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Ulam model for the sine-Gordon soliton system

F. Kh. Abdullaev, S. A. Darmanyan, and B. A. Umarov
Department of Thermophysics, Uzbek Academy of Sciences, Tashkent 700 135,
"Pravda" str.28 Union of Soviet Socialist Republics
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The Ulam model analog of the Fermi acceleration for the sine-Gordon solitons is investigated. The motion of soliton between two potential walls, one of which is periodically oscillating, is studied. With the help of the Melnikov method the expression for the threshold of stochasticity in soliton motion is found.

The investigation of stochastic dynamics of various autostructures in nonlinear media (solitons, bions, autowaves, etc.) represents one of the most interesting problems in modern theory of nonlinear waves. A great number of papers deals with the numerical and analytical studies of chaotic soliton behavior under the action of the external periodic fields.[1–3] In this case there is an analogy between solitons and particles moving under the action of forces.

Ulam's model describing the particle motion between two walls, one of which is oscillating, represents one of the main models for the investigation of particle chaotic motion. It has been applied to the description of the Fermi mechanism of stochastic acceleration of particles. In Refs. 4 and 5 it was shown that at some conditions the particle motion becomes stochastic, and stochastic acceleration of the particle up to some boundary velocity v_0 takes place. In Ref. 4 it was considered the particle motion on the oscillating plate in gravitational field, i.e., the unmoving wall is changed by gravitational field. As shown, the acceleration is boundless in this case.

In this paper we study the analog of the Ulam model for solitons. We make use of the sine-Gordon (SG) model. This generalization of the solitonic case can be very useful for the investigation of the stochastic behavior of solitons in oscillating potential.

Let us consider the problem of the SG soliton motion between two potential barriers, one of which is periodically oscillating. The wave equation takes the form

$$\phi_{tt}-\phi_{xx}+\sin\phi = -\epsilon[\delta(x-L)+\delta(x+L+a\sin(\omega t))]\sin\phi\ , \quad (1)$$

where $\epsilon \ll 1$ and $a \ll 1$. This perturbed sine-Gordon equation arises, for example, in the problem of the magnetic soliton motion between two impurities, one of which is oscillating. Another example can be presented by the problem of kink motion in the long Josephson junction with the nonstationary micro-short-circuit.[6]

Let us investigate the dynamics of a single soliton

$$\phi_s(x,t)=4\tan^{-1}\left[\exp\left[\sigma\frac{x-\xi(t)}{(1-v^2)^{1/2}}\right]\right]\ . \quad (2)$$

$\sigma=\pm1$ corresponds to kink(+) or antikink(−).

Using the perturbation theory for the SG solitons, one obtains the equation system for the velocity $v(t)$ and coordinate $\xi(t)$ of the soliton center[7]

$$\frac{dv}{dt}=\frac{\epsilon}{2}(1-v^2)\left[\operatorname{sech}^2\left[\frac{\xi-L}{(1-v^2)^{1/2}}\right]\tanh\left[\frac{\xi-L}{(1-v^2)^{1/2}}\right]+\operatorname{sech}^2\left[\frac{\xi+L+a\sin(\omega t)}{(1-v^2)^{1/2}}\right]\tanh\left[\frac{\xi+L+a\sin(\omega t)}{(1-v^2)^{1/2}}\right]\right]\ , \quad (3)$$

$$\frac{d\xi}{dt}=v-\frac{\epsilon}{2}v\left[\operatorname{sech}^2\left[\frac{L-\xi}{(1-v^2)^{1/2}}\right]\tanh\left[\frac{L-\xi}{(1-v^2)^{1/2}}\right](L-\xi) +[L+\xi+a\sin(\omega t)]\operatorname{sech}^2\left[\frac{\xi+L+a\sin(\omega t)}{(1-v^2)^{1/2}}\right]\tanh\left[\frac{\xi+L+a\sin(\omega t)}{(1-v^2)^{1/2}}\right]\right]\ . \quad (4)$$

Then we consider the case of small soliton velocities $v^2 \ll 1$. Equations (3) and (4) take the form

$$\frac{dv}{dt}=\frac{\epsilon}{2}[\operatorname{sech}^2(\xi-L)\tanh(\xi-L)+\operatorname{sech}^2(\xi+L)\tanh(\xi+L)]+\frac{\epsilon a}{2}[1-2\sinh^2(\xi+L)]\operatorname{sech}^4(\xi+L)\sin(\omega t)\ ,$$
$$\frac{d\xi}{dt}=v\ . \quad (5)$$

This system is analogous to the equations system describing the motion of particles with the unit mass in the anharmonic potential $U(\xi)$,

$$U=\frac{\epsilon}{4}[\mathrm{sech}^2(\xi+L)+\mathrm{sech}^2(\xi-L)] . \tag{6}$$

Analyzing the behavior of the system (5) and (6) at $a=0$, we obtain

$$\frac{d\xi}{dt}=\pm\left[2H-2\epsilon\frac{(\alpha y+1)}{(y+\alpha)^2}\right]^{1/2} , \tag{7}$$

where H is unperturbed Hamiltonian ($a=0$), $y=\cosh 2\xi$, and $\alpha=\cosh 2L$.

Let us further investigate the behavior of the solutions near the separatrix. The separatrix is defined by the condition

$$H_c=\frac{\epsilon}{4}\frac{\alpha^2}{(\alpha^2-1)} , \tag{8}$$

where H_c is the value of total Hamiltonian on the separatrix. From (7) we obtain

$$(2H_c)^{1/2}(t-t_0)=\xi-\frac{\alpha+y_0}{2(y_0^2-1)^{1/2}}\ln\left[\frac{y_0-1+(y_0^2-1)^{1/2}\tanh\xi}{y_0-1-(y_0^2-1)^{1/2}\tanh\xi}\right] ,$$
$$y_0=\alpha-2/\alpha . \tag{9}$$

It is convenient to make use of the Melnikov method[8] in order to investigate the threshold of appearance chaosticity and characteristics of the SG soliton random dynamics. For this purpose we calculate the Melnikov function $M(t_0)$ characterizing the width of stochastic layer being formed near the separatrix

$$M(t_0)=\frac{\epsilon a}{2}\int_{-\infty}^{\infty}v(t)\sin(\omega t)\{1-2\sinh^2[\xi(t)+L]\}\times\mathrm{sech}^4[\xi(t)+L]dt . \tag{10}$$

Introducing the new variable of integration ξ we obtain

$$M(t_0)=\frac{\epsilon a}{2}\int_{-\xi_0}^{\xi_0}\sin[\omega t(\xi)]\mathrm{sech}^4(\xi+L)\times[1-2\sinh^2(\xi+L)]d\xi ,$$
$$t(\xi)=f(\xi)+t_0,\quad \xi_0=\tfrac{1}{2}\mathrm{arccosh}y_0 . \tag{11}$$

Here $f(\xi)$ is defined by the right-hand side of (9) multiplied by $1/(2H_c)^{1/2}$. Considering $L\gg1$ and applying (8) and (11), we obtain the following expression for the Melnikov function:

$$M(t_0)=\frac{a\epsilon\omega_1^2}{4\sqrt{\pi}}\mathrm{Im}[\exp(i\omega_1L+i\omega t_0)\times\Gamma(i\omega_1/2)\Gamma(\tfrac{1}{2}-i\omega_1/2)] ,$$

where $\omega_1=\omega/(2H_c)^{1/2}\approx\omega\sqrt{2/\epsilon}$ and $\Gamma(z)$ is the γ function. Using the formula for the asymptotic Γ function we find that at $\omega_1\gg1$,

$$M(t_0)\approx a\epsilon\omega_1^{3/2}\exp(-\pi\omega_1/2)\sin(\omega t_0+\omega_1 L) .$$

The chaotic motion arises if $M(t_0)$ changes sign at arbitrary t_0. It is obvious that this condition is performed and the width of chaotic layer is defined by the value Δ,

$$\Delta\approx a\epsilon\omega_1^{3/2}\exp(-\pi\omega_1/2) . \tag{12}$$

In this part we consider the soliton motion between two potential walls with regard to the damping. The account of the damping leads to the addition into Eq. (1) of the perturbation of type $\Gamma\phi_t$, where $\Gamma\ll1$ is the damping constant. Performing the calculation analogous to the one given above, we define for $M_\Gamma(t_0)$ the following expression:

$$M_\Gamma(t_0)=M(t_0)-\Gamma L\sqrt{2\epsilon} . \tag{13}$$

It is shown from (13) the random motion may exist at the conditions

$$\Gamma<\Delta/L\sqrt{2\epsilon} .$$

Let us consider the case when the soliton moves between two walls

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\epsilon[\delta(x-L-a\sin(\omega t))+\delta(x+L+a\sin(\omega t+\psi))]\sin\phi , \tag{14}$$

where ψ is the difference of the phases of the oscillating walls. The calculation shows that the Melnikov function

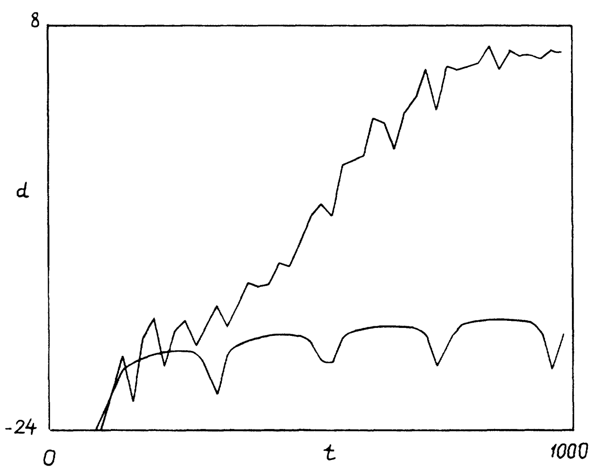

FIG. 1. Dependence $\ln D(t)$, where $D(t)$ is the distance between two trajectories versus time: (a) $\omega=0.6$, $a=0.1$, $v_{01}=0.211\,519\,3$, $v_{02}=0.211\,519\,4$; (b) $\omega=1$, $a=0.1$, $v_{01}=0.151$, $v_{02}=0.151\,000\,1$.

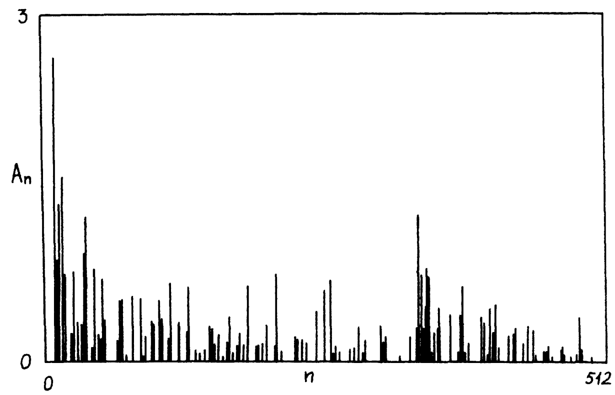

FIG. 2. Fourier transform of v^2 in chaotic region. A_n are the coefficients of Fourier series, $\omega_n = 2\pi n/N$, and $n = 1\ldots N, N=512$.

is equal to

$$M(t_0) = 2\epsilon a \omega_1^{3/2} \exp(-\pi\omega_1/2) \times \sin(\omega_1 L + \psi/2) \sin(\omega t_0 - \psi/2) .$$

As followed, the chaos arises at the next condition

$$\omega_1 L + \psi/2 \neq n\pi, \quad (n = 0, 1, 2, \ldots) .$$

Below we briefly discuss the numerical modeling results. We perform numerical integration of the (3) and (4) system at $\omega = 0.6$, $a = 0.1$, and $\epsilon = 0.1$. We have used the usual methods finding of the stochastical motion: a calculation of the local instabilities of the orbits and spectral power density for v^2.

Begin with the time evolution of distance between closed trajectories

$$D(t) = \{[\xi_1(t) - \xi_2(t)]^2 + [v_1(t) - v_2(t)]^2\}^{1/2} . \quad (15)$$

The results for the initial distance $D(0) = 10^{-7}$ and the times $\sim 10^3$ are presented in Fig. 1. It is shown that $\ln D(t)$ grows; this corresponds to the exponential divergence of trajectories.

We find the Fourier spectrum for v^2, which is shown in Fig. 2, is continuum. All the above-mentioned numerical results lead us to the conclusion that the regions of chaotic motion of solitons exist. We also observed the growth of the stochastic layer width at ω decreasing.

[1] I. Aranson, K. A. Gorshkov, and M. I. Rabinovich, Zh. Eksp. Teor. Fiz. **84**, 929 (1984).

[2] F. Kh. Abdullaev, S. A. Darmanyan, and B. A. Umarov, Phys. Lett. **108A**, 51 (1985).

[3] A. R. Bishop, *Dynamical Problem in Soliton Systems* (Springer-Verlag, Heidelberg, 1987).

[4] G. M. Zaslavsky and B. V. Chirikov, Dokl. Akad. Nauk SSSR **159**, 306 (1964).

[5] A. Lichtenberg and M. Liberman, *Regular and Stochastic Dynamics* (Springer, Berlin, 1984).

[6] M. Scgeneman *et al.*, Phys. Rev. Lett. **50**, 74 (1983).

[7] A. C. Scott and D. W. McLaughlin, Phys. Rev. **18A**, 1652 (1978).

[8] V. K. Melnikov, Trans. Moscow Math. Soc. **12**, 1 (1963).