega^2=\frac{4V_\infty a^2}{m}+\frac{4k_1}{m}\sin^2\left[\frac{qd}{2}\right]+\frac{4k_2}{m}\sin^2(qd) , \quad (19)$$

and reduces to Eq. (9) when $k_2=0$. The group velocity $v_g=d\omega/dq$ and the parametrization is the same as above.

The multiple-scale expansion in the semidiscrete approximation can be carried out again, in essentially the same way as before. A rather long calculation shows that one still obtains, quite remarkably, the nonlinear Schrödinger equation, Eq. (5), for the envelope function F_1, but the coefficients P and Q are now given by

$$P=\frac{k_1d^2}{2m\omega}\cos(qd)+\frac{2k_2d^2}{m\omega}\cos(2qd)-\frac{k^2d^2}{2m^2\omega^3}\sin^2(qd) , \quad (20)$$

$$Q=\frac{8V_\infty a^4}{m\omega}\left[\frac{6+(11k_1/V_\infty a^2)\sin^4(qd/2)+(11k_2/V_\infty a^2)\sin^4(qd)}{3+(4k_1/V_\infty a^2)\sin^4(qd/2)+(4k_2/V_\infty a^2)\sin^4(qd)}-6\lambda\sin^4\left[\frac{qd}{2}\right]\right] , \quad (21)$$

and reduce to the coefficients obtained previously for $k_2=0$ and $\lambda=0$. The solution is still given by Eq. (12) and Eqs. (15)–(17) remain unchanged. In particular, there are again allowed and forbidden bands for q and the amplitude and width of the solution exhibit a behavior similar to that shown in Fig. 2 (Fig. 3). The presence of next-nearest-neighbor interactions and cubic interaction terms hardly affects the solution; the amplitude is virtually unchanged whereas the envelope becomes somewhat broader. The allowed bands for q have also become narrower. Again, the semidiscrete limit does not provide a good description of the system.

It is interesting to see how exactly the additional terms in the equation of motion, Eq. (18), affect the propagation of pulses along the DNA chain. To this end, we proceeded to a numerical integration of Eq. (18). Reference 3 contains the details of the algorithm used as well as the results of a detailed study of the simpler model. As in

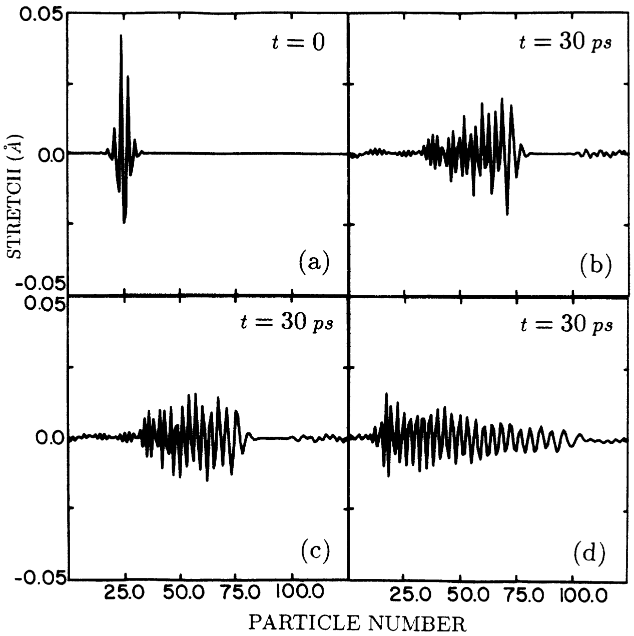

FIG. 4. Propagation of a solution of the nonlinear Schrödinger equation along the DNA chain. (a) $t=0$ ps, (b) $t=30$ ps with $k_2=0$, (c) $t=30$ ps with $k_2=0.1k_1$, (d) $t=30$ ps with $k_2=0.4k_1$. In all cases, $\lambda=0$.

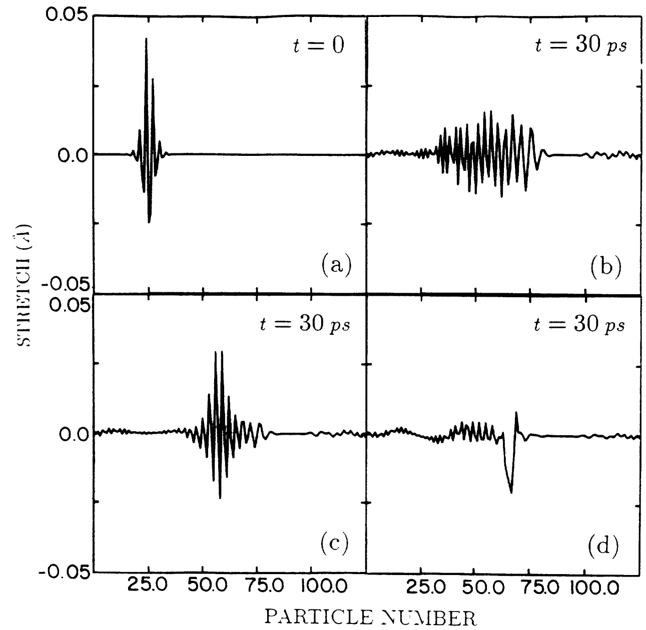

FIG. 5. Propagation of a solution of the nonlinear Schrödinger equation along the DNA chain. (a) $t=0$ ps, (b) $t=30$ ps with $\lambda=0$, (c) $t=30$ ps with $\lambda=26.6$ eV/Å^4, (d) $t=30$ ps with $\lambda=88.8$ eV/Å^4. In all cases, $k_2=0.1k_1$.

Ref. 3, we study the propagation along the chain of small-amplitude solutions of the nonlinear Schrödinger equation. This choice was motivated by the results of the multiple-scale analysis. There is, however, no good reason why a solution of this form should propagate. Other pulse shapes with comparable amplitudes and widths do not propagate (except for extremely small Morse potentials, but then the effect is purely harmonic). This is an indication that a nonlinear Schrödinger type of solution is probably closely related to the exact solution of the (discrete) system. Even though, strictly speaking, the multiple-scale analysis breaks down, it remains a useful guide.

First, if λ is set equal to zero, i.e., if we consider the effect of the next-nearest-neighbor interaction only, it can be seen from Fig. 4 that the initial pulse propagates, but disperses rather rapidly, and the larger k_2, the higher the dispersion rate. The cubic term in the equation of motion has exactly the opposite effect, namely, it limits the dispersion rate (Fig. 5). Compare in particular Figs. 4(c) and 5(c); they correspond to identical situations, but $\lambda=0$ in the former and $\lambda=26.6$ eV/Å^4 in the latter.

Finally, it is worth mentioning that for other types of pulses (unrelated to the nonlinear Schrödinger equation, say a hyperbolic secant), which under Eq. (1) tend to remain localized, the presence of the cubic interaction term makes the localization more difficult. This term tends to force an initial pulse, even with zero initial velocity, to propagate. This phenomenon is typical[9] of situations where a nonlinear force acts *between* two particles in the chain as opposed to a nonlinear force acting on each particle individually. Hence, should the stacking interaction in DNA include a sufficiently large nonlinear term, the propagation of energy packets would be favored and could pay a role in energy transport along the double helix. On the other hand, the presence of long-range linear interactions tends to disperse the pulses rapidly. Given the uncertainty on the exact numerical values of these effects in real DNA, it is difficult to make a conclusion on what situation should be expected in DNA.

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